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A pseudospectral Sinc method for numerical investigation of the nonlinear time-fractional Klein-Gordon and sine-Gordon equations

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Abstract

In this paper, a pseudospectral method is proposed for solving the nonlinear time-fractional Klein-Gordon and sine-Gordon equations. The method is based on the Sinc operational matrices. A finite difference scheme is used to discretize the Caputo time-fractional derivative, while the spatial derivatives are approximated by the Sinc method. The convergence of the full discretization of the problem is studied. Some numerical examples are presented to confirm the accuracy and efficiency of the proposed method. The numerical results are compared with the analytical solution and the reported results in the literature.

Keywords. Fractional differential equation, Nonlinear Klein-Gordon and sine-Gordon equations, Sinc operational matrices, Pseudospectral method, Convergence.

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1. Introduction

Differential equations of fractional order are the generalization of the classical integer-order differential equations. Fractional calculus is an important tool for modeling complex problems and has attracted the interest of the scientist in various areas [9, 14, 31, 34]. The key advantage of the fractional models over the classical models is their nonlocal properties. Here, we consider the following time-fractional nonlinear equation:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+\lambda_1 u(x,t)+\lambda_2 g\left(u(x,t)\right)=\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t), a\leq x\leq b, t\geq 0, \tag{1.1}$$

subject to initial conditions

$$u(x,0)=\phi_0(x), \frac{\partial u(x,0)}{\partial t}=\phi_1(x), a\leq x\leq b, \tag{1.2}$$

and the boundary conditions

$$u(a,t)=\psi_1(t), u(b,t)=\psi_2(t), t\geq 0, \tag{1.3}$$

where $1<\alpha\leq 2$, g, f are continuous functions and λ_1, λ_2 are some positive constants. The $\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}$ is the Caputo fractional derivative of order α with respect t expressed by

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}\frac{\partial^2 u(x,t)}{\partial s^2}ds, \tag{1.4}$$

where $\Gamma(.)$ is the Gamma function. When the nonlinear term is $g(u)=u^{\beta}$, (1.1) corresponds to the Klein-Gordon equation of fractional order with quadratic or cubic nonlinearity. The Klein-Gordon equation arises in many branches of sciences such as optics, fluid dynamics, DNA dynamics, quantum mechanics, solid mechanics and so on [3, 11, 29]. The (1.1) becomes the time fractional sine-Gordon equation when $g(u)=sin(u)$. The sine-Gordon equation arises in the various areas of physics such as classical lattice dynamics in the continuum media and the electromagnetic

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wave propagation [6, 7, 36]. Several numerical methods have been proposed to obtain the approximate solution of fractional Klein-Gordon and sine-Gordon equations in the literature. For example, we can mention to the shifted Legendre operational matrices [36], Haar wavelets method [8], the implicit radial basis function meshless method [5], the collocation method based on the combination of the Sinc function and the second kind shifted Chebyshev polynomials [22], Legendre spectral element method [12], local radial basis functions finite difference method [23], compact finite difference and DIRKN methods [17], homotopy-perturbation method [4] and so on. In recent years, the use of the Sinc methods in solving various type of differential equations have attracted the attention of many scientists and researchers [1, 2, 15, 16, 18–21, 24, 26, 27]. These methods have been used in solving many scientific and engineering problems such as fractional convection-diffusion equation [28], Thomas-Fermi equation [25], fractional order boundary value problems [30], and so on. The main objective of this work is to introduce a numerical method based on the Sinc approximation in the framework of the pseudospectral method for the nonlinear time-fractional Klein-Gordon and sine-Gordon equations. A finite difference scheme is used to discretize the Caputo time-fractional derivative, while the spatial derivatives are approximated by the Sinc method. In addition, the convergence of the full discretization of the problem is studied. The main advantages of the proposed method are as follows; first, it is easy to implement and program; second, it eliminates the treatment of the boundary conditions, using the basis functions which satisfy the boundary conditions exactly; thirdly, using the Sinc operational matrices we transform the main nonlinear problem into a linear system of algebraic equations; fourthly, it does not need any linearization process for nonlinear time-dependent problems. Some numerical examples are provided to confirm the applicability and accuracy of the proposed method. The numerical results are compared with the analytical solution and the reported results in the literature. The numerical results and comparisons exhibit that the proposed method is powerful and accurate.

2. Preliminaries

In this section, a review of the Sinc function and notation is presented. The reader can refer to [13, 32] for more details.

For any $-\infty < x < \infty$, the Sinc function is given by,

$$Sinc(x) = \begin{cases} \frac{sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0, \end{cases} \tag{2.1}$$

and the translated Sinc functions are given as follows for any $h > 0$

$$S(j,h)(x) = Sinc\left(\frac{x - jh}{h}\right), \qquad j = 0, \pm 1, \pm 2, ... \tag{2.2}$$

Lemma 2.1. *[13] Let $S(j,h)$ is defined as (2.2), then*

$$\delta_{jk}^{(0)} = S(j,h)(kh) = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{2.3}$$

and

$$\delta_{jk}^{(1)} = h\frac{d}{dx}\left(S(j,h)(x)\right)(kh) = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases} \tag{2.4}$$

and

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{dx^2}\left(S(j,h)(x)\right)(kh) = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k. \end{cases} \tag{2.5}$$

Theorem 2.2. *[33] Let f be defined on $\mathbb{R}$, and let Fourier transform $\tilde{f}$, be such that for some positive constant d,*

$$|\tilde{f}(y)| = O\left(e^{-d|y|}\right), \qquad y \to \pm\infty. \tag{2.6}$$

Then, as $h \to 0$,

$$\|f(x) - \sum_{j=-\infty}^{\infty} f(jh)S(j,h)\|_\infty = O\left(e^{-\frac{\pi d}{h}}\right). \tag{2.7}$$

Definition 2.3. Let D_s denote the infinite strip domain of width $2d, d > 0$ and $N^p(f, D_s)$ for $1 \le p < \infty$ is defined as follows:

$$N^p(f, D_s) \equiv lim_{y\to d^-}\left\{\left(\int_{-\infty}^{\infty}|f(t+iy)|^p dt\right)^{1/p} + \left(\int_{-\infty}^{\infty}|f(t-iy)|^p dt\right)^{1/p}\right\}.$$

Then consider $B^p(D_s)$ as the set of analytic functions such that

$$\int_{-d}^{d}|f(t+iy)|dy = O\left(|t|^a\right), t \to \pm\infty, 0 \le a < 1,$$

and

$$N^p(f, D_s) < \infty.$$

The exponential convergence of the truncated Sinc expansion is presented in the following theorem.

Theorem 2.4. *[13] Assume* $f \in B^p(D_s)$ $(p = 1$ *or* $2)$ *and that there are positive constant* α, β*, and* C *such that*

$$|f(x)| \le C\left\{\begin{array}{l} exp(\alpha|x|), x \in (-\infty, 0), \\ exp(-\beta|x|), x \in [0, \infty). \end{array}\right. \tag{2.8}$$

Then for $N = [|\frac{\alpha}{\beta}M + 1|]$ *and* $h = \sqrt{\pi d/(\alpha M)}$ *we have*

$$\|f - \sum_{j=-M}^{N} f(jh)S(j,h)\|_\infty \le K\sqrt{M}exp(-\sqrt{\pi d\alpha M}), \tag{2.9}$$

where K *is a constant depending on* f, p *and* d.

3. The numerical scheme

In this section, a finite difference scheme is combined with the Sinc method to discretize the nonlinear time-fractional Klein-Gordon and sine-Gordon equations (1.1)-(1.3). Before utilizing the Sinc approximation on the nonhomogeneous problems, we convert the nonhomogeneous boundary conditions to a homogeneous ones, since the Sinc functions satisfy the homogeneous conditions $\lim_{x\to\pm\infty} Sinc(x) = 0$. In fact, it eliminates the treatment of the boundary conditions, using the basis functions which satisfy the boundary conditions exactly. So, in order to utilize the Sinc method, we homogenize the boundary condition of (1.1)-(1.3). Let $H(x,t) = \frac{b-x}{b-a}\psi_1(t) + \frac{x-a}{b-a}\psi_2(t)$ $v(x,t) = u(x,t) - H(x,t)$ then (1.1)-(1.3) is convert to the following problem

$$\left\{\begin{array}{l} \frac{\partial^\alpha v(x,t)}{\partial t^\alpha} + \lambda_1 v(x,t) + \lambda_2 g\left(v(x,t) + H(x,t)\right) = \frac{\partial^2 v(x,t)}{\partial x^2} + h(x,t), a \le x \le b, t \ge 0, \\ v(x,0) = \Phi_0(x), \frac{\partial v(x,0)}{\partial t} = \Phi_1(x), a \le x \le b, \\ v(a,t) = 0, v(b,t) = 0, t \ge 0, \end{array}\right. \tag{3.1}$$

where $h(x,t) = -\frac{\partial^\alpha H(x,t)}{\partial t^\alpha} - \lambda_1 H(x,t) + f(x,t)$, $\Phi_0(x) = \phi_0(x) - H(x,0)$ and $\Phi_1(x) = \phi_1(x) - \frac{\partial H(x,0)}{\partial t}$. Now, since the Sinc function is defined on the $(-\infty, \infty)$, we need to use a change of variable to transform the domain of the problem to the whole real line. For this purpose, we use the following map

$$x = \psi(\zeta) = \frac{a + be^\zeta}{1 + e^\zeta}, \tag{3.2}$$

which transforms the problem (3.1) to a new problem on the whole real line as follows:

$$\begin{cases} \frac{\partial^\alpha v(\psi(\zeta),t)}{\partial t^\alpha} + \lambda_1 v(\psi(\zeta),t) + \lambda_2 g(v(\psi(\zeta),t) + H(\psi(\zeta),t)) = \\ \qquad \frac{\partial^2 v(\psi(\zeta),t)}{\partial x^2} + h(\psi(\zeta),t), -\infty \le \zeta \le \infty, t \ge 0, \\ v(\psi(\zeta),0) = \Phi_0(\psi(\zeta)), \frac{\partial v(\psi(\zeta),0)}{\partial t} = \Phi_1(\psi(\zeta)), -\infty \le \zeta \le \infty, \\ lim_{\zeta\to\pm\infty} v(\psi(\zeta),t) = 0. \end{cases} \tag{3.3}$$

By placing $U(\zeta,t) = v(\psi(\zeta),t)$ and using the chain rule of differentiation we have

$$\begin{cases} \frac{\partial^\alpha U(\zeta,t)}{\partial t^\alpha} + \lambda_1 U(\zeta,t) + \lambda_2 g\left(U(\zeta,t) + \mathcal{H}(\zeta,t)\right) = \\ \left(\frac{1}{\psi'(\zeta)}\right)^2 \frac{\partial^2 U(\zeta,t)}{\partial \zeta^2} - \frac{\psi''(\zeta)}{(\psi'(\zeta))^3} \frac{\partial U(\zeta,t)}{\partial \zeta} + \mathfrak{h}(\zeta,t), -\infty \le \zeta \le \infty, t \ge 0, \\ U(\zeta,0) = \Phi_0(\psi(\zeta)), \frac{\partial U(\zeta,0)}{\partial t} = \Phi_1(\psi(\zeta)), -\infty \le \zeta \le \infty, \\ lim_{\zeta\to\pm\infty} U(\zeta,t) = 0, \end{cases} \tag{3.4}$$

where $\mathcal{H}(\zeta,t) = H(\psi(\zeta),t)$ and $\mathfrak{h}(\zeta,t) = h(\psi(\zeta),t)$.

3.1. **Temporal discretization.** In order to discretize the time-fractional derivative in (3.4), we define:

$$\begin{aligned} &\delta t = \tfrac{T}{M}, t_n = n\delta t, n = 0,1,2,...,M, \\ &U^n(\zeta) = U(\zeta,t_n), U^n_\zeta(\zeta) = \tfrac{\partial U(\zeta,t_n)}{\partial \zeta}, U^n_{\zeta\zeta}(\zeta) = \tfrac{\partial^2 U(\zeta,t_n)}{\partial \zeta^2}, \\ &b_j = (j+1)^{2-\alpha} - j^{2-\alpha}, j = 0,1,2,..., \end{aligned}$$

which could be easily shown that $b_j, j = 0,1,2,...$ have the following properties

$$\begin{aligned} &b_0 = 1, \\ &b_0 > b_1 > b_2 > ... > b_n, b_n \to 0 as j \to \infty, \\ &\textstyle\sum_{j=0}^{n-1}(b_j - b_{j+1}) + b_n = 1. \end{aligned} \tag{3.5}$$

Lemma 3.1. *[10, 35] Let $1 < \alpha < 2$ and $\alpha_0 = \frac{\delta t^{-\alpha}}{\Gamma[3-\alpha]}$. Then*

$$\frac{\partial^\alpha U(\zeta,t_{n+1})}{\partial t^\alpha} = \alpha_0 \sum_{j=0}^{n} b_j \left(U^{n-j+1}(\zeta) - 2U^{n-j}(\zeta) + U^{n-j-1}(\zeta)\right) + r^{n+1}_{\delta t}. \tag{3.6}$$

The truncation error bound is given by

$$|r^{n+1}_{\delta t}| \le C\delta t^{3-\alpha}, \tag{3.7}$$

where C is a constant.

Using Lemma 3.1, the main equation of (3.4) can be discretized as

$$\begin{aligned} &\alpha_0 \textstyle\sum_{j=0}^{n} b_j \left(U^{n-j+1} - 2U^{n-j} + U^{n-j-1}\right) + \lambda_1(\frac{U^{n+1}+U^n}{2}) \\ &\qquad + \lambda_2 g(U^n + \mathcal{H}^n) = \left(\tfrac{1}{\psi'}\right)^2 U^{n+1}_{\zeta\zeta} - \tfrac{\psi''}{(\psi')^3} U^{n+1}_\zeta + \mathfrak{h}^{n+1} + O(\delta t^{3-\alpha}). \end{aligned} \tag{3.8}$$

To replace the term U^{-1} which appears in the above summation, we can use the initial condition $U_t(\zeta,0) = \Phi_1(\psi(\zeta))$ to obtain $U^0_t = \frac{U^1 - U^{-1}}{2\delta t}$ which gives $U^{-1} = U^1 - 2\delta t \Phi_1(\psi(\zeta))$. Therefor the summation in the left hand side of (3.8) can be rewritten as

$$\begin{aligned} &\textstyle\sum_{j=0}^{n} b_j \left(U^{n-j+1} - 2U^{n-j} + U^{n-j-1}\right) = U^{n+1} - U^n - \\ &\textstyle\sum_{j=0}^{n-1}(b_j - b_{j+1})\left(U^{n-j} - U^{n-j-1}\right) + b_n U^1 - 2b_n \delta t \Phi_1(\psi(\zeta)) - b_n U^0. \end{aligned} \tag{3.9}$$

Finally, the time-discretized form of the main equation of the problem (3.4) with the given initial conditions will be as the following iterative scheme

$$\begin{aligned} &\left(\alpha_0 + \tfrac{\lambda_1}{2}\right) U^{n+1} - \left(\tfrac{1}{\psi'}\right)^2 U^{n+1}_{\zeta\zeta} + \tfrac{\psi''}{(\psi')^3} U^{n+1}_\zeta = \\ &\left(\alpha_0 - \tfrac{\lambda_1}{2}\right) U^n + \alpha_0 \textstyle\sum_{j=0}^{n-1}(b_j - b_{j+1})\left(U^{n-j} - U^{n-j-1}\right) - \alpha_0 b_n U^1 \\ &+ 2\alpha_0 b_n \delta t \Phi_1\left(\psi(\zeta)\right) + \alpha_0 b_n U^0 - \lambda_2 g(U^n + \mathcal{H}^n) + \mathfrak{h}^{n+1} + O(\delta t^{3-\alpha}). \end{aligned} \tag{3.10}$$

The equation (3.10) can be rewritten in a operator form as follows

$$\mathcal{L}U^{n+1}(\zeta) = F^n(\zeta) + O(\delta t^{3-\alpha}), \tag{3.11}$$

where

$$\mathcal{L}U^{n+1}(\zeta) = \left(\alpha_0 + \frac{\lambda_1}{2}\right) U^{n+1} - \left(\frac{1}{\psi'}\right)^2 U^{n+1}_{\zeta\zeta} + \frac{\psi''}{(\psi')^3} U^{n+1}_{\zeta} \tag{3.12}$$

and

$$\begin{aligned} F^n(\zeta) &= \left(\alpha_0 - \tfrac{\lambda_1}{2}\right) U^n + \alpha_0 \textstyle\sum_{j=0}^{n-1} (b_j - b_{j+1}) \left(U^{n-j} - U^{n-j-1}\right) \\ &-\alpha_0 b_n U^1 + 2\alpha_0 b_n \delta t \Phi_1(\psi(\zeta)) + \alpha_0 b_n U^0 - \lambda_2 g\left(U^n + \mathcal{H}^n\right) + \mathfrak{h}^{n+1}. \end{aligned} \tag{3.13}$$

3.2. **Full discretization.** The Sinc method is applied to the discretization of the problem (3.4) in space direction. We approximate the solution $U^{n+1}(\zeta)$ of (3.10) as follows

$$U_N^{n+1}(\zeta) = \sum_{i=-N}^{N} c_i^{n+1} Sinc\left(\frac{\zeta - ih}{h}\right) = \sum_{i=-N}^{N} c_i^{n+1} S(i,h)(\zeta), \tag{3.14}$$

which satisfies the homogenous boundary condition given in (3.4). In (3.14), h is a fixed step size and c_i^{n+1} are unknown coefficients that must be determined. If we evaluate (3.14) at the grid point $\zeta_j = jh, j = 0, \pm1, \pm2, ..., \pm N$, then we get

$$U_N^{n+1}(\zeta_j) = \sum_{i=-N}^{N} c_i^{n+1} S(i,h)(\zeta_j), j = 0, \pm1, \pm2, ..., \pm N, \tag{3.15}$$

and in the matrix notation

$$\boldsymbol{U}_N^{n+1} = A\boldsymbol{C}^{n+1}, \tag{3.16}$$

where $\boldsymbol{U}_N^{n+1} = [U_N^{n+1}(\zeta_{-N}), ..., U_N^{n+1}(\zeta_N)]^T$, $\boldsymbol{C}^{n+1} = [c_{-N}^{n+1}, ..., c_N^{n+1}]^T$ and the elements of matrix A are $A_{ji} = S(i,h)(\zeta_j)$. Let $\mathcal{L}$ be the defined linear operator in (3.12), then we have

$$\mathcal{L}U_N^{n+1}(\zeta) = \sum_{i=-N}^{N} c_i^{n+1} \mathcal{L}S(i,h)(\zeta). \tag{3.17}$$

We evaluate (3.17) at the grid point $\zeta_j = jh, j = 0, \pm1, \pm2, ..., \pm N$, then we get the following matrix system

$$\boldsymbol{L}\boldsymbol{U}_N^{n+1} = A_L \boldsymbol{C}^{n+1}, \tag{3.18}$$

where $\boldsymbol{U}_N^{n+1}$ and $\boldsymbol{C}^{n+1}$ are the same as before and the matrix A_L has entries $\mathcal{L}S(i,h)(\zeta_j)$, Which can be obtained using Lemma 2.1 as follows

$$\mathcal{L}S(i,h)(\zeta_j) = \left(\alpha_0 + \frac{\lambda_1}{2}\right) \delta_{ij}^{(0)} - \left(\frac{1}{h\psi'(\zeta_j)}\right)^2 \delta_{ij}^{(2)} + \frac{\psi''}{h(\psi')^3} \delta_{ij}^{(1)}. \tag{3.19}$$

By substituting $\boldsymbol{C}^{n+1}$ form (3.16) into (3.18) we get

$$\boldsymbol{L}\boldsymbol{U}_N^{n+1} = A_L A^{-1} \boldsymbol{U}_N^{n+1}. \tag{3.20}$$

Therefore, we obtained the operational matrix $\boldsymbol{L} = A_L A^{-1}$ corresponding to the linear operator $\mathcal{L}$. where A and A_L are the following matrices:

$$A = \begin{bmatrix} S(-N,h)(\zeta_{-N}) & S(-N+1,h)(\zeta_{-N}) & \cdots & S(N,h)(\zeta_{-N}) \\ S(-N,h)(\zeta_{-N+1}) & S(-N+1,h)(\zeta_{-N+1}) & \cdots & S(N,h)(\zeta_{-N+1}) \\ & \ddots & \ddots & \\ S(-N,h)(\zeta_{N}) & S(-N+1,h)(\zeta_{N}) & \cdots & S(N,h)(\zeta_{N}) \end{bmatrix},$$

and

$$A_L = \begin{bmatrix} \mathcal{L}S(-N,h)(\zeta_{-N}) & \mathcal{L}S(-N+1,h)(\zeta_{-N}) & \cdots & \mathcal{L}S(N,h)(\zeta_{-N}) \\ \mathcal{L}S(-N,h)(\zeta_{-N+1}) & \mathcal{L}S(-N+1,h)(\zeta_{-N+1}) & \cdots & \mathcal{L}S(N,h)(\zeta_{-N+1}) \\ & \ddots & \ddots & \\ \mathcal{L}S(-N,h)(\zeta_{N}) & \mathcal{L}S(-N+1,h)(\zeta_{N}) & \cdots & \mathcal{L}S(N,h)(\zeta_{N}) \end{bmatrix}.$$

Now using the operational matrix $\boldsymbol{L}$ and the zero vector as the initial guess $\boldsymbol{U}_N^0$, an approximate solution of the (3.11) at the points $\zeta_j = jh, j = 0, \pm 1, \pm 2, ..., \pm N$ can be obtained by solving the following linear system at each iteration

$$\boldsymbol{L}\boldsymbol{U}_N^{n+1} = \boldsymbol{F}^n, \tag{3.21}$$

where $\boldsymbol{F}^n = [F^n(\zeta_{-N}), ..., F^n(\zeta_N)]^T$. The function F^n is defined in the equation (3.13), which depends on the values of $\boldsymbol{U}_N^0, ..., \boldsymbol{U}_N^n$ in the previous iterations.

3.3. **Convergence analysis.** In the following theorem, we investigate the convergence of the full discretization iterative scheme (3.21).

Theorem 3.2. *Suppose that the conditions of the Theorem 2.4 are satisfied and $g(u)$ satisfies the Lipschitz condition*

$$|g(u) - g(v)| \leq \mathfrak{L}|u - v|, \qquad \forall u, v, \tag{3.22}$$

where $\mathfrak{L}$ is the Lipschitz constant. Then the proposed scheme (3.21) is convergent to the solution of problem (3.4).

Proof. Let $\|\boldsymbol{u}\|_\infty = \max_{-N\leq i\leq N} |u_i|$ for any vector $\boldsymbol{u} = [u_{-N}, ..., u_N]$. From the Lipschitz condition for g, it can be seen that

$$\|g(\boldsymbol{u}) - g(\boldsymbol{v})\|_\infty \leq \mathfrak{L}\|\boldsymbol{u} - \boldsymbol{v}\|_\infty, \qquad \forall u, v. \tag{3.23}$$

Let $\widehat{U}_N^{n+1}(\zeta) = \sum_{i=-N}^{N} \hat{c}_i^{n+1} S(i,h)(\zeta)$ be the Sinc expansion of the exact solution of the problem (3.4) at a specific time t_{n+1}. From the triangle inequality we have

$$\|\boldsymbol{U}^{n+1} - \boldsymbol{U}_N^{n+1}\|_\infty \leq \|\boldsymbol{U}^{n+1} - \widehat{\boldsymbol{U}}_N^{n+1}\|_\infty + \|\widehat{\boldsymbol{U}}_N^{n+1} - \boldsymbol{U}_N^{n+1}\|_\infty. \tag{3.24}$$

Let $\boldsymbol{E}^{n+1} = \widehat{\boldsymbol{U}}_N^{n+1} - \boldsymbol{U}_N^{n+1}$, then from (3.10), (3.21) and $\boldsymbol{E}^0 = 0$ we have

$$\begin{aligned} \boldsymbol{L}\boldsymbol{E}^{n+1} &= \left(\alpha_0 - \tfrac{\lambda_1}{2}\right)\boldsymbol{E}^n + \alpha_0 \textstyle\sum_{j=0}^{n-1}(b_j - b_{j+1})\left(\boldsymbol{E}^{n-j} - \boldsymbol{E}^{n-j-1}\right) \\ &\quad -\alpha_0 b_n \boldsymbol{E}^1 - \lambda_2\left(g\left(\widehat{\boldsymbol{U}}_N^n + \boldsymbol{\mathcal{H}}_N^n\right) - g\left(\boldsymbol{U}_N^n + \boldsymbol{\mathcal{H}}_N^n\right)\right) + O(\delta t^{3-\alpha}), \end{aligned} \tag{3.25}$$

where $\boldsymbol{\mathcal{H}}_N^n = [\mathcal{H}(\zeta_{-N}, t_n), ..., \mathcal{H}(\zeta_N, t_n)]^T$. We use the mathematical induction to prove the result. For $n = 1$ we have

$$\boldsymbol{L}\boldsymbol{E}^1 = -\alpha_0 b_0 \boldsymbol{E}^1 + O(\delta t^{3-\alpha}),$$

which easily gives the following result

$$\|\boldsymbol{E}^1\|_\infty \leq C_1 \delta t^{3-\alpha},$$

where C_1 is a constant. Now let $\|\boldsymbol{E}^i\|_\infty \leq C_i\delta t^{3-\alpha}, i = 1,2,3,...,n$. Then, from triangle inequality, the Lipschitz condition for g and (3.25) we have

$$\begin{aligned}\|\boldsymbol{E}^{n+1}\|_\infty &\leq \|\boldsymbol{L}^{-1}\| \Big[\big(\alpha_0 - \tfrac{\lambda_1}{2}\big)\|\boldsymbol{E}^n\|_\infty \\ &+\alpha_0 \textstyle\sum_{j=0}^{n-1}(b_j - b_{j+1})\|\boldsymbol{E}^{n-j} - \boldsymbol{E}^{n-j-1}\|_\infty \\ &+\alpha_0 b_n\|\boldsymbol{E}^1\|_\infty + \lambda_2\mathfrak{L}\|\boldsymbol{E}^n\|_\infty + O(\delta t^{3-\alpha})\Big] \\ &\leq \|\boldsymbol{L}^{-1}\| \Big[\big(\alpha_0 - \tfrac{\lambda_1}{2}\big) + \alpha_0 \textstyle\sum_{j=0}^{n-1}(b_j - b_{j+1}) + \alpha_0 b_n + \lambda_2\mathfrak{L}\Big] C_{n+1}\delta t^{3-\alpha}.\end{aligned} \tag{3.26}$$

Then from (3.5) we have

$$\|\boldsymbol{E}^{n+1}\|_\infty \leq \|\boldsymbol{L}^{-1}\| \left[\left(2\alpha_0 - \frac{\lambda_1}{2}\right) + \lambda_2\mathfrak{L}\right] C_{n+1}\delta t^{3-\alpha}. \tag{3.27}$$

From Theorem 2.4, (3.24), and (3.27), we conclude that

$$\|\boldsymbol{U}^{n+1} - \boldsymbol{U}_N^{n+1}\|_\infty \leq K\sqrt{N} exp\left(-\sqrt{\pi d\alpha N}\right) + \mathcal{C}\delta t^{3-\alpha}, \tag{3.28}$$

where K and $\mathcal{C}$ are some positive constant. The first part of the obtained error estimate vanishes as $N \to \infty$ and is the Sinc approximation error. The second part of the obtained error estimate vanishes as $\delta t \to 0$ and is the error of temporal discretization. The proof is complete. □

4. Numerical experiments

In this section, we present some numerical examples to show the efficiency and accuracy of the proposed method. The results obtained by the proposed method are compared with the analytical solution and the reported results in the literature.

Example 4.1. We consider the following nonlinear fractional sine-Gordon equation as the first example

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - sin(u(x,t)) + f(x,t), 0 \leq x,t \leq 1, \\ u(x,0) = 0, \frac{\partial u(x,0)}{\partial t} = 0, 0 \leq x \leq 1, \\ u(0,t) = 0, u(1,t) = t^2 sin1, \end{cases} \tag{4.1}$$

where $f(x,t) = \left(\frac{2t^{2-\alpha}}{\Gamma(2-\alpha)} + t^2\right) sin(x) + sin\left(t^2 sin(x)\right)$. The exact solution to the problem is $u(x,t) = t^2 sin(x)$. Table 1 shows the maximum absolute errors and CPU time of the obtained approximate solutions for example 4.1 with $2N+1 = 61$ grid points and various values of $\delta t = 0.01, 0.005, 0.001, 0.0007$ and $\alpha = 1.15, 1.5, 1.75, 1.85$. Table 2 shows the maximum absolute errors and CPU time of the obtained approximate solutions for example 4.1 with $\delta t = 0.001$ and various values of grid points $2N+1$ and fractional derivative order α. We observed that the absolute errors are decreased by increasing N or by decreasing δt. Figure 1 presents the graph of the absolute errors of the approximate solutions for example 4.1 with $\delta t = 0.0007$, $N = 30$ and various values of α. For comparison, the best results reported in [5] and [35] have 10^{-4} maximum absolute errors.

Table 1. Maximum absolute error for example 4.1 with $N = 30$ and various values of δt and α.

	$\alpha = 1.15$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.85$	CPU time(s)
$\delta t = 0.01$	1.44021×10^{-4}	1.31085×10^{-4}	1.46578×10^{-4}	1.6068×10^{-4}	11.37
$\delta t = 0.005$	7.19266×10^{-5}	6.41681×10^{-5}	6.91503×10^{-5}	7.7651×10^{-5}	24.17
$\delta t = 0.001$	1.38586×10^{-5}	1.24987×10^{-5}	1.27514×10^{-5}	1.36377×10^{-5}	366.81
$\delta t = 0.0007$	9.51033×10^{-6}	8.75809×10^{-6}	8.92473×10^{-6}	9.19039×10^{-6}	888.46

Example 4.2. Let us consider the following nonlinear fractional Klein-Gordon equation with quadratic nonlinearity

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + (u(x,t))^2 = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), 0 \leq x,t \leq 1, \\ u(x,0) = 0, \frac{\partial u(x,0)}{\partial t} = 0, 0 \leq x \leq 1, \\ u(0,t) = t^{\frac{3}{2}}, u(1,t) = 0, \end{cases} \tag{4.2}$$

TABLE 2. Maximum absolute error for example 4.1 with $\delta t = 0.001$ and various values of N and α.

	$\alpha = 1.15$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.85$	CPU time(s)
$N = 10$	1.74645×10^{-4}	1.74638×10^{-4}	1.74626×10^{-4}	1.74618×10^{-4}	131.23
$N = 20$	6.87852×10^{-5}	6.20042×10^{-5}	6.16707×10^{-5}	6.07852×10^{-5}	253.53
$N = 30$	1.38586×10^{-5}	1.24987×10^{-5}	1.27514×10^{-5}	1.36377×10^{-5}	366.81

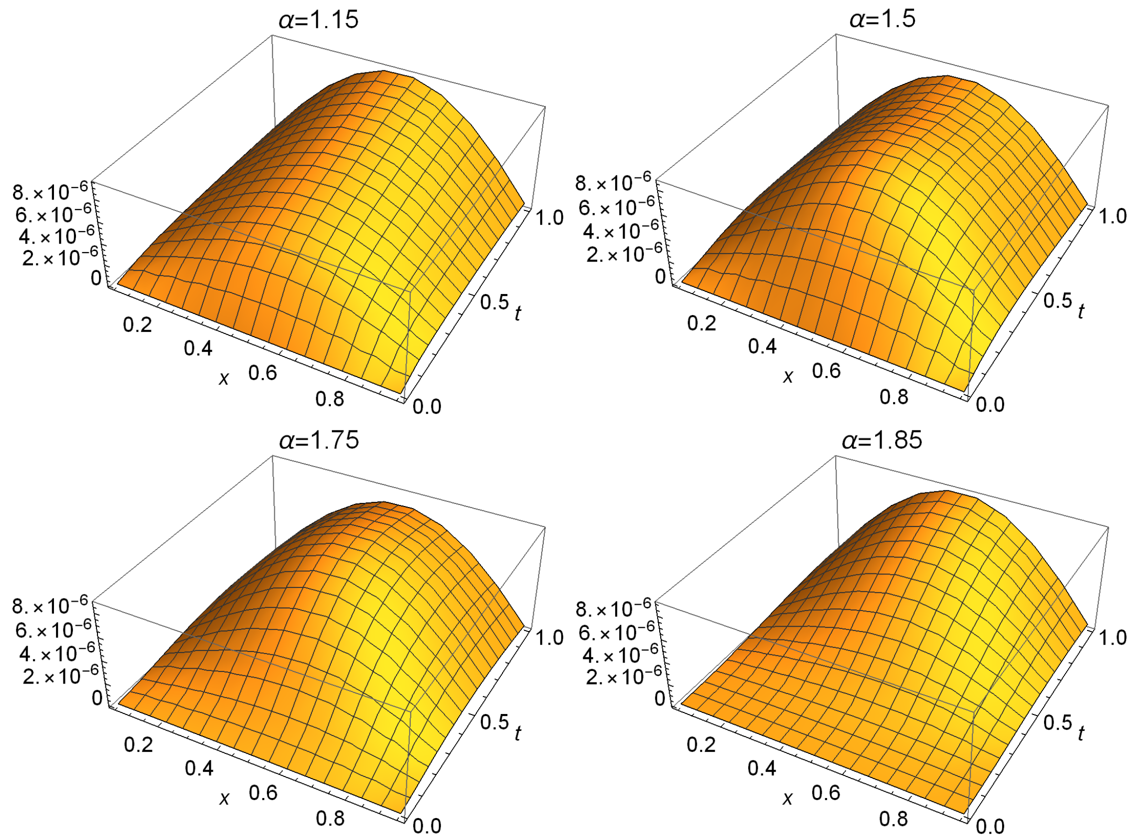

FIGURE 1. Graph of absolute error for example 4.1 with $\delta t = 0.0007$, $N = 30$ and various values of α.

where $f(x,t) = \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2}-\alpha)}(1-x)^{\frac{5}{2}}t^{\frac{3}{2}-\alpha} - \frac{15}{4}(1-x)^{\frac{1}{2}}t^{\frac{3}{2}} + (1-x)^5 t^3$. The exact solution to the problem is $u(x,t) = (1-x)^{\frac{5}{2}}t^{\frac{3}{2}}$. Tables 3 and 4 show the maximum absolute errors and CPU time of the obtained approximate solutions for Example 4.2 with various values of $N, \delta t$ and α. For comparison, the best results reported in [22] and [35] are given in Table 3. It is obvious that the proposed method gives us more accurate approximations by increasing the number of grid points or by decreasing the time step. Figure 2 presents the graph of the absolute errors of the approximate solutions for Example 4.2 with $\delta t = 0.0005$, $N = 30$ and various values of α.

Example 4.3. Let us consider the following nonlinear fractional Klein-Gordon equation with qubic nonlinearity

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t) + \frac{3}{2}(u(x,t))^3 = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), 0 \le x, t \le 1, \\ u(x,0) = 0, \frac{\partial u(x,0)}{\partial t} = 0, 0 \le x \le 1, \\ u(0,t) = 0, u(1,t) = 0, \end{cases} \tag{4.3}$$

where $f(x,t) = \frac{\Gamma(3-\alpha)}{2} sin(\pi x)t^2 + \left(1+\pi^2\right) sin(\pi x)t^{2+\alpha} + \frac{3}{2}(u(x,t))^3$. The exact solution to the problem is $u(x,t) = sin(\pi x)t^{2+\alpha}$. Tables 5 and 6 show the maximum absolute errors and CPU time of the obtained approximate solutions

Table 3. Maximum absolute error for example 4.2 with $N = 30$ and various values of δt and α.

	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$	CPU time(s)
$\delta t = 0.01$	1.4236×10^{-3}	4.15844×10^{-3}	1.10772×10^{-2}	7.40
$\delta t = 0.005$	7.23734×10^{-4}	2.49251×10^{-3}	7.59888×10^{-3}	16.98
$\delta t = 0.001$	1.47711×10^{-4}	7.32791×10^{-4}	4.39273×10^{-3}	318.71
$\delta t = 0.0005$	7.41767×10^{-5}	4.28641×10^{-4}	8.40538×10^{-4}	1275.63
Method [22]	8.7105×10^{-4}	6.2045×10^{-4}	9.4248×10^{-4}	—
Method [35]	2.6018×10^{-4}	2.1305×10^{-4}	9.0333×10^{-4}	2127.98

Table 4. Maximum absolute error for example 4.2 with $\delta t = 0.001$ and various values of N and α.

	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$	CPU time(s)
$N = 10$	8.71076×10^{-4}	9.70736×10^{-4}	9.04048×10^{-3}	121.67
$N = 20$	6.47747×10^{-4}	8.32939×10^{-4}	7.03614×10^{-3}	233.55
$N = 30$	1.47711×10^{-4}	7.32791×10^{-4}	4.39273×10^{-3}	318.71

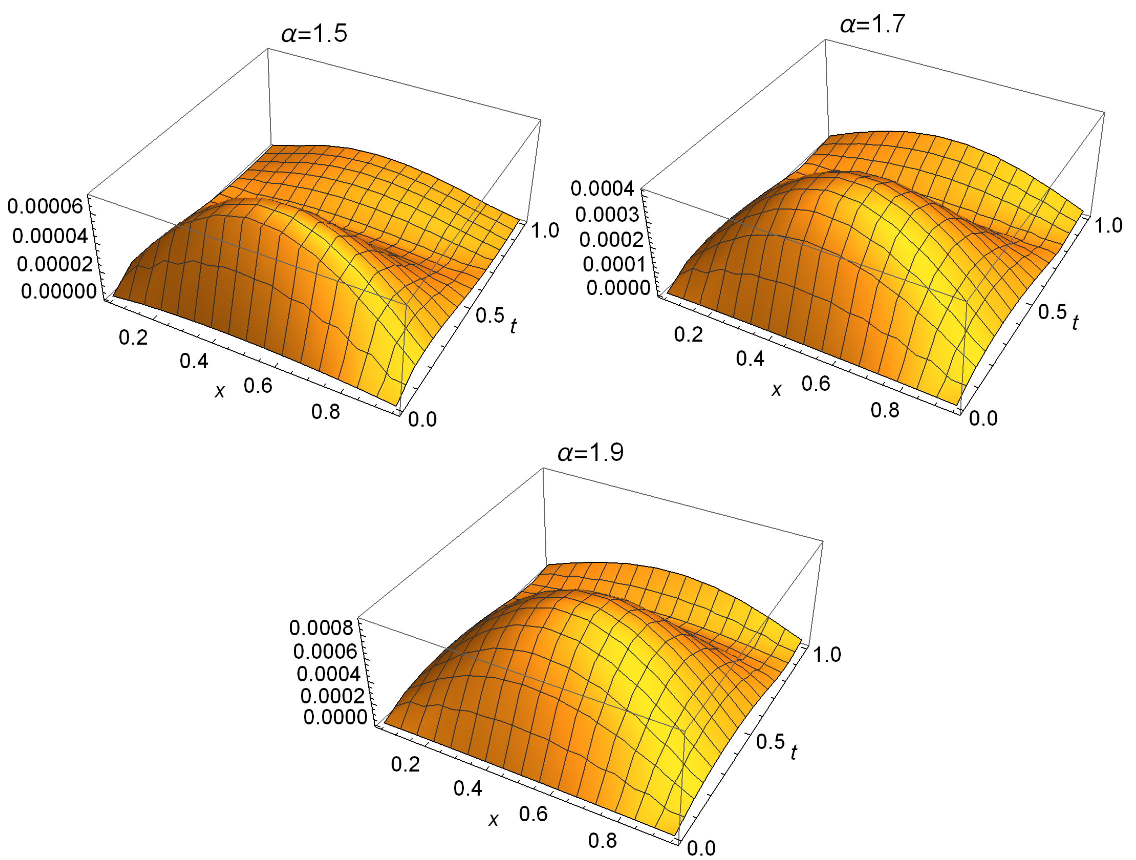

Figure 2. Graph of absolute error for example 4.2 with $\delta t = 0.0005$, $N = 30$ and various values of α.

for Example 4.3 with various values of $N, \delta t$ and α. For comparison, the best results reported in [22] and [35] are given in Table 5. We observed that the absolute errors are decreased by increasing N or by decreasing δt. Figure 3 presents the graph of the absolute errors of the approximate solutions for Example 4.3 with $\delta t = 0.0007$, $N = 30$ and various values of α.

TABLE 5. Maximum absolute error for example 4.3 with $N = 30$ and various values of δt and α.

	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$	CPU time(s)
$\delta t = 0.01$	9.14409×10^{-3}	1.13218×10^{-2}	1.77801×10^{-2}	6.79
$\delta t = 0.005$	4.45119×10^{-3}	5.46873×10^{-3}	8.63986×10^{-3}	16.67
$\delta t = 0.001$	8.60429×10^{-4}	1.03633×10^{-3}	1.63348×10^{-3}	313.75
$\delta t = 0.0005$	4.27425×10^{-4}	5.10939×10^{-4}	8.01469×10^{-4}	1181.34
Method [22]	1.6396×10^{-3}	1.5471×10^{-3}	1.4380×10^{-3}	—
Method [35]	8.1773×10^{-4}	1.0003×10^{-3}	1.6051×10^{-3}	3484.98

TABLE 6. Maximum absolute error for example 4.3 with $\delta t = 0.001$ and various values of N and α.

	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$	CPU time(s)
$N = 10$	1.80611×10^{-3}	1.80539×10^{-3}	1.95394×10^{-3}	115.30
$N = 20$	8.21644×10^{-4}	1.20756×10^{-3}	1.77606×10^{-3}	226.83
$N = 30$	8.60429×10^{-4}	1.03633×10^{-3}	1.63348×10^{-3}	313.75

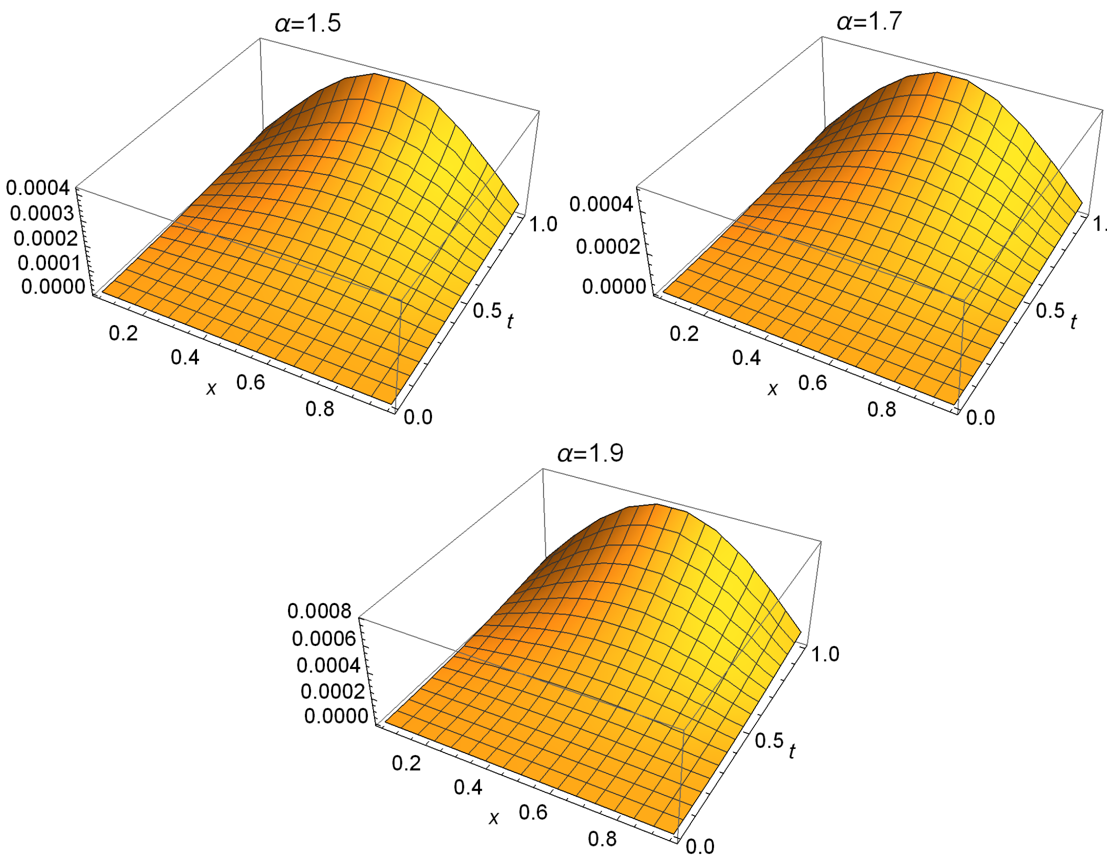

FIGURE 3. Graph of absolute error for example 4.3 with $\delta t = 0.0005$, $N = 30$ and various values of α.

From the results presented in this section, it can be seen that by increasing the number of node points or by decreasing the time step size, we can obtain more accurate approximations which confirms the presented convergence analysis. In fact, even with a small number of grid points, we can obtain approximate solutions with acceptable accuracy.

5. Conclusions

In this paper, the numerical solution of nonlinear time-fractional Klein-Gordon and sine-Gordon equations based on the Sinc operational matrices are investigated. A finite difference scheme is combined with the Sinc method to discretize the nonlinear fractional problem. Furthermore, the convergence of the combined method is proven. Some numerical examples are provided to confirm the applicability and accuracy of the proposed method. The results of numerical experiments are compared with the analytical solution and the reported results in the literature. The numerical results confirm the convergence analysis and exhibit that the proposed method is powerful and accurate.

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