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Improved residual method for approximating Bratu problem

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Abstract

In this study, firstly, the residual method, which was developed for initial value problems, is improved to find unknown coefficients without requiring for any system solution. Later, the adaptation of improved residual method is given to find approximate solutions of boundary value problems. Finally, the method improved and adapted for boundary value problems is used to find both critical eigenvalue and eigenfunctions of the one-dimensional Bratu problem. The most significant advantage of the method is finding approximate solutions of nonlinear problems without any linearization or solving any system of equations. Error analysis of the adapted method is given and an upper bound on the approximation error is derived for the eigenfunctions. The numerical results obtained are compared with the theoretical findings. Comparisons and theoretical observations show that the improved and adapted method is very convenient and successful in solving boundary value problems and eigenvalue problems approximately with high accuracy.

Keywords. Bratu Problem, Bezier curves, Residual method, Error analysis.
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1. Introduction

Consider the following boundary value problem

$$\begin{cases} y'' + \lambda e^y = 0, & 0 < x < 1, \\ y(0) = 0, \quad y(1) = 0, \end{cases} \tag{1.1}$$

which is referred to as the one-dimensional Bratu problem. This nonlinear eigenvalue problem has known two-bifurcated solution for $\lambda < \lambda_c$ and only one solution for $\lambda = \lambda_c$, but no solution for $\lambda > \lambda_c$.

The Bratu problem

$$\begin{cases} \Delta u + \lambda e^u = 0, \quad \Omega : \{(x,y) \in 0 \le x \le 1, 0 \le y \le 1\}, \\ u = 0, \quad \partial\Omega, \end{cases} \tag{1.2}$$

is a nonlinear elliptical partial differential equation and comes out in a various fields such as the fuel ignition model found in thermal combustion theory [8], the Chandrasekhar model of the expansion of the universe [7], chemical reactor theory, and nanotechnology. The problem arises via the study of the solid fuel ignition model

$$\begin{aligned} v_t &= \Delta v + \lambda(1-\epsilon v)^m e^{v/(1+\epsilon v)} \\ v &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.3}$$

where λ is a Frank-Kamanetskii parameter, v is a dimensionless temperature and $1/\epsilon$ is the activation energy. Non-trivial solutions of the Bratu problem (1.2) arise as steady-state solutions of the solid fuel ignition model within the approximation $\epsilon \le 1$. The brief history and the importance of the Bratu problem is given in [12] by Jacobsen and Schmitt. The problem is also a nonlinear eigenvalue problem that is often used as a comparison tool for numerical methods owing to the bifurcation nature of the solution for $\lambda < \lambda_c$. Various numerical techniques such as Taylor's

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decomposition method [3], Mickens Finite difference scheme [5], the special case of Hermite interpolation technique [13], the Taylor wavelets method [14], the collocation method based on Genocchi polynomials [10], modified perturbation method [1], weighed residual method [15], iterative differential quadrature method [17], an iterative numerical scheme based on the Newton-Raphson-Kantorovich Method [18], the successive differentiation method [19], Chebyshev wavelet [20] have been adapted independently to overcome the Bratu model numerically.

In [6], the exact solution of one dimensional Bratu problem (1.1) is given as

$$y(x) = -2\ln\left[\frac{\cosh((x-\frac{1}{2})\frac{\theta}{2})}{\cosh(\frac{\theta}{4})}\right], \tag{1.4}$$

where θ and λ satisfy

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right). \tag{1.5}$$

Equation (1.5) has two solutions for $0 < \lambda < \lambda_c$, only one solution for $\lambda = \lambda_c$ which is called critical eigenvalue, no solution for $\lambda > \lambda_c$. The mentioned critical eigenvalue λ_c solves

$$1 = \sqrt{2\lambda_c}\sinh\left(\frac{\theta_c}{4}\right)\frac{1}{4}. \tag{1.6}$$

From (1.5) and (1.6), one may obtain the following

$$\theta/4 = \coth(\theta/4) \quad \text{and} \quad \theta_c = 4.798714561030. \tag{1.7}$$

As a result, from (1.6), the exact value of critical eigenvalue λ_c is evaluated as

$$\lambda_c = \frac{8}{\sinh^2(\theta_c/4)} = 3.51383071912516.$$

In this work, the residual method is improved. It is based on the construction of the approximate solution of the differential equation using Bezier curves which are used in various studies, such as, [4, 9, 11]. We also propose the improved residual method to approximate the critical eigenvalue and eigenfunctions of the Bratu problem. The residual method was first developed in [2] to approximate initial value problems. In the method, approximate solutions are written as Bezier curves. Then, the unknown control points are calculated from the system that is encountered by minimizing the Taylor's series expansion of the residual function at certain points. In this study, it is aimed to calculate the control points without the need for any system solution. For this purpose, an explicit formula for control points is obtained. Then the improved method is adapted to boundary value problems and used to find approximate eigenfunctions and critical eigenvalue of the Bratu problem. The most significant advantage of the method is to find approximate results of nonlinear problems without using any linearization or solving any system of equations.

A brief description of the proposed method for initial value problems, improvement and adaptation of the method for boundary value problems is given in section 2. The computation of eigenvalues and eigenfunctions is given in section 3. The upper bound of the error caused by the proposed method for boundary value problems is given in section 4. Numerical results and comparisons for different λ values are presented in section 5. In conclusion, we summarize our study and present our suggestions regarding future works.

2. Improved Residual Method for Boundary Value Problems

2.1. **Improvement of residual method.** In this section, we first give a brief description of the residual method developed for initial value problems in [2]. Consider the non-linear initial value problems

$$y'' = F(x, y, y'), \tag{2.1}$$

with the initial conditions

$$y(a) = \alpha, \quad y'(a) = \beta, \tag{2.2}$$

where $F \in C^{n-2}[a,b] \times C(D_1) \times C(D_2)$, D_1, and D_2 are the closed intervals in $\mathbb{R}$, α and β are finite constants and n is the degree of Bézier curves.

In the beginning, the following initial value problems for each subinterval S_i are obtained as

$$y_i''(x) = F(x, y_i(x), y_i'(x)), \quad x \in S_i = [a_{i-1}, a_i], \quad \text{for} \quad 1 \le i \le N, \tag{2.3}$$

$$\text{and} \quad y_1(a_0) = \alpha, \qquad y_1'(a_0) = \beta, \tag{2.4}$$

$$y_i(a_{i-1}) = y_{i-1}(a_{i-1}), \qquad y_i'(a_{i-1}) = y_{i-1}'(a_{i-1}), \tag{2.5}$$

for $2 \le i \le N$, by dividing the interval $[a, b]$ into N equally spaced subintervals $S_i = [a_{i-1}, a_i]$, where $a_i = a + ih$, $i = 0, 1, ..., N$, $h = (b - a)/N$ and N is a positive integer. It can be seen from (2.5) that the derivative of piecewise function created by the solutions obtained in all sub-intervals and itself are continuous. The mentioned funtion will be the solution of the initial value problem (2.1) and (2.2), since it satifies both the equation and conditions of the initial value problem.

In each subsequent interval S_i, the approximate solution is constructed as nth degree Bézier curve

$$u_i(x) = \sum_{j=0}^{n} c_j^i B_j^n\left(x; [a_{i-1}, a_i]\right), \tag{2.6}$$

where

$$B_j^n\left(x; [a_{i-1}, a_i]\right) = \binom{n}{j} \frac{1}{h^n} (x - a_{i-1})^j (a_i - x)^{n-j},$$

are the Bernstein polynomials and c_j^i are $(n+1)$ unknown control points over each subinterval S_i to be determined later. Since $u_i(x)$ are the approximate solutions of $y_i(x)$, they must satisfy the initial conditions (2.4) and (2.5), that is,

$$u_1(a_0) = \alpha, \qquad u_1'(a_0) = \beta, \tag{2.7}$$

$$u_i(a_{i-1}) = u_{i-1}(a_{i-1}), \qquad u_i'(a_{i-1}) = u_{i-1}'(a_{i-1}), \tag{2.8}$$

for $2 \le i \le N$. From the properties of Bézier curves, equations (2.7) and (2.8) become

$$\begin{aligned} &c_0^1 = \alpha, \quad c_1^1 = \beta \frac{h}{n} + \alpha, \\ &c_0^i = c_n^{i-1}, \quad c_1^i = 2c_n^{i-1} - c_{n-1}^{i-1}, \end{aligned} \tag{2.9}$$

for $i = 2, \ldots, N$, which means we have $(n-1)$ unknown control points for each S_i.

If we write the approximate solutions obtained in each sub-interval as a piecewise function as follows

$$u(x) = \begin{cases} u_1(x), & x \in S_1, \\ u_2(x), & x \in S_2, \\ \vdots \\ u_N(x), & x \in S_N, \end{cases} \tag{2.10}$$

we get an approximate solution for the initial value problem (2.1) and (2.2).

When the approximate solutions (2.6) are substituted into the differential equation (2.3), for $i = 1, \ldots, N$, we have the piecewise residual function

$$R(x) = R_i(x), \qquad x \in S_i, \quad \text{where} \quad R_i(x) = u_i''(x) - F(x, u_i(x), u_i'(x)), \quad x \in S_i.$$

The aim is to identify unknown control points c_j^i that will make sufficiently differentiable residual function $R_i(x)$ minimum in S_i. For this, the first $(n-1)$ terms in Taylor's series expansion of $R_i(x)$ at $x = a_{i-1}$ are forced to be zero. So, the unknown control points c_j^i are obtained from

$$R_i^{(k)}(a_{i-1}) = 0, \quad \text{for} \quad k = 0, \dots, n-2, \quad \text{which yields} \tag{2.11}$$

$$R_i^{(k)}(a_{i-1}) = u_i^{(k+2)}(a_{i-1}) - F^{(k)}\left(a_{i-1}, u_i(a_{i-1}), u_i'(a_{i-1})\right) = 0, \quad k = 0, \dots, n-2. \tag{2.12}$$

From derivative property of Bézier curves at the end points, (2.12) becomes

$$\frac{n(n-1)\dots(n-k-1)}{h^{k+2}}\Delta^{k+2}c_0^i - F^{(k)}\left(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\right) = 0, \tag{2.13}$$

where

$$F^{(k)}(x,y,z) = \frac{\partial^k}{\partial x^k}F(x, y(x), z(x)) \quad \text{and} \quad \Delta^{k+2}c_0^i = \sum_{j=0}^{k+2}\binom{k+2}{j}(-1)^{k+2-j}\, c_j^i.$$

The $(n-1)$ unknown control points are determined by solving linear system of equations (2.13) for $k = 0, \dots, n-2$ in [2]. What has been mentioned so far in this subsection is a summary of the residual method developed in [2]. Now we will discuss about the improvements made in the residual method in this study.

Equation (2.13) is obtained using endpoint property of Bézier curves and the control points are obtained forcing $(n-1)$ terms of Taylor's series expansion of residual function to be zero at the left endpoint of the subinterval S_i. Similarly, we repeat this process for the right endpoint to have the following linear system of equations

$$\frac{n(n-1)\dots(n-k+1)}{h^k}\Delta^k c_{n-k}^i - F^{(k-2)}\left(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i)\right) = 0, \tag{2.14}$$

where $F^{(k)}(x,y,z) = \dfrac{\partial^k}{\partial x^k}F(x, y(x), z(x))$ and $\Delta^k c_{n-k}^i = \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\, c_{n-k+j}^i$. In this work, instead of solving the control points from the linear system of equations as in [2], we formulate the control points c_k^i explicitly in terms of c_0^i and c_1^i or c_{n-1}^i and c_n^i as in Theorem 2.1.

Theorem 2.1. *Let n be the degree and c_k^i be the control points of the Bézier curves (2.6) that are the approximate solutions of (2.3) and (2.8) in interval S_i and $F(x,y,z)$ be the function given in (2.1), then*

$$c_k^i = kc_1^i - (k-1)c_0^i + \sum_{j=2}^{k}\binom{k}{j}\frac{(n-j)!}{n!}h^j F^{(j-2)}(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)), \quad k = 2, 3, ..., n \tag{2.15}$$

$$c_k^i = (n-k)c_{n-1}^i - (n-k-1)c_n^i + \sum_{j=2}^{n-k}(-1)^j h^j\binom{n-k}{j}\frac{(n-j)!}{n!}F^{(j-2)}\left(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i)\right), \tag{2.16}$$

for $k = 0, 1, ..., n-2$ and $i = 1, 2, \dots, N$, where $F^{(k)}(x, y(x), z(x)) = \dfrac{\partial^k}{\partial x^k}F(x, y(x), z(x))$ and h is the stepsize.

Proof. To prove this theorem, first we prove the following equations by using some classical binomial identities.
i)

$$\sum_{j=1}^{k}\binom{k}{j-1}(-1)^{k+1-j} = -1, \quad \text{for any } k \geq 1. \tag{2.17}$$

ii)

$$\sum_{j=m}^{k} \binom{k+1}{j}\binom{j}{m}(-1)^{k+1-j} = -\binom{k+1}{m}, \quad \text{for } 0 \le m \le k. \tag{2.18}$$

iii)

$$\sum_{j=2}^{k}(-1)^{j}\binom{k-1}{j-1} = 1, \quad \text{for any } k \geqslant 2. \tag{2.19}$$

iv)

$$\sum_{j=2}^{k+2-p}(-1)^{j}\binom{k+1}{j-1}\binom{k+2-j}{p} = \binom{k+1}{p}, \quad \text{for any } p = 0, 1, ..., k. \tag{2.20}$$

We prove equations (2.15) and (2.16) by induction on k. For $k = 2$ by using (2.13) and (2.14), we obtain

$$c_2^i = \frac{(n-2)!}{n!}h^2 F\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) + 2c_1^i - c_0^i,$$

$$c_{n-2}^i = 2c_{n-1}^i - c_n^i + \sum_{j=2}^{2}(-1)^j h^j F^{(j-2)}\binom{2}{j}\frac{(n-j)!}{n!},$$

which show that given equatins hold for $k = 2$. By the induction hypothesis, we suppose that the equations hold for k. Then we need to prove them for $k+1$. The inductive steps are proved after huge calculations using the above identities (2.17), (2.18), (2.19), and (2.20) respectively. □

2.2. **Adaptation of improved residual method to boundary value problems.** In this subsection, by following the steps given in [16], we will give an adaptation of the improved residual method to the boundary value problems in the form

$$y''(x) = F(x, y(x), y'(x)), \tag{2.21}$$

with the boundary conditions

$$y(a) = \alpha, \qquad y(b) = \beta, \tag{2.22}$$

where $F \in C^{n-2}([a,b] \times D_1 \times D_2)$, D_1 and D_2 are closed intervals in $\mathbb{R}$, a, b, α, β are finite constants and n be the degree of the Bézier curves.

Consider the following initial value problem

$$\begin{aligned} y'' &= F(x, y, y'), \quad x \in [a,b], \\ y(a) &= \alpha, \qquad y'(a) = s, \end{aligned} \tag{2.23}$$

where s is the root of equation

$$y(b; s) - \beta = 0. \tag{2.24}$$

Thus, the solutions of the initial value problem (2.23) that satisfy the condition (2.24) are also the solutions of the boundary value problem (2.21) and (2.22). In this subsection, it is aimed to find the approximate value s denoted by s^* and approximate solution $u(x; s^*)$ of $y(x; s)$ using improved residual method. To achieve this goal, the improved residual method is applied to (2.23) with high degree approximating polynomial and large step-size $h < 1$. So, the unknown coefficients c_k^i of the approximate polynomial $u(x; s)$ are computed in terms of the parameter s. In order to be an approximate solution for the boundary value problem (2.21) and (2.22), the approximate solution $u(x; s)$, which is obtained depending on the parameter s, is substituted into (2.24),

$$u(x; s) - \beta = 0. \tag{2.25}$$

Equation (2.25) is solved approximately using Newton's method with an initial value α. Thus, the approximate value s^* of s is found. Finally, the following perturbed initial value problem

$$\begin{aligned} &y''(x) = F(x, y(x), y'(x)), \qquad x \in [a, b], \\ &y(a) = \alpha \qquad y'(a) = s^*, \end{aligned} \tag{2.26}$$

where s^* is the approximate root of (2.25), is obtained. Now, the improved residual method can be applied to (2.26) in order to find more accurate approximate solution of (2.21) and (2.22).

3. Computation of Eigenvalues and Eigenfunctions of the Bratu Problem by Improved Residual Method

We divide the interval $[0, 1]$ into two equal pieces [0,1/2] and [1/2,1] to find approximate critical eigenvalue and initial slopes of Bratu problem. Then, we define the following problems

$$\begin{aligned} &u_1'' + \lambda e^{u_1} = 0, \quad 0 < x < 1/2, \\ &u_1(0) = 0 \quad \text{and} \quad u_1'(0) = s_1, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &u_2'' + \lambda e^{u_2} = 0, \quad 1/2 < x < 1, \\ &u_2(1) = 0 \quad \text{and} \quad u_2'(1) = s_2. \end{aligned} \tag{3.2}$$

By the continuity of the approximate solution, we have

$$u_1(1/2, s_1) = u_2(1/2, s_2), \tag{3.3}$$

which implies

$$c_0^2 - c_n^1 = 0, \tag{3.4}$$

where c_n^1 and c_0^2 are the last and the first control points of $u_1(x, s_1)$ and $u_2(x, s_2)$, respectively.

Using $c_0^1 = 0$, $c_1^1 = \dfrac{s_1}{2n}$ and Theorem 2.1, we have

$$c_n^1 = \frac{s_1}{2} + \sum_{k=2}^{n} \frac{1}{k!}(1/2)^k F^{(k-2)}(0, 0, s_1). \tag{3.5}$$

Similarly, using $c_n^2 = 0$, $c_{n-1}^2 = -\dfrac{s_2}{2n}$ and Theorem 2.1, we get

$$c_0^2 = -\frac{s_2}{2} + \sum_{k=2}^{n} (-1)^k \frac{1}{k!}(1/2)^k F^{(k-2)}(1, 0, s_2). \tag{3.6}$$

Substituting equations (3.5) and (3.6) into equation (3.4), we obtain

$$c_0^2 - c_n^1 = -\frac{s_1 + s_2}{2} + \sum_{k=2}^{n} \frac{1}{k!}(1/2)^k \Big((-1)^k F^{(k-2)}(1, 0, s_2) - F^{(k-2)}(0, 0, s_1) \Big) = 0. \tag{3.7}$$

For the difference in the above sum, we will use the following lemmas which are given and proved in [3].

Lemma 3.1. *For $j = 1, \ldots, n$, let $F^{(j)}(x, y, z)$ satisfies the recurrence relation*

$$F^{(j)}(x, y, z) = z \frac{\partial F^{(j-1)}(x, y, z)}{\partial y} - \lambda e^y \frac{\partial F^{(j-1)}(x, y, z)}{\partial z}, \tag{3.8}$$

with $F^{(0)}(x, y(x), z(x)) = -\lambda e^y$ and $z(x) = y'(x)$. Then

$$F^{(j-1)}(x, y(x), z(x)) = \sum_{i=0}^{b_j} (-1)^{i+1} a_{j,i} \lambda^{i+1} (e^y)^{i+1} z^{j-2i-1}, \tag{3.9}$$

where $b_j = \lfloor \frac{j-1}{2} \rfloor$ *and* $a_{j,i} = \begin{cases} 1, & i = 0 \\ (i+1)a_{j-1,i} + (j-2i)a_{j-1,i-1} & 1 \le i \le b_{j-1}\,. \\ 0 & else. \end{cases}$

Lemma 3.2. *If $F^{(j)}(x,y,z)$ satisfies (3.9), then for $j = 2,\dots,n+1$, the following relations hold:*
(i)

$$(-1)^k F^{(k-2)}(1,0,s_2) - F^{(k-2)}(0,0,s_1) = (s_1+s_2)(-1)^k \sum_{i=0}^{b_{k-1}} (-1)^{i+1} a_{k-1,i} \lambda^{i+1} \left(\sum_{j=0}^{k-2i-3} (-1)^j s_2^{k-2i-3-j} s_1^j \right), \tag{3.10}$$

for any fixed s_1 and s_2,
(ii)

$$(-1)^k F^{(k-1)}(1,0,s_1) - F^{(k-1)}(0,0,s_2) \quad = -2F^{(k-1)}(0,0,s_2), \quad for \quad s_2 = -s_1. \tag{3.11}$$

So (3.7) becomes

$$c_0^2 - c_n^1 = -\frac{s_1+s_2}{2} + (s_1+s_2) \times \sum_{k=2}^{n} \frac{1}{k!}(1/2)^k \left(-1)^k \sum_{i=0}^{b_{k-1}} (-1)^{i+1} a_{k-1,i} \lambda^{i+1} \left[\sum_{j=0}^{k-2i-3} (-1)^j s_2^{k-2i-3-j} s_1^j \right] \right),$$

which yields

$$c_0^2 - c_n^1 = -(s1+s2) \times \left(\frac{1}{2} + \sum_{k=2}^{n} \frac{1}{k!}(1/2)^k (-1)^{k-1} \sum_{i=0}^{b_{k-1}} (-1)^{i+1} a_{k-1,i} \lambda^{i+1} \left[\sum_{j=0}^{k-2i-3} (-1)^j s_2^{k-2i-3-j} s_1^j \right] \right).$$

Therefore to satisfy the equation (3.4) for all n, we must have $s_1 + s_2 = 0$ that is;

$$s_1 = -s_2. \tag{3.12}$$

To satisfy the continuity of the first derivative of the approximate solution, we have

$$u_1'(1/2) = u_2'(1/2),$$

which implies that

$$c_n^1 - c_{n-1}^1 = c_1^2 - c_0^2. \tag{3.13}$$

We need to compute c_{n-1}^1 and c_1^2 by using $s_1 = -s_2$, (3.11) and Theorem 2.1 for c_{n-1}^1 and c_1^2. Thus equation (3.13) becomes

$$s_1 + \sum_{k=1}^{n-1} (\frac{1}{k!})(1/2)^{k+1}(2F^{(k-1)}(0,0,s_1)) = 0. \tag{3.14}$$

We name the left-hand side of (3.14) as

$$G(s_1,\lambda) = s_1 + \sum_{k=1}^{n-1} (\frac{1}{k!})(1/2)^k F^{(k-1)}(0,0,s_1). \tag{3.15}$$

Drawing the implicit equation $G(s_1,\lambda) = 0$ gives Figure 1 which demonstrates that λ has a maximum value. In order to find the maximum value, the following equation must be satisfied

$$\frac{d\lambda}{ds_1} = -\frac{\partial G/\partial s_1}{\partial G/\partial \lambda} = 0, \quad \text{that is} \quad \frac{\partial G}{\partial s_1} = 0.$$

The nonlinear equations

$$G(s_1,\lambda) = 0, \quad \text{and} \quad \frac{\partial G}{\partial s_1} = 0,$$

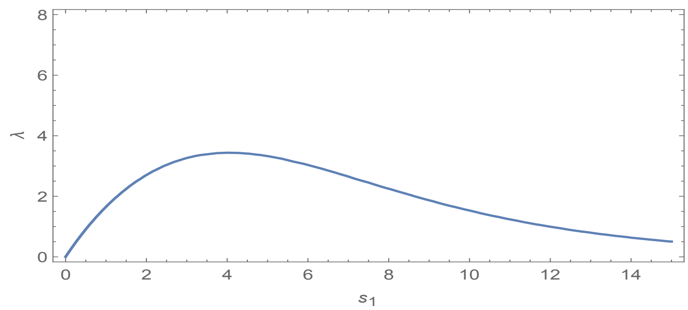

Figure 1. Graph of the equation $G(s_1, \lambda) = 0$.

are solved approximately by Newton's method, so the critical eigenvalue $\lambda_c = 3.51383$ and the corresponding initial value $s_1 \simeq y'(0)$ are found approximately.

Table 1. The approximate initial slopes s_1^1 and s_1^2 corresponding to various $\lambda \leq \lambda_c$ obtained from $G(s_1, \lambda) = 0$

λ	s_1^1	s_1^2
0.5	0.261277	13.008
1	0.549353	10.8467
2	1.24822	8.26876
3	2.3196	6.10338
3.513830719225065	4.	-

From Figure 1 and Table 1, it can be seen that there isn't any s_1 for $\lambda > \lambda_c$, there is a unique solution corresponding to the initial value $s_1^1 = s_1$ for $\lambda = \lambda_c$, and there are two solutions corresponding to the initial values s_1^1 and s_1^2 for $\lambda < \lambda_c$ as in the theoretical knowledge of the Bratu problem.

The approximate initial slopes of the Bratu problem for the corresponding eigenvalues $\lambda \leq \lambda_c$ are given in Table 1.

4. Error Analysis of Improved Residual Method

In this section, using some lemmas and theorems from [2, 3, 16], we emphasize that to use approximate value for initial slope does not change order of convergence of the improved residual method. At the end of this section, we give an upper bound for the error $|y(x, s) - u(x, s^*)|$, where $y(x, s)$ represents the exact solution of the Bratu problem and $u(x, s^*)$ is the corresponding approximate solution obtained using the improved residual mathod.

Lemma 4.1. *For $j = 1, \ldots, n$, let $F^{(j)}(y, z)$ satisfies the recurrence relation*

$$F^{(j)}(y, z) = z\frac{\partial F^{(j-1)}(y, z)}{\partial y} - \lambda e^y \frac{\partial F^{(j-1)}(y, z)}{\partial z}, \tag{4.1}$$

with $F^{(0)}(y(x), z(x)) = -\lambda e^y$ and $z(x) = y'(x)$. Then

$$|F^{(j)}(y(x), z(x))| \leq \xi M^{j+1-b_{j+1}} \frac{1}{|M-1|}, \tag{4.2}$$

where $M = \max_{(y,z)\in D}\{|F^{(0)}(y, z)|, |z|\}$, D is $2-$dimensional box in R^2 and $\xi = max\{a_{j,i}\}$.

Proof of this lemma is given in [3] in the proof of Lemma 2.

Lemma 4.2. *Let the function $F(x, v, z)$ in (2.21) satisfy $0 < \frac{\partial F}{\partial v} < L_1$ for some positive constant L_1. Then*

$$|s^* - y'(a)| \leq \widetilde{K} h^n, \tag{4.3}$$

where $\widetilde{K}$ is a positive constant, y is the exact solution of the boundary value problem (2.21) and (2.22), s^* is the approximate root of (2.25) and h is the step size of the improved residual method.

This lemma can be proved using Theorem A.2.1, Lemma A.2.2, Theorem A.2.3, and Theorem A.2.4 given in [16].

Before giving the error bound of the improved residual method, we define the following initial value problems

$$y''_{1,2} + \lambda e^{y_{1,2}} = 0, \quad 0 < x < 1, \tag{4.4}$$

with the the conditions

$$y_1(0) = 0, \quad y'_1(0) = s, \tag{4.5}$$

and

$$y_2(0) = 0, \qquad y'_2(0) = s^*, \tag{4.6}$$

where s^* is the approximate value of s with the condition (4.3).

The proofs of the following lemma and theorem are given in [2].

Lemma 4.3. *Let $y_1(x)$ be the exact solution of (4.4)-(4.5) and $u_1(x)$ be the approximate solution of (4.4)-(4.6) on $x \in [0, h]$ with $|s - s^*| \leq \widetilde{K} h^n$, where s^* is the approximate root of (2.25) and $\widetilde{K}$ is a positive constant. Then*

$$|y_1(x) - u_1(x)| \leq \widetilde{K} h^{n+1} + O(h^{n+2}), \tag{4.7}$$

and

$$|y'_1(x) - u'_1(x)| \leq (n+1)\overline{K} h^n + O(h^{n+1}), \tag{4.8}$$

where $\overline{K} = \widetilde{K} + K$, $K = \dfrac{1}{(n+1)!}\xi \dfrac{M^{(n/2)+1}}{|M-1|}$.

Theorem 4.4. *Let $y(x)$ be the exact solution of the second order non-linear boundary value problem (1.1), and $u(x; s^*)$ be the nth degree approximate solution of (4.4)-(4.6), then*

$$|y(x) - u(x; s^*)| \leq \overline{M} h^{n-1}, \qquad x \in [a, b], \tag{4.9}$$

where s^* is the approximate root of (2.25) with the condition (4.3), $\overline{M} = \overline{K}(n+1)$ and $\overline{K}$ is defined in Lemma 4.3.

5. Numerical Results

The improved residual method is applied to the one dimensional Bratu Problem. In this section, we give graphs of the errors of the obtained numerical results. We also give the tables of the maximum error moduli and observed orders of the errors. From tables and figures, it can be seen that proposed method has high order accuracy for large step size. It can be also seen that the obtained numerical orders are well confirmed with the theoretical findings.

The observed orders are computed using the following formula

$$ord(h) = \frac{\log \frac{e_h}{e_{h/2}}}{\log 2}, \tag{5.1}$$

where e_h and $e_{h/2}$ are the maximum error moduli of the global errors when the problem is solved with stepsize h and $h/2$, respectively.

TABLE 2. Maximum error moduli and observed errors for $\lambda = 1$ and $s_1^1 = 0.549353$.

	n=2	n=3	n=4	n=5
N=8	2.96128×10^{-3}	2.6025×10^{-4}	1.48175×10^{-5}	5.06533×10^{-7}
N=16	1.49719×10^{-3}	7.17721×10^{-5}	1.93077×10^{-6}	3.96614×10^{-8}
N=32	7.50268×10^{-4}	1.8823×10^{-5}	2.45018×10^{-7}	2.52492×10^{-9}
		Observed Orders		
ord(1/8)	0.983959	1.8584	2.94005	3.67485
ord(1/16)	0.996781	1.93092	2.97822	3.97343

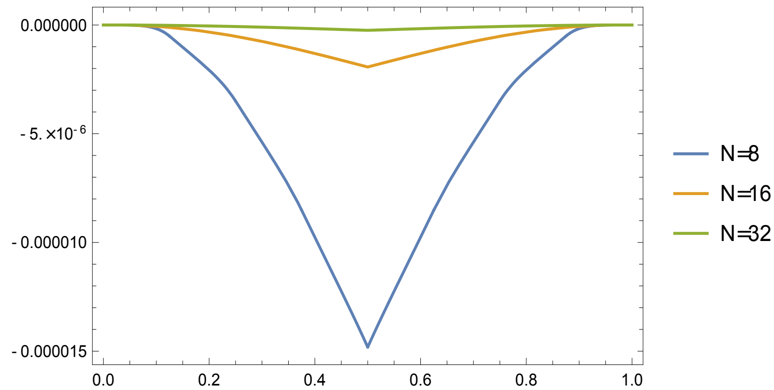

FIGURE 2. Errors for s_1^1 when $\lambda = 1$.

TABLE 3. Maximum error moduli and observed errors for $\lambda = 1$ and $s_1^2 = 10.8466$.

	n=2	n=3	n=4	n=5
N=8	0.433665	0.0839156	5.10122×10^{-3}	3.48684×10^{-3}
N=16	0.226312	0.0209262	6.19774×10^{-4}	4.87454×10^{-4}
N=32	0.113062	0.00491807	5.83269×10^{-5}	6.85824×10^{-5}
		Observed Orders		
ord(1/8)	0.938268	2.00362	3.04103	2.83858
ord(1/16)	1.0012	2.08915	3.40951	2.82936

TABLE 4. Maximum error moduli and observed errors for $\lambda = 3$ and $s_1^1 = 2.31960$.

	n=2	n=3	n=4	n=5
N=8	0.0453978	9.03418×10^{-4}	6.26397×10^{-4}	4.27077×10^{-5}
N=16	0.022926	3.56031×10^{-4}	8.81698×10^{-5}	2.2744×10^{-6}
N=32	0.0114629	1.07455×10^{-4}	1.1604×10^{-5}	1.16448×10^{-7}
		Observed Orders		
ord(1/8)	0.985636	1.34339	2.82872	4.23094
ord(1/16)	1.00002	1.72826	2.92566	4.28772

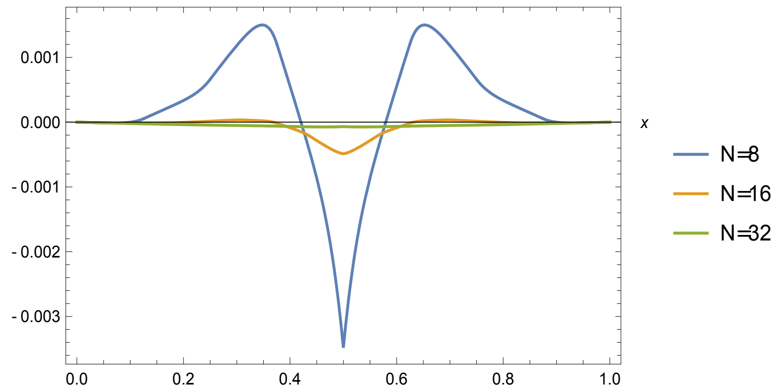

FIGURE 3. Errors for s_1^2 when $\lambda = 1$.

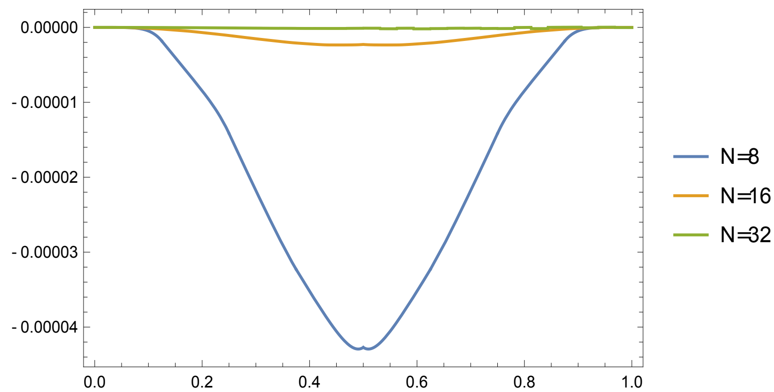

FIGURE 4. Errors for s_1^1 when $\lambda = 3$.

TABLE 5. Maximum error moduli and observed errors for $\lambda = 3$ and $s_1^2 = 6.10338$.

	n=2	**n=3**	**n=4**	**n=5**
N=8	0.218647	0.0204631	2.19945×10^{-3}	1.07686×10^{-3}
N=16	0.111056	0.00448602	4.82462×10^{-4}	7.73086×10^{-5}
N=32	0.0553462	0.00101242	7.34435×10^{-5}	4.73904×10^{-6}
Observed Orders				
ord(1/8)	0.977321	2.18951	2.18865	3.80006
ord(1/16)	1.00473	2.14763	2.71571	4.02796

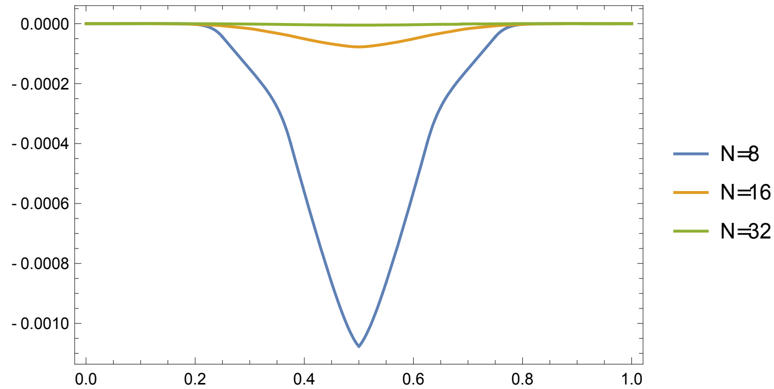

FIGURE 5. Errors for s_1^2 when $\lambda = 3$.

TABLE 6. Maximum error moduli and observed errors for $\lambda = 3.513830719$ and $s = 4$

	n=2	**n=3**	**n=4**	**n=5**
N=8	0.11534	4.95501×10^{-3}	1.69011×10^{-3}	3.20816×10^{-4}
N=16	0.0583163	1.12227×10^{-3}	2.68653×10^{-4}	2.02561×10^{-5}
N=32	0.0291081	2.66845×10^{-4}	3.71078×10^{-5}	1.23786×10^{-6}
Observed Orders				
ord(1/8)	0.983921	2.14247	2.6533	3.98532
ord(1/16)	1.00248	2.07235	2.85595	4.03244

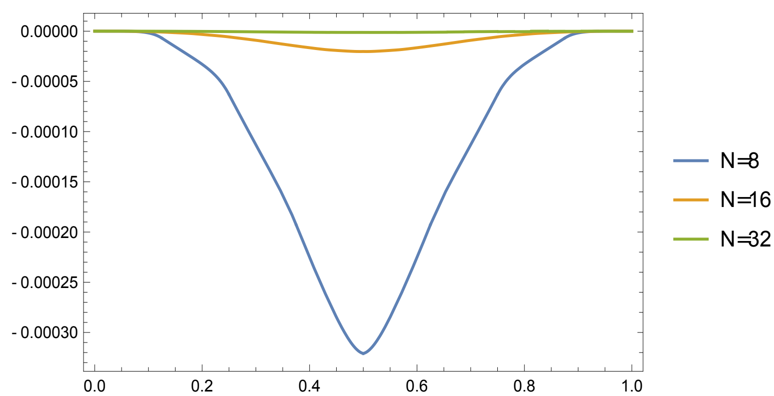

FIGURE 6. Errors for $\lambda_c = 3.513830719$.

6. CONCLUSION

In this work, residual method is improved to find approximate solutions of boundary value problems. The improved residual method is applied to solve one-dimensional Bratu problem numerically. The obtained numerical results are compared with the theoretical findings. Comparisons and theoretical observations show that the improved and adapted method is very convenient and successful in solving boundary value problems and eigenvalue problems approximately with high accuracy. The proposed method can be applied to singularly perturbed Sturm-Liouville eigenvalue problems.

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