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New approximations of space-time fractional Fokker-Planck equations

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Abstract

The present study focuses on the two new hybrid methods: variational iteration $\mathbb{J}$-transform technique ($\mathbb{J}$-VIT) and $\mathbb{J}$-transform method with optimal homotopy analysis ($_O$HA$\mathbb{J}$TM) for analytical assessment of space-time fractional Fokker-Planck equations (STF-FPE), appearing in many realistic physical situations, e.g., in ultra-slow kinetics, Brownian motion of particles, anomalous diffusion, polymerases of Ribonucleic acid, deoxyribonucleic acid, continuous random movement, and formation of wave patterns. $_O$HA$\mathbb{J}$TM is developed via optimal homotopy analysis after implementing the properties of $\mathbb{J}$-transform while ($\mathbb{J}$-VIT) is produced by implementing properties of the $\mathbb{J}$-transform and the theory of variational iteration.
Banach approach is utilized to analyze the convergence of these methods. In addition, it is demonstrated that $\mathbb{J}$-VIT is T-stable. Computed new approximations are reported as a closed form expression of the Mittag-Leffler function, and in addition, the effectiveness/validity of the proposed new approximations is demonstrated via three test problems of STF-FPE by computing the error norms: L_2 and absolute errors. The numerical assessment demonstrates that the developed techniques perform better for STF-FPE and for a given iteration, and $_O$HA$\mathbb{J}$TM produces new approximations with better accuracy as compared to $\mathbb{J}$-VIT as well as the techniques developed recently.

Keywords. Fractional Fokker-Planck equations(STF-FPE), $\mathbb{J}$- transform, Optimal homotopy analysis $\mathbb{J}$- transform method ($_O$HA$\mathbb{J}$TM), Variational calculus, Variational iteration technique, $\mathbb{J}$-VIT.

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1. Introduction

The idea of fractional order derivative came into existence when in 1965 a query was raised to Leibnitz by L'Hôpital about the meaning of the derivative of order half. Thereafter, derivatives/integrals of fractional orders are defined in several ways and originated a new branch of mathematical analysis known as F-Cal(fractional calculus), which got a lot of attention of a huge family of researchers/scientists in the fields of science and technology. In the past decades, F-Cal turns into the most ground-breaking weapon to trace/analyze nonlinear activities because of its vital features like memory index in time, shadows of past, homotopic mapping to the derivatives of integer order over time and the virtue of non-locality [19, 30, 35, 36, 39]. Fractional order differential equations(FDEs), more general forms of classical differential equations, are well-known choices for demonstrating the complex physical activities having memory/hereditary effects, e.g., the nonlinear FDEs are suitable choices in demonstrating sensible circumstances: such as propagation of earthquake, volcanic ejection, astronomy, population growth, nonlinear organic framework, electrical system, etc. [2, 5, 10, 47, 63].

During the past decades, several techniques (analytical/numerical) for evaluation of the behavior of partial differential equations [43, 49, 53, 59, 61] and FDEs [11, 16, 17] have been utilized/developed, some of them are listed as variational iteration method [9, 11–13, 33], Variational iterative Laplace transform method [32], Homotopy perturbation transform method [54], homotopy perturbation method [17, 56, 71], homotopy analysis transform method [18],

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the Adomian decomposition [16], Homotopy perturbation new integral transform method [45], and fractional reduced differential transform method [44, 46, 49, 50, 52, 55, 58, 62].

The study of temporal - space fractional BlochTorrey equation is done by collocation method with Legendre basis with temporal accuracy of order $O(\delta\tau^{2-\alpha})$ [29], diffusion model in porous media [6], space-time fractional advection-diffusion equation by collocation scheme with temporal accuracy of order $O(\delta\tau^{3-\alpha})$ [7] and the stability analysis of the scheme for the temporal-fractional Black-Scholes equation is analyzed in [28]. Space fractional-order diffusion-equation studied utilizing compact finite difference scheme [40], and time fractional Burgers equation via trigonometric tension B-spline based collocation scheme [60]. For more details the interested readers may go through [6, 7, 28, 40, 60] and the papers therein.

The main idea of a fractional model of the Fokker-Plank equation was introduced by Tarasov [66] which describes the Brownian like motion of particles in a medium [3, 14, 26, 67]. The more general representation of space and - time fractional model of the Fokker-Plank equation is given in (1.1), which got a number of applications in the fields of sciences like in ultra-slow kinetics [4, 65], solid-state physics [21], anomalous diffusion [27, 64], polymerases of Ribonucleic acid and deoxyribonucleic acid, continuous random movement, wave propagation, movement of ribosomes along the messenger Ribonucleic acid and formation of wave pattern [23, 72]. In the current article, two new hybrid methods so-called $_{O}$HA$\mathbb{J}$TM and $\mathbb{J}$-VIT have been proposed for analytical assessment of the space-time fractional Fokker-Planck equations (1.1).

$$ {}_{\tau}\mathcal{D}^{C}_{\alpha}\varphi(\sigma,\tau) = \mathcal{T}[\varphi(\sigma,\tau)], \qquad \varphi(\sigma,0) = g(\sigma), \quad \sigma > 0,\ \tau > 0, \tag{1.1} $$

where $\mathcal{T}[\varphi(\sigma,\tau)] = \left[-{}_{\sigma}\mathcal{D}^{C}_{\beta}P(\sigma,\tau,\varphi) + {}_{\sigma}\mathcal{D}^{C}_{2\beta}Q(\sigma,\tau,\varphi)\right]\varphi(\sigma,\tau)$, $\varphi(\sigma,\tau)$ a unknown function to evaluate, $\alpha,\beta(0 < \alpha,\beta \leq 1)$ denote the orders of time and space Caputo derivatives, respectively; $P(\sigma,t,\phi)$ is the drift coefficient while $Q(\sigma,\tau,\varphi)$ the diffusion coefficient, g-smooth function, and the operator $\mathcal{D}^{C}_{\alpha}$ used for fractional differential operator of Caputo type (C-FDO), defined as

Definition 1.1. [30, 36, 47, 58, 62] The C-FDO of $\varphi \in C_\mu,\ \mu \geq -1$ of order $\kappa - 1 < \alpha \leq \kappa$ is denoted as ${}_{\tau}\mathcal{D}^{\alpha}_{C}\varphi(\tau)$ and is defined as: for $\alpha = \kappa$, ${}_{\tau}\mathcal{D}^{\kappa}_{C}\varphi(\tau) := \frac{\partial^\kappa \varphi(\tau)}{\partial \tau^\kappa}$ and for $\kappa - 1 < \alpha < \kappa$,
${}_{\tau}\mathcal{D}^{\alpha}_{C}\varphi(\tau) = {}_{\tau}\mathcal{D}^{-(\kappa-\alpha)}_{C}\,{}_{\tau}\mathcal{D}^{\kappa}_{C}\varphi(\tau) = \frac{1}{\Gamma(\kappa-\alpha)}\int_0^\tau (\kappa-\epsilon)^{\kappa-(\alpha+1)}\frac{\partial^\kappa\varphi(\epsilon)}{\partial\epsilon^\kappa}d\epsilon,$
where ${}_{\tau}\mathcal{D}^{-\alpha}_{C}\varphi(\tau)$ is αth-order Riemann-Liouville fractional intergral operator on $\varphi \in C_\mu,\ \mu \geq -1$ is defined by ${}_{\tau}\mathcal{D}^{0}_{C}\varphi(\tau) := \varphi(\tau)$ and ${}_{\tau}\mathcal{D}^{-\alpha}_{C}\varphi(\tau) = \frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-\epsilon)^{\alpha-1}\varphi(\epsilon)d\epsilon, \quad \alpha > 0, \quad \tau > 0,$

Because of wide utilization, the study of STF-FPE becomes important. The STF-FPE with Caputo-type fractional derivatives has already been analyzed via many rigorous techniques like - fractional variational iteration method (VIM), FRDTM [41], Adomian decomposition method (ADM) [34], fractional implicit trapezoidal method [69], homotopy perturbation method [26], iterative Laplace transform method [71], finite element method [68], residual power series method [27] and so forth.

2. BASIC LITERATURE

Some basic concepts of $\mathbb{J}$-transform as well as Banach-fixed-point theorem are revisited in this section for better understanding of remaining part of work.

Definition 2.1 (Contraction [22])**.** A contraction map on a metric space $\Pi = (\Pi, d)$ is a self map T on Π such that for some nonnegative reals $\gamma(0 < \gamma < 1)$ it satisfies

$$ \forall \sigma_1,\sigma_2 \in \Pi, \quad d(\mathrm{T}\sigma_1,\mathrm{T}\sigma_2) \leq \gamma d(\sigma_1,\sigma_2). \tag{2.1} $$

Theorem 2.2 (Fixed Point Theorem & Error Estimates [22])**.** *Banach's fixed point theorem states that if* T *be a contraction map on Banach space* $\Pi \equiv (\Pi, d)$*, then it will consist of a unique fixed point.*

Moreover, if Picard iterative sequence $\{\sigma_\lambda\}_{\lambda=1}^\infty$*, obtained via process* $\sigma_\lambda = \mathrm{T}\sigma_{\lambda-1}$ *where* σ_0 *is arbitrary point in* Π*, then* $\{\sigma_\lambda\}_{\lambda=1}^\infty$ *converges to a fixed point* σ *of* T *uniquely. Also*
i) Prior error estimate of process is:

$d(\sigma, \sigma_\lambda) \leq \frac{\gamma^\lambda}{1-\gamma} d(\sigma_1, \sigma_0)$, *and*
ii) posterior error estimate of the process is:
$d(\sigma, \sigma_\lambda) \leq \frac{\gamma}{1-\gamma} d(\sigma_0, \sigma_{\lambda-1})$.

Theorem 2.3 (Picard T-Stable, [20, 38]). *A Picard T-stable map on a Banach space* Π *is a self map* T *on* Π *satisfying the property that* $\forall \sigma, \sigma_1 \in \Pi$, $d(\mathrm{T}\sigma_1, \mathrm{T}\sigma) \leq \gamma d(\sigma_1, \sigma) + K d(\sigma_1, \mathrm{T}\sigma_1)$, *for some* $K \geq 0$ *and* $0 \leq \gamma < 1$.

2.1. $\mathbb{J}$-Transform: Definition and basic properties.

Definition 2.4 ([73]). If $\mathcal{F}$ be the set of functions of exponential order:

$$\mathcal{F} = \left\{ h(\tau) : |h(\tau)| \leq K \exp\left(\frac{|\tau|}{r_\ell}\right) \text{ for some } 0 < K < \infty, \text{ and } r_\ell > 0, \text{ if } \tau \in (-1)^\ell \times [0, \infty), \ \ell = 1, 2 \right\},$$

then the $\mathbb{J}$-transform of the function $h(\tau) \in \mathcal{F}$ is denoted and defined as

$$\mathbb{J}[h(\tau)](s, \vartheta) = \vartheta \int_0^\infty e^{\left(\frac{-s\tau}{\vartheta}\right)} h(\tau) d\tau := \mathcal{H}(s, \vartheta), \tag{2.2}$$

providing integral (2.2) is convergent. ϑ and s are transformed variables,

Theorem 2.5 ([73]). *For the crucial understanding of the methods the basic properties of* $\mathbb{J}$*-transform are reported as:*
i) $\mathbb{J}\left[\frac{\tau^{\lambda\beta+\kappa}}{\Gamma(\lambda\beta+\kappa+1)}\right](s, \vartheta) = \frac{\vartheta^{\lambda\beta+\kappa+2}}{s^{\lambda\beta+\kappa+1}}, \quad \kappa, \ \lambda = 0, \ 1, \ \ldots.$
ii) $\mathbb{J}$*-transform of derivative* $\frac{\partial^m h(\tau)}{\partial \tau^m} \in \mathcal{F}$ *is given as*

$$\mathbb{J}\left[\frac{\partial^m h(\tau)}{\partial \tau^m}\right](s, \vartheta) = \frac{s^m}{\vartheta^m} \mathcal{H}(s, \vartheta) - \sum_{\ell=1}^{m} \frac{s^{m-\ell}}{\vartheta^{m-(\ell+1)}} \frac{\partial^{\ell-1} h(0^+)}{\partial \tau^{\ell-1}}, \quad m \geq 1.$$

iii) *If* $h_1(\tau), h_2(\tau) \in \mathcal{F}$, *then*
- (a) *For constants* c_1 *and* c_2- $\mathbb{J}[c_1 \ h_1(\tau) + c_2 h_2(\tau)](s, \vartheta) = c_1 \ \mathbb{J}[h_1(\tau)](s, \vartheta) + c_2 \ \mathbb{J}[h_2(\tau)](s, \vartheta)$, *(Linearity property of* $\mathbb{J}$*-transform),*
- (b) $\mathbb{J}[(h_1 * h_2)(\tau)](s, \vartheta) = \frac{1}{\vartheta} \mathbb{J}[h_1(\tau)](s, \vartheta) \ \mathbb{J}[h_1(\tau)](s, \vartheta)$, *where* $h_1 * h_2$ *referred to the convolution of 2-functions* $h_1(\tau)$ *and* $h_2(\tau)$. *This property is known as convolution theorem of* $\mathbb{J}$*-transform.*

3. PROCEDURE OF VARIATIONAL ITERATION TECHNIQUE

Variational iteration technique (VIT), developed by He [11] for PDEs of integer order, is one of the most effective/efficient technique . Thereafter, many researchers utilized VIT and its modified form: mVIT to get the efficient/effective solutions of various types of nonlinear problems of integer orders [1, 9, 12, 13, 15, 33] and fractional order, see [31, 51, 57] and inside articles.

To understand the basics of mVIT, the correction functional of (1.1) as in [1] is written as

$$\varphi_{\lambda+1}(\sigma, \tau) = \varphi_\lambda(\sigma, \tau) + \int_0^\tau \theta(\tau, \epsilon) \left[{}_\tau\mathcal{D}_C^\alpha \varphi(\sigma, \epsilon) - \mathcal{T} \tilde{\varphi}_\lambda(\sigma, \epsilon)\right] d\epsilon, \tag{3.1}$$

where the term $\theta(\tau, \epsilon)$ is well known Lagrange multiplier, and its evaluation is a key step in solving the NPDEs, φ_λ is the λth-iterative result, and $\tilde{\varphi}_\lambda$ is the restricted variational of φ_λ [8]. After implementing stationary property of functional on (3.1), we get

$$\delta\varphi_{\lambda+1}(\sigma, \tau) = \delta\varphi_\lambda(\sigma, \tau) + \delta \int_0^t \theta(\tau, \epsilon) \ {}_\tau\mathcal{D}_C^\alpha \varphi_\lambda(\sigma, \epsilon) \, d\epsilon = 0. \tag{3.2}$$

Now at that key point, Lagrange multiplier $\theta(\tau, \epsilon)$ can be evaluated from (3.2) via integrating (3.2) and using the properties of variational theory. When α is an integer, the integration of (3.2) can be easily done while for the non-integer case the process of integrating the last term of (3.2) is quite tough [15]. In this direction, variational iteration technique with utilization of Laplace transform and it's properties have been used in [32, 42] for rigorous problems in integer order problems, where the key step of computing Lagrange multiplier is done quite easily.

Now in the following section VIT with variational theory and properties of $\mathbb{J}$-transform are utilized to develop so called $\mathbb{J}$-VIT for the study of STF-FPE.

3.1. **PROCEDURE of $\mathbb{J}$-VIT FOR STF-FPE.** First of all, for fractional order operator some properties of $\mathbb{J}$-transform are investigated, which is essential part of the methodology

Theorem 3.1. *If $h(\sigma,\tau) \in \mathcal{F}$, and $\mathcal{H}(\sigma,s,\vartheta)$ is it's $\mathbb{J}$-transform, then*

(a) $\mathbb{J}\left[{}_\tau\mathcal{D}_C^{-\alpha}h(\sigma,\tau)\right](s,\vartheta) = \frac{\vartheta^\alpha}{s^\alpha}\mathcal{H}(\sigma,s,\vartheta)$,

(b) $\mathbb{J}\left[{}_\tau\mathcal{D}_C^{\alpha}h(\sigma,\tau)\right](s,\vartheta) = \left(\frac{s}{\vartheta}\right)^\alpha\mathcal{H}(\sigma,s,\vartheta) - \sum_{\ell=1}^{\kappa}\frac{s^{\alpha-\ell}}{\vartheta^{\alpha-(\ell+1)}}\frac{\partial^{\ell-1}h(\sigma,0^+)}{\partial\tau^{\ell-1}}, \kappa-1<\alpha\le\kappa\in\mathbb{N}$.

where ${}_\tau\mathcal{D}_C^{-\alpha}h(\sigma,\tau)$ and ${}_\tau\mathcal{D}_C^{\alpha}h(\sigma,\tau)$ are Riemann-Liouville FIO and C-FDO of order α, respectively.

Proof. (a) Using Theorem 2.5(i,iii), we have

$$\mathbb{J}\left[{}_\tau\mathcal{D}_C^{-\alpha}h(\sigma,\tau)\right](s,\vartheta) = \mathbb{J}\left[\frac{\tau^{\alpha-1}}{\Gamma(\alpha)} * h(\sigma,\tau)\right](s,\vartheta) = \frac{\vartheta^\alpha}{s^\alpha}\ \mathcal{H}(\sigma,s,\vartheta).$$

(b) Theorem 2.5(ii) with Case (a), we have

$$\mathbb{J}\left[{}_\tau\mathcal{D}_C^{\alpha}h(\sigma,\tau)\right](s,\vartheta) = \mathbb{J}\left[{}_\tau\mathcal{D}_C^{-(\kappa-\alpha)}\ {}_\tau\mathcal{D}_C^{\kappa}h(\sigma,\tau)\right](s,\vartheta) = \left(\frac{s}{\vartheta}\right)^\alpha\mathcal{H}(\sigma,s,\vartheta) - \sum_{\ell=1}^{\kappa}\frac{s^{\alpha-\ell}}{\vartheta^{\alpha-(\ell+1)}}\left.\frac{\partial^{\ell-1}h}{\partial\tau^{\ell-1}}\right|_{\tau=0}.$$

□

Now, on imposing the $\mathbb{J}$-transform on the space-time fractional Fokker-Plank equation (1.1) and utilizing the property of $\mathbb{J}$-transform of ${}_\tau\mathcal{D}_C^{\alpha}\varphi(\sigma,\tau)$ of from Theorem 3.1(b)

$$\mathbb{J}\left[{}_\tau\mathcal{D}_C^{\alpha}\varphi(\sigma,\tau)\right](s,\vartheta) = \left(\frac{s}{\vartheta}\right)^\alpha\varphi(\sigma,s,\vartheta) - \sum_{\ell=1}^{\kappa}\frac{s^{\alpha-\ell}}{\vartheta^{\alpha-(\ell+1)}}\left.\frac{\partial^{\ell-1}\varphi}{\partial\tau^{\ell-1}}\right|_{\tau=0},$$

we have

$$\left(\frac{s}{\vartheta}\right)^\alpha\varphi(\sigma,s,\vartheta) - \frac{s^{\alpha-1}}{\vartheta^{\alpha-2}}\varphi(\sigma,0) - \mathbb{J}\left[\mathcal{T}\varphi\left(\sigma,\epsilon\right)\right]\theta(s,\vartheta) = 0. \tag{3.3}$$

A correction functional for (3.3) is constructed in (3.4) by using the analogous process of mVIM.

$$\varphi_{\lambda+1}(\sigma,s,\vartheta) = \varphi_\lambda(\sigma,s,\vartheta) + \left(\left(\frac{s}{\vartheta}\right)^\alpha\varphi_\lambda(\sigma,s,\vartheta) - \frac{s^{\alpha-1}}{\vartheta^{\alpha-2}}\varphi_\lambda(\sigma,0)\right)\theta(s,\vartheta) - \theta(s,\vartheta)\mathbb{J}\left[\tilde{\mathcal{T}}\varphi_\lambda\left(\sigma,\tau\right)\right](s,\vartheta). \tag{3.4}$$

where $\tilde{\varphi}_\lambda$ and $\tilde{\mathcal{T}}$ restricted variations, and so, $\delta\tilde{\varphi}_\lambda = 0$ and $\delta\tilde{\mathcal{T}} = 0$.
Utilizing variational operator δ on (3.4) with the above property, we get

$$\delta\varphi_{\lambda+1}(\sigma,s,\vartheta) = \delta\varphi_\lambda(\sigma,s,\vartheta)\ \left(1+\theta(s,\vartheta)\left(\frac{s}{\vartheta}\right)^\alpha\right). \tag{3.5}$$

On implementing the stationary condition: $\delta\varphi_{\lambda+1}(\sigma,s,\vartheta) = 0$ for (3.4) in (3.5), we have

$$1+\theta(s,\vartheta)\left(\frac{s}{\vartheta}\right)^\alpha = 0 \to \theta(s,\vartheta) = -\left(\frac{\vartheta}{s}\right)^\alpha. \tag{3.6}$$

Now, Utilizing this Lagrange multiplier in (3.4), we have

$$\varphi_{\lambda+1}(\sigma,s,\vartheta) = \frac{\vartheta^2}{s}\varphi_\lambda(\sigma,0) + \left(\frac{\vartheta}{s}\right)^\alpha\mathbb{J}\left[\mathcal{T}\varphi_\lambda\left(\sigma,\tau\right)\right](s,\vartheta). \tag{3.7}$$

Implementing inverse $\mathbb{J}$-transform operator on both side of (3.7), we have

$$\varphi_{\lambda+1}(\sigma,\tau) = \mathrm{T}\varphi_\lambda(\sigma,\tau), \tag{3.8}$$

where

$$\mathrm{T}\varphi_\lambda(\sigma,\tau) = \varphi_\lambda(\sigma,0) + \mathbb{J}^{-1}\left[\left(\frac{\vartheta}{s}\right)^\alpha \mathbb{J}\left[\mathcal{T}\varphi_\lambda(\sigma,\tau)\right](s,\vartheta)\right]. \tag{3.9}$$

which is required $(\lambda+1)$th $\mathbb{J}$-VIT iterative formula for the iterative solution of STF-FPE (1.1).

4. STABILITY ANALYSIS & CONVERGENCE OF $\mathbb{J}$-VIT

Theorem 4.1 (Stability analysis of $\mathbb{J}$-VIT)**.** *Let $(\mathbf{B},\|\cdot\|)$ be a Banach space and T be a self-map on $(\mathbf{B}$. The results evaluated from the iterative process: $\varphi_{\lambda+1}(\sigma,\tau) = \mathrm{T}\varphi_\lambda(\sigma,\tau)$ as obtained in (3.8) and (3.9) will be Picard T-stable whenever there exists $\eta_0 > 0$ for which the following axioms hold good for every τ;*

a) $\|\mathcal{T}(\varphi_p(\sigma,\tau)) - \mathcal{T}(\varphi_n(\sigma,\tau))\| \le \|\mathcal{T}(\varphi_p(\sigma,\tau) - \varphi_n(\sigma,\tau))\| \le \eta_1\|\varphi_p(\sigma,\tau) - \varphi_n(\sigma,\tau)\|$, *and*

b) $\theta = \eta_0\left\|\frac{\tau^\alpha}{\Gamma(1+\alpha)}\right\| < 1.$

Proof. Let $p,n \in \mathbb{N}$.

$$\mathrm{T}\varphi_p - \mathrm{T}\varphi_n = \mathbb{J}^{-1}\left[\left(\frac{\vartheta}{s}\right)^\alpha \mathbb{J}\left[\mathcal{T}\varphi_p(\sigma,\tau) - \mathcal{T}\varphi_n(\sigma,\tau)\right](s,\vartheta)\right]. \tag{4.1}$$

Taking norm of both sides of (4.1), we have

$$\|\mathrm{T}\varphi_p - \mathrm{T}\varphi_n\| \le \mathbb{J}^{-1}\left[\left(\frac{\vartheta}{s}\right)^\alpha \mathbb{J}\left[\|\mathcal{T}\varphi_p(\sigma,\tau) - \mathcal{T}\varphi_n(\sigma,\tau)\|\right](s,\vartheta)\right].$$

On implementing the condition (a), we have

$$\begin{aligned}\|\mathrm{T}\varphi_p - \mathrm{T}\varphi_n\| &\le \eta_0\|\varphi_p(\sigma,\tau) - \varphi_n(\sigma,\tau)\|\left(\mathbb{J}^{-1}\left[\left(\frac{\vartheta}{s}\right)^\alpha \mathbb{J}[1](s,\vartheta)\right]\right)\\ &= \eta_0\|\varphi_p(\sigma,\tau) - \varphi_n(\sigma,\tau)\|\left(\mathbb{J}^{-1}\left[\frac{\vartheta^{2+\alpha}}{s^\alpha}\right]\right) \le \theta\|\varphi_p(\sigma,\tau) - \varphi_n(\sigma,\tau)\|.\end{aligned} \tag{4.2}$$

Thus, we get

$$\|\mathrm{T}\varphi_p - \mathrm{T}\varphi_n\| \le \theta\|\varphi_p(\sigma,\tau) - \varphi_n(\sigma,\tau)\| \le \beta\|\varphi_p(\sigma,\tau) - \mathrm{T}\varphi_p\| + \theta\|\varphi_p(\sigma,\tau) - \varphi_n(\varphi,\tau)\|, \quad \text{for } \beta \ge 0 \tag{4.3}$$

which concludes that proposed $\mathbb{J}$-VIT is Picard T-stable whenever $\theta < 1$, Theorem 2.3. □

Convergence analysis of $\mathbb{J}$-VIT and estimate of its error are presented in following theorem. For simplicity, we read $\varphi_n(\sigma,\tau)$ as φ_n.

Theorem 4.2 (Convergence analysis)**.** *Let $\mathbf{B} = (C[\Omega\times(0,T)],\|\cdot\|)$ be a Banach space and $\{\varphi_n\}_1^\infty$ be the sequence on $\mathbf{B}$ obtained from the $\mathbb{J}$-VIT iteration formula $\varphi_{\lambda+1} = \mathrm{T}\varphi_\lambda$ as in (3.8), where T be the self map on $\mathbf{B}$. Then*
***(i)** A fix point of T exist uniquely in $\mathbf{B}$.*
***(ii)** $\{\varphi_n\}_1^\infty$ with $\varphi_0 \in \mathbf{B}$ will converge to that fix point.*

Proof. **(i)** Proof is direct consequence of Theorem 2.2 as T be a contraction on $\mathbf{B}$.

(ii) Since $\varphi_{\lambda+1} = \mathrm{T}\varphi_\lambda$ with initial value $\varphi_0 \in \mathbf{B}$, we have

$$\varphi_1 = \mathrm{T}\varphi_0, \qquad \varphi_2 = \mathrm{T}\varphi_1 = \mathrm{T}^2\varphi_0, \qquad \cdots, \qquad \varphi_k = \mathrm{T}^k\varphi_0 \qquad \cdots. \tag{4.4}$$

Let $\kappa,\ \lambda$ are natural numbers such that $\lambda > \kappa$. Then Equations (4.2) and (4.4) imply that

$$\begin{aligned}&\|\varphi_2 - \varphi_1\| = \|\mathrm{T}\varphi_1 - \mathrm{T}\varphi_0\| \le \theta\|\varphi_1 - \varphi_0\|, \quad \|\varphi_3 - \varphi_2\| = \|\mathrm{T}\varphi_2 - \mathrm{T}\varphi_1\| \le \theta\|\varphi_2 - \varphi_1\| \le \theta^2\|\varphi_1 - \varphi_0\|, \quad \ldots\\ &\qquad \|\varphi_{\lambda+1} - \varphi_\lambda\| \le \theta^\lambda\|\varphi_1 - \varphi_0\|.\end{aligned} \tag{4.5}$$

Utilizing the Cauchy-Schwarz inequality in (4.5), we have

$$\begin{aligned}\|\varphi_\lambda-\varphi_\kappa\| &= \|\sum_{\ell=0}^{\lambda-\kappa-1}(\varphi_{\kappa+\ell+1}-\varphi_{\kappa+\ell})\| \le \sum_{\ell=0}^{\lambda-\kappa-1}\|\varphi_{\kappa+\ell+1}-\varphi_{\kappa+\ell}\| \\ &\le \sum_{\ell=0}^{\lambda-\kappa-1}\theta^{\kappa+\ell}\|\varphi_1-\varphi_0\| = \theta^\kappa\|\varphi_1-\varphi_0\|\sum_{\ell=0}^{\lambda-\kappa-1}\theta^\ell \\ &= \theta^\kappa\|\varphi_1-\varphi_0\|\frac{1-\theta^{\lambda-\kappa}}{1-\theta} \le \|\varphi_1-\varphi_0\|\frac{\theta^\kappa}{1-\theta},\end{aligned} \tag{4.6}$$

as $0<1-\theta^{\lambda-\kappa}<1$ whenever $0<\theta<1$.

Moreover, $\|\varphi_1-\varphi_0\|$ is fixed finite value. Thus, for any given positive $\epsilon, \exists$ a natural number κ_0 such that

$$\|\varphi_\lambda-\varphi_\kappa\|<\epsilon, \qquad \kappa,\ \lambda\ge\kappa_0. \tag{4.7}$$

This confirms that $\{\varphi_\lambda\}_{\lambda=1}^{\infty}$ is Cauchy sequence and consequently, it converges to $\varphi\in\mathbf{B}$, i.e

$$\lim_{\lambda\to\infty}\varphi_\lambda=\varphi.$$

Now utilizing the triangle inequality and equation (4.2), we get

$$\begin{aligned}\|\varphi-\mathrm{T}\varphi\| &= \|\varphi-\varphi_\kappa+\varphi_\kappa-T\varphi\| \\ &\le \|\varphi-\varphi_\kappa\|+\|\mathrm{T}\varphi_{\kappa-1}-\mathrm{T}\varphi\| \le \|\varphi-\varphi_\kappa\|+\theta\|\varphi_{\kappa-1}-\varphi\|=0 \qquad \text{as} \quad \kappa\to\infty.\end{aligned} \tag{4.8}$$

This confirms that $\varphi=\mathrm{T}\varphi$ i.e $\varphi\in\mathbf{B}$ is a fix point of T. ☐

Corollary 4.3. *The prior and the posterior error estimate of maximum absolute error at λth iterative result computed using iteration formula (3.8) are*

$\|\varphi-\varphi_\lambda\|\le\frac{\theta^\lambda}{1-\theta}\|\varphi_1-\varphi_0\|$, *and* $\|\varphi-\varphi_\lambda\|\le\frac{1}{1-\theta}\|\varphi_1-\varphi_0\|$, *here* $\theta\in[0,1)$.
respectively.

Proof. It is direct consequence of (4.6) with $\lambda\to\infty$. ☐

5. BASIC PROCEDURE OF $_O$HA𝕁TM

In this section we illustrate basic solution process of $_O$HA𝕁TM for nonlinear STF-FPE (1.1). By implementing the 𝕁-transform on equation STF-FPE (1.1) and use the feature of 𝕁-transform to obtain equation (3.3), the following can be written as

$$\mathbb{J}[\varphi(\sigma,\tau)](s,\vartheta)-\frac{\vartheta^2}{s}\varphi(\sigma,0)+\left(\frac{\vartheta}{s}\right)^\alpha\mathbb{J}\left[\mathcal{T}\varphi\left(\sigma,\tau\right)\right](s,\vartheta)=0.$$

Assume the nonlinear operator defined as follows:

$$\Xi[\Theta(\sigma,\tau;\omega)]=\mathbb{J}[\Theta(\sigma,\tau;\omega)](s,\vartheta)-\frac{\vartheta^2}{s}\varphi(\sigma,0)+\left(\frac{\vartheta}{s}\right)^\omega\mathbb{J}\left[\mathcal{T}\Theta(\sigma,\tau;\omega)\right](s,\vartheta).$$

where $\Theta(\sigma,\tau;\omega)$ is the function of real variables $\omega,\sigma,\tau;\ \omega\in[0,1]$ denotes the standard embedding parameter.

According to [25], the deformation expression for order zero is:

$$(1-\omega)\mathbb{J}\left[\Theta(\sigma,\tau;\omega)-\varphi_0(\sigma,\tau)\right](s,\vartheta)=\omega\hbar H(\sigma,\tau)\Xi[\Theta(\sigma,\tau;\omega)], \tag{5.1}$$

where the parameters $\hbar\ne0,\ H(\sigma,\tau)$ denotes the auxiliary things ; $\varphi_0(\sigma,\tau)$ be the initial solution for function $\varphi(\sigma,\tau)$. Remarkably noticed that $_O$HA𝕁TM offers a wide range of options for selecting auxiliary items in the procedure.
It is observable that:

$\Theta(\sigma,\tau;0)=\varphi(\sigma,0)$ and $\Theta(\sigma,\tau;1)=\varphi(\sigma,\tau)$ is valid for $\omega=0,1$, respectively.

specifying that as ω evolve from 0 towards 1, the solution $\Theta(\sigma,\tau;\omega)$ evolve simultaneously from initial solution: $\varphi_0(\sigma,\tau)$ towards the exact solution: $\varphi(\sigma,\tau)$. In addition, the function $\varphi(\sigma,\tau;\omega)$ recorded in series form of ω by utilizing Taylor's formula as:

$$\Theta(\sigma,\tau;\omega) = \varphi_0(\sigma,\tau) + \sum_{\lambda=1}^{\infty} \varphi_\lambda(\sigma,\tau)\omega^\lambda, \tag{5.2}$$

where as $\varphi_\lambda(\sigma,\tau) = \frac{1}{\lambda\,!}\frac{\partial^\lambda \Theta}{\partial \omega^\lambda}\Big|_{\omega=0}$, and $\hbar$-parameter governs the convergence range to obtain more accurate solution (5.2). While convergence criteria for the solution (5.2) at $\omega = 1$ can be proven by opting the appropriate auxiliary parameters: $\hbar, H(\sigma,\tau) \neq 0$ and the initial solution. Therefore,

$$\varphi(\sigma,\tau) = \varphi_0(\sigma,\tau) + \sum_{\lambda=1}^{\infty} \varphi_\lambda(\sigma,\tau). \tag{5.3}$$

Set

$\overrightarrow{\varphi}_\lambda(\sigma,\tau) = (\varphi_0(\sigma,\tau),\ \varphi_1(\sigma,\tau),\ \varphi_2(\sigma,\tau),\ \ldots,\ \varphi_\lambda(\sigma,\tau))$.

Hereafter, the λth order-deformation equation is constructed as follows:

$$\mathbb{J}\left[\varphi_\lambda(\sigma,\tau) - \chi_\lambda \varphi_{\lambda-1}(\sigma,\tau)\right](s,\vartheta) = \hbar\omega H(\sigma,\tau)\mathbb{P}_\lambda(\overrightarrow{\varphi}_{\lambda-1}(\sigma,\tau)), \tag{5.4}$$

where

$$\begin{cases} \chi_\lambda = 0, & \text{if } \lambda \leq 1, \\ \chi_\lambda = 1, & \text{otherwise }. \end{cases}$$

In above equation (5.4), we obtain the following by applying the inverse operator of $\mathbb{J}$-transform and assign $\omega = 1, H(\sigma,\tau) = 1$, one achieve the following:

$$\varphi_\lambda(\sigma,\tau) = \chi_\lambda \varphi_{\lambda-1}(\sigma,\tau) + \hbar\mathbb{J}^{-1}\left[\mathbb{P}_\lambda(\overrightarrow{\varphi}_{\lambda-1}(\sigma,\tau)),\right] \tag{5.5}$$

while

$\mathbb{P}_\lambda\left(\overrightarrow{\varphi}_{\lambda-1}(\sigma,\tau)\right) = \frac{1}{(\lambda-1)!}\ \frac{\partial^{\lambda-1}\mathcal{T}[\Theta(\sigma,\tau,\omega)]}{\partial\omega^{\lambda-1}}\Big|_{\omega=0}$.

After the evaluation of approximate truncated series solution of equation (1.1) of order m, it is written as follows:

$$S_m(\sigma,\tau) = \sum_{\lambda=0}^{m} \varphi_\lambda(\sigma,\tau), \tag{5.6}$$

The solution (5.6) approaches toward the exact solution $\varphi(\sigma,\tau)$ of the equation (1.1) precisely as $m \to \infty$.

Theorem 5.1 (CONVERGENCE & ERROR ESTIMATES in $_O$HA$\mathbb{J}$TM)**.** *If $\exists\ \lambda$ with $0 < \lambda < 1$, for which the condition $\|\varphi_{\ell+1}(\sigma,\tau)\| \leq \lambda\|\varphi_\ell(\sigma,\tau)\|$, $\ell \geq 1$ holds true, then*

***a)** The Mth-order $_O$HA$\mathbb{J}$TM results $S_M(\sigma,\tau)$ in (5.6) for STF-FPE (1.1) enumerated from (5.3), converges as $M \to \infty$.*

***b)** According to [24], the maximum absolute error in $S_M(\sigma,\tau)$ is as follows*

$$\|\varphi(\sigma,\tau) - S_M(\sigma,\tau)\| \leq \frac{\lambda^{M+1}}{1-\lambda}\|\varphi_0(\sigma,\tau)\|. \tag{5.7}$$

***c)** Furthermore, whenever the result in (5.3) converges, where $\varphi_\lambda(\sigma,\tau)$'s are evaluated from (5.5). Then the result gained from (5.3) must be the exact solution to STF-FPE (1.1).*

Proof. The assumption leads to

$$\|\varphi_1(\sigma,\tau)\| \leq \lambda\|\varphi_0(\sigma,\tau)\|, \quad \|\varphi_2(\sigma,\tau)\| \leq \lambda\|\varphi_1(\sigma,\tau)\| \leq \lambda^2\|\varphi_0(\sigma,\tau)\|, \quad \ldots \quad \|\varphi_\ell(\sigma,\tau)\| \leq \lambda^\ell\|\varphi_0(\sigma,\tau)\|.$$

In consequence, and so, for $M, N \in \mathbb{N}$ with $N > M$, we get

$$\|S_M(\sigma,\tau) - S_N(\sigma,\tau)\| = \left\| \sum_{j=M}^{N} \varphi_j(\sigma,\tau) \right\| \leq \|\varphi_0(\sigma,\tau)\| \sum_{j=M}^{N} \lambda^j = \|\varphi_0(\sigma,\tau)\| \frac{(1-\lambda^{N-M+1})\lambda^M}{1-\lambda}.$$

Moreover, $1 - \lambda^{N-M+1} < 1$ as $0 < \lambda < 1$, and so, the above inequality reduces to

$$\|S_M(\sigma,\tau) - S_N(\sigma,\tau)\| \leq \|\varphi_0(\sigma,\tau)\| \frac{\lambda^M}{1-\lambda} \to 0 \qquad \text{as} \qquad M \to \infty. \tag{5.8}$$

implying that $\{S_M(\sigma,\tau)\}_{M=1}^{\infty}$ is cauchy sequence, and so, it is convergent.
Part **b)** is obtained direct by taking $N \to \infty$ in (5.8) as follows

$$\|S_M(\sigma,\tau) - \varphi(\sigma,\tau)\| \leq \|\varphi_0(\sigma,\tau)\| \frac{\lambda^M}{1-\lambda}. \tag{5.9}$$

c) In special case, when $N - 1 = M = \ell$. Then from equation (5.8), we get

$(*)$ $\lim_{\ell\to\infty} \varphi_\ell(\sigma,\tau) = 0$.

Since

$(**)$ $\varphi_\ell(\sigma,\tau) = \sum_{j=1}^{\ell} \left[\varphi_j(\sigma,\tau) - \chi_\lambda \varphi_{j-1}(\sigma,\tau)\right].$

Utilize the condition $(*)$ and $(**)$ in (5.5) with property $\omega \neq 0$, we have

$\lim_{\ell\to\infty} \sum_{j=1}^{\ell} \mathbb{P}_j(\overrightarrow{\varphi}_{j-1}(\sigma,\tau)) = \sum_{j=1}^{\infty} \mathbb{P}_j(\overrightarrow{\varphi}_{j-1}(\sigma,\tau)) = 0.$

Consequently,

$$\begin{aligned}
\sum_{j=1}^{\infty} \mathbb{P}_j\left(\overrightarrow{\varphi}_{j-1}(\sigma,\tau)\right) &= \sum_{j=1}^{\infty} \left[\mathbb{J}[\varphi_{j-1}(\sigma,\tau)](s,\vartheta) - (1-\chi_j)\frac{\vartheta^2}{s}\varphi(\sigma,0) - \left(\frac{\vartheta}{s}\right)^{\alpha} \mathbb{J}\left[\mathcal{T}\tilde{\varphi}_{j-1}(\sigma,\tau)\right](s,\vartheta)\right] \\
&= \left(\frac{\vartheta}{s}\right)^{\alpha} \mathbb{J}\left[{}_{\tau}\mathcal{D}_{\alpha}^{C}(\varphi(\sigma,\tau)) - \mathcal{T}[\phi(\sigma,t)]\right] = 0, \\
&\Rightarrow_{\tau} \mathcal{D}_{\alpha}^{C}(\varphi(\sigma,\tau)) - \mathcal{T}[\varphi(\sigma,\tau)] = 0,
\end{aligned}$$

This confirms that the exact solution to STF-FPE (1.1) is $\varphi(\sigma,\tau)$ in (5.3). □

APPROXIMATION FOR THE OPTIMAL $\hbar$

The accuracy, efficiency, and validity of ${}_{O}$HA$\mathbb{J}$TM is tested by evaluating the residual error and L_2 norms [24] in the evaluated series solution. The squared residual error [25] in the Mth order series solution $S_M(\sigma,\tau)$ as given in (5.6) is defined as

$$\Delta_M(\hbar) = \left(\int_{a_1}^{a_2}\int_{b_1}^{b_2} \left(\mathcal{R}^{\alpha}\left[S_M(x,\tau)\right]\right)^2 dx d\tau\right)^{\frac{1}{2}}, \tag{5.10}$$

here $\mathcal{R}^{\alpha}\left[S_M(\sigma,\tau)\right]$ denotes residual error. $\hbar$, the optimal parameter, plays a key role for the rapid convergence of the series solution. One can adjust the value of $\hbar$ so that the residual error and convergence rate is optimal. The optimal value of $\hbar$ can be chosen by minimizing the $\Delta_M(\hbar)$. In order to controlling the computational cost in the computation of optimal value of $\hbar$, we utilize the following formula instead of (5.10)

$$\Delta_M(\hbar) = \left(\frac{1}{c_1\ c_2}\sum_{j=0}^{c_1}\sum_{l=0}^{c_2}\left(\mathcal{R}^{\alpha}\left[S_M(j\delta\sigma, l\delta t)\right)^2\right)\right)^{\frac{1}{2}}. \tag{5.11}$$

here $\delta\sigma = \frac{b_1-a_1}{c_1}$ and $\delta t = \frac{b_2-a_2}{c_2}$. We set $c_1 = c_2 = 10$.

6. TEST EXAMPLES TO VALIDATE $_O$HAJTM AND $\mathbb{J}$-VIT

This section report the numerical study of three different test examples to examine the effectiveness, accuracy, and validity of both the proposed techniques: $_O$HAJTM and $\mathbb{J}$-VIT in terms of absolute error and L_2-error.

6.1. Example 6.1.

Example 6.1. The first test example deals with linear STF-FPE of the form

$$ {}_\tau\mathcal{D}^C_\alpha \varphi(\sigma,\tau) = {}_\sigma \mathcal{D}^C_{2\beta}\left\{\frac{\sigma^{2\beta}\varphi(\sigma,\tau)}{12\beta^2}\right\} - {}_\sigma \mathcal{D}^C_\beta\left\{\frac{\sigma^{\beta}\varphi(\sigma,\tau)}{6\beta}\right\},\ \varphi(\sigma,0)=\sigma^{2\beta},\quad \sigma,\tau>0,\ 0<\alpha,\beta\le 1. \tag{6.1}$$

and the exact solution of (6.1) is

$$\varphi(\sigma,\tau) = \sigma^{2\beta}\ E_{\alpha,1}\left(\frac{\tau^\alpha}{2}\right). \tag{6.2}$$

6.1.1. $\mathbb{J}$-*VIT SOLUTIONS OF EXAMPLE 6.1.* In view of the equation (3.7), $(\lambda+1)$th iterative result for example 6.1 is evaluated from the following recurrence relation

$$\varphi_{\lambda+1}(\sigma,\tau) = \varphi_\lambda(\sigma,0) + \mathbb{J}^{-1}\left\{\left(\frac{\vartheta}{s}\right)^\alpha \mathbb{J}\left\{{}_\sigma\mathcal{D}^C_{2\beta}\left\{\frac{\sigma^{2\beta}\varphi(\sigma,\tau)}{12\beta^2}\right\} - {}_\sigma \mathcal{D}^C_\beta\left\{\frac{\sigma^{\beta}\varphi(\sigma,\tau)}{6\beta}\right\}\right\}\right\}. \tag{6.3}$$

On solving the equation (6.3) with $\varphi_0(\sigma,0)=\sigma^{2\beta}$, we get

$$\begin{aligned}
\varphi_1(\sigma,\tau) =& \sigma^{2\beta}\left(1+\frac{\tau^\alpha}{2\Gamma(1+\alpha)}\right),\\
\varphi_2(\sigma,\tau) =& \sigma^{2\beta}\left(1+\frac{\tau^\alpha}{2\Gamma(1+\alpha)}+\frac{\tau^{2\alpha}}{4\Gamma(1+2\alpha)}\right),\\
\varphi_3(\sigma,\tau) =& \sigma^{2\beta}\left(1+\frac{\tau^\alpha}{2\Gamma(1+\alpha)}+\frac{\tau^{2\alpha}}{4\Gamma(1+2\alpha)}+\frac{\tau^{3\alpha}}{8\Gamma(1+3\alpha)}\right),\\
\varphi_4(\sigma,\tau) =& \sigma^{2\beta}\left(1+\frac{\tau^\alpha}{2\Gamma(1+\alpha)}+\frac{\tau^{2\alpha}}{4\Gamma(1+2\alpha)}+\frac{\tau^{3\alpha}}{8\Gamma(1+3\alpha)}+\frac{\tau^{4\alpha}}{16\Gamma(1+4\alpha)}\right),\\
\vdots\quad & \vdots\\
\varphi_M(\sigma,\tau) =& \sigma^{2\beta}\sum_{j=0}^{M}\frac{\tau^{j\alpha}}{2^j\Gamma(j\alpha+1)}.
\end{aligned}$$

It is seen that for $M\to\infty$, we get

$$\varphi(\sigma,\tau) = \sigma^{2\beta}\sum_{j=0}^{\infty}\frac{\tau^{j\alpha}}{2^j\Gamma(j\alpha+1)} = \sigma^{2\beta}\ E_{\alpha,1}\left(\frac{\tau^\alpha}{2}\right). \tag{6.4}$$

6.1.2. $_O$*HAJTM SOLUTION OF EXAMPLE 6.1.* Consider the equation (6.1) and impose $\mathbb{J}$ -transform on both side to obtain

$$\mathbb{J}\left[\varphi(\sigma,\tau)\right](s,\vartheta) = \frac{\vartheta^2}{s}\varphi(\sigma,0) + \left(\frac{\vartheta}{s}\right)^\alpha \mathbb{J}\left[{}_\sigma\mathcal{D}^C_{2\beta}\left(\frac{\sigma^{2\beta}\varphi(\sigma,\tau)}{12\beta^2}\right) - {}_\sigma \mathcal{D}^C_\beta\left(\frac{\sigma^{\beta}\varphi(\sigma,\tau)}{6\beta}\right)\right](s,\vartheta). \tag{6.5}$$

The nonlinear operator is computed as

$$\Xi[\Theta(\sigma,\tau;\omega)] = \mathbb{J}[\Theta(\sigma,\tau;\omega)](s,\vartheta) - \frac{\vartheta^2}{s}\Theta(\sigma,0) - \left(\frac{\vartheta}{s}\right)^\alpha \mathbb{J}\left[{}_\sigma\mathcal{D}^C_{2\beta}\left(\frac{\sigma^{2\beta}\Theta(\sigma,\tau;\omega)}{12\beta^2}\right) - {}_\sigma \mathcal{D}^C_\beta\left(\frac{\sigma^{\beta}\Theta(\sigma,\tau;\omega)}{6\beta}\right)\right](s,\vartheta).$$

This yields the following recursive equation from (5.5) with aid of above nonlinear operator:

$$\varphi_\lambda(\sigma,\tau) = \chi_\lambda\varphi_{\lambda-1}(\sigma,\tau) + \hbar\mathbb{J}^{-1}\left[\mathbb{P}_\lambda(\overrightarrow{\varphi}_{\lambda-1}(\sigma,\tau))\right], \tag{6.6}$$

here

$$\begin{aligned}\mathbb{P}_\lambda(\overrightarrow{\varphi}_{\lambda-1}(\sigma,\tau)) &= \mathbb{J}\left[\varphi_{\lambda-1}(\sigma,\tau)\right](s,\vartheta)\\ &- (1-\chi_\lambda)\frac{\vartheta^2}{s}\varphi(\sigma,0) - \left(\frac{\vartheta}{s}\right)^\alpha \mathbb{J}\left[{}_\sigma\mathcal{D}^C_{2\beta}\left(\frac{\sigma^{2\beta}\varphi_{\lambda-1}(\sigma,\tau)}{12\beta^2}\right) -{}_\sigma \mathcal{D}^C_\beta\left(\frac{\sigma^\beta\varphi_{\lambda-1}(\sigma,\tau)}{6\beta}\right)\right](s,\vartheta).\end{aligned} \tag{6.7}$$

With aid Mathematica 10.0 software, the enumeration $\varphi_\lambda(\sigma,\tau)$ for $\lambda\geq 1$ from the relation (6.11) is as

$$\begin{aligned}
\varphi_1(\sigma,\tau) =& \frac{h\tau^\alpha\sigma^{2\beta}}{2\Gamma(1+\alpha)},\\
\varphi_2(\sigma,\tau) =& -\frac{h^2\tau^{2\alpha}\sigma^{2\beta}}{4\Gamma(1+2\alpha)} + \frac{\beta h^2\tau^{2\alpha}\sigma^{2\beta}\Gamma(2\beta)}{\Gamma(1+2\alpha)\Gamma(1+2\beta)} - \frac{h^2\tau^\alpha\sigma^{2\beta}}{\Gamma(1+\alpha)} + \frac{\beta h^2\tau^\alpha\sigma^{2\beta}\Gamma(2\beta)}{\Gamma(1+\alpha)\Gamma(1+2\beta)} - \frac{h\tau^\alpha\sigma^{2\beta}}{\Gamma(1+\alpha)}\\
&+\frac{\beta h\tau^\alpha\sigma^{2\beta}\Gamma(2\beta)}{\Gamma(1+\alpha)\Gamma(1+2\beta)},\\
\varphi_3(\sigma,\tau) =& \sigma^{2\beta}\Bigg(\frac{6\beta^2h^3\tau^{2\alpha}\Gamma(3\beta)\Gamma(4\beta)}{\Gamma(1+2\alpha)\Gamma(1+3\beta)\Gamma(1+4\beta)} - \frac{3\beta^2h^3\tau^{3\alpha}\Gamma(3\beta)\Gamma(4\beta)}{\Gamma(1+3\alpha)\Gamma(1+3\beta)\Gamma(1+4\beta)} + \frac{\beta h^3\tau^{2\alpha}\Gamma(2\beta)}{\Gamma(1+2\alpha)\Gamma(1+2\beta)}\\
&-\frac{3\beta h^3\tau^{2\alpha}\Gamma(3\beta)}{4\Gamma(1+2\alpha)\Gamma(1+3\beta)} + \frac{3\beta h^3\tau^{3\alpha}\Gamma(3\beta)}{8\Gamma(1+3\alpha)\Gamma(1+3\beta)} + \frac{\beta h^3\tau^\alpha\Gamma(2\beta)}{\Gamma(1+\alpha)\Gamma(1+2\beta)} - \frac{h^3\tau^{2\alpha}}{4\Gamma(1+2\alpha)} - \frac{h^3\tau^\alpha}{\Gamma(1+\alpha)}\\
&+\frac{6\beta^2h^2\tau^{2\alpha}\Gamma(3\beta)\Gamma(4\beta)}{\Gamma(1+2\alpha)\Gamma(1+3\beta)\Gamma(1+4\beta)} + \frac{\beta h^2\tau^{2\alpha}\Gamma(2\beta)}{\Gamma(1+2\alpha)\Gamma(1+2\beta)} - \frac{3\beta h^2\tau^{2\alpha}\Gamma(3\beta)}{4\Gamma(1+2\alpha)\Gamma(1+3\beta)} + \frac{2\beta h^2\tau^\alpha\Gamma(2\beta)}{\Gamma(1+\alpha)\Gamma(1+2\beta)}\\
&-\frac{h^2\tau^{2\alpha}}{4\Gamma(1+2\alpha)} - \frac{2h^2\tau^\alpha}{\Gamma(1+\alpha)} + \frac{\beta h\tau^\alpha\Gamma(2\beta)}{\Gamma(1+\alpha)\Gamma(1+2\beta)} - \frac{h\tau^\alpha}{\Gamma(1+\alpha)}\Bigg).
\end{aligned}$$

In the similar fashion, more terms of $\varphi_\lambda(\sigma,\tau)$, for $\lambda\geq 4$ can be enumerated. In addition, $5th$-order numeric-solution of (6.1) with the aid of equation (5.6) is given as

$$\begin{aligned}
S_5(\sigma,\tau) &= \sum_{\lambda=0}^{5}\varphi_\lambda(\sigma,\tau)\\
&= \sigma^{2\beta}\Bigg(1 - \frac{2h\tau^\alpha}{\Gamma(1+\alpha)} - \frac{h^2\tau^\alpha}{\Gamma(1+\alpha)} - \frac{h^2\tau^{2\alpha}}{4\Gamma(1+2\alpha)} + \frac{2\beta h\tau^\alpha\Gamma(2\beta)}{\Gamma(1+\alpha)\Gamma(1+2\beta)}\\
&\quad+\frac{\beta h^2\tau^\alpha\Gamma(2\beta)}{\Gamma(1+\alpha)\Gamma(1+2\beta)} + \frac{\beta h^2\tau^{2\alpha}\Gamma(2\beta)}{\Gamma(1+2\alpha)\Gamma(1+2\beta)}\Bigg).
\end{aligned}$$

and exclusively for $\hbar=-1$ the above equation reduces as

$$S_5(\sigma,\tau) = \sigma^{2\beta}\sum_{j=0}^{5}\frac{\tau^{j\alpha}}{2^j\Gamma(1+j\alpha)}, \tag{6.8}$$

above series in the compact form can be written as $\varphi(\sigma,\tau) = \sigma^{2\beta}\ E_{\alpha,1}\left(\frac{\tau^\alpha}{2}\right)$. which is exact solution.

6.2. **Example 6.2.**

Example 6.2. This test example deals with nonlinear TF-FPE of the form

$${}_\tau\mathcal{D}^C_\alpha\varphi(\sigma,\tau) = \frac{\partial^2\varphi^2(\sigma,\tau)}{\partial\sigma^2} + \frac{\partial}{\partial\sigma}\left(\frac{\sigma\ \varphi(\sigma,\tau)}{3} - \frac{4\ \varphi^2(\sigma,\tau)}{\sigma}\right),\ \varphi(\sigma,0)=\sigma^2,\ 0<\alpha\leq 1,\quad \sigma,\ \tau>0. \tag{6.9}$$

The exact solution of (6.9) with $0<\alpha\leq 1$ is

$$\varphi(\sigma,\tau) = \sigma^2E_{\alpha,1}(\tau^\alpha). \tag{6.10}$$

where $E_{\alpha,1}(z) = \sum_{\kappa=0}^{\infty}\frac{z^\kappa}{\Gamma(1+\kappa\alpha)}$ is the well known Mittag Leffler function [30, 36].

6.2.1. $\mathbb{J}$-*VIT SOLUTIONS OF EXAMPLE 6.2.* The iteration formula of $\mathbb{J}$VIT as mentioned in (3.7) for equation(6.9) is given by

$$\varphi_{\lambda+1}(\sigma,\tau)=\varphi_\lambda(\sigma,0)+\mathbb{J}^{-1}\left\{\left(\frac{\vartheta}{s}\right)^\alpha \mathbb{J}\left\{\frac{\partial^2}{\partial\sigma^2}\varphi^2(\sigma,\tau)+\frac{\partial}{\partial\sigma}\left(\frac{\sigma}{3}\varphi_\lambda(\sigma,\tau)-\frac{4}{\sigma}\varphi_\lambda^2(\sigma,\tau)\right)\right\}\right\}. \tag{6.11}$$

Solving relation (6.11), we get

$$\begin{aligned}
\varphi_1(\sigma,\tau)=&\sigma^2\left(1+\frac{\tau^\alpha}{\Gamma(1+\alpha)}\right),\\
\varphi_2(\sigma,\tau)=&\sigma^2\left(1+\frac{\tau^\alpha}{\Gamma(1+\alpha)}+\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)}\right),\\
\varphi_3(\sigma,\tau)=&\sigma^2\left(1+\frac{\tau^\alpha}{\Gamma(1+\alpha)}+\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\tau^{3\alpha}}{\Gamma(1+3\alpha)}\right),\\
\varphi_4(\sigma,\tau)=&\sigma^2\left(1+\frac{\tau^\alpha}{\Gamma(1+\alpha)}+\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\tau^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{\tau^{4\alpha}}{\Gamma(1+4\alpha)}\right),\\
\varphi_5(\sigma,\tau)=&\sigma^2\left(1+\frac{\tau^\alpha}{\Gamma(1+\alpha)}+\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\tau^{3\alpha}}{\Gamma(1+3\alpha}+\frac{\tau^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{\tau^{5\alpha}}{\Gamma(1+5\alpha)}\right),\\
\vdots\quad&\vdots\\
\varphi_M(\sigma,\tau)=&\sigma^2\sum_{j=0}^{M}\frac{\tau^{j\alpha}}{\Gamma(j\alpha+1)},
\end{aligned}$$

which shows that iterative results are converging to the compact form of exact solution (6.10).

6.2.2. $_O$*HA*$\mathbb{J}$*TM SOLUTION OF EXAMPLE 6.2.* Consider the equation (6.9) and impose J -transform on both side to obtain

$$\mathbb{J}\left[\varphi(\sigma,\tau)\right](s,\vartheta)=\frac{\vartheta^2}{s}\varphi(\sigma,0)+\left(\frac{\vartheta}{s}\right)^\alpha\mathbb{J}\left[\frac{\partial^2\varphi^2(\sigma,\tau)}{\partial\sigma^2}+\frac{\partial}{\partial\sigma}\left(\frac{\sigma\ \varphi(\sigma,\tau)}{3}-\frac{4\ \varphi^2(\sigma,\tau)}{\sigma}\right)\right](s,\vartheta),$$

the nonlinear operator is computed as

$$\Xi[\Theta(\sigma,\tau;\omega)]=\mathbb{J}[\Theta(\sigma,\tau;\omega)](s,\vartheta)-\frac{\vartheta^2}{s}\Theta(\sigma,0)-\left(\frac{\vartheta}{s}\right)^\alpha\mathbb{J}\left[\frac{\partial^2\Theta^2(\sigma,\tau;\omega)}{\partial\sigma^2}+\frac{\partial}{\partial\sigma}\left(\frac{\sigma\ \Theta(\sigma,\tau;\omega)}{3}-\frac{4\ \Theta^2(\sigma,\tau;\omega)}{\sigma}\right)\right](s,\vartheta).$$

On substituting the above nonlinear equation into equation (5.5), we obtain the following recurrence formula

$$\varphi_\lambda(\sigma,\tau)=\chi_\lambda\varphi_{\lambda-1}(\sigma,\tau)+\hbar\mathbb{J}^{-1}\left[\mathbb{P}_\lambda(\overrightarrow{\varphi}_{\lambda-1}(\sigma,\tau))\right], \tag{6.12}$$

here

$$\begin{aligned}
\mathbb{P}_\lambda(\overrightarrow{\varphi}_{\lambda-1}(\sigma,\tau))=&\ \mathbb{J}\left[\varphi_{\lambda-1}(\sigma,\tau)\right](s,\vartheta)-(1-\chi_\lambda)\frac{\vartheta^2}{s}\varphi(\sigma,0)-\left(\frac{\vartheta}{s}\right)^\alpha\mathbb{J}\left[\frac{\partial^2}{\partial\sigma^2}\left(\sum_{\kappa=0}^{\lambda-1}\varphi_\kappa(\sigma,\tau)\varphi_{\lambda-1-\kappa}(\sigma,\tau)\right)\right.\\
&\left.+\frac{\partial}{\partial\sigma}\left(\frac{\sigma\ \varphi_{\lambda-1}(\sigma,\tau)}{3}-\frac{4}{\sigma}\left(\sum_{\kappa=0}^{\lambda-1}\varphi_\kappa(\sigma,\tau)\varphi_{\lambda-1-\kappa}(\sigma,\tau)\right)\right)\right](s,\vartheta).
\end{aligned}$$

On solving recurrence formula (6.12), we get

$$\begin{aligned}
\varphi_1(\sigma,\tau) &= -\frac{h\sigma^2\tau^\alpha}{\Gamma(1+\alpha)}, \qquad (6.13)\\
\varphi_2(\sigma,\tau) &= \sigma^2\left(\frac{h^2\tau^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{h^2\tau^\alpha}{\Gamma(1+\alpha)}-\frac{h\tau^\alpha}{\Gamma(1+\alpha)}\right),\\
\varphi_3(\sigma,\tau) &= \sigma^2\left(\frac{2h^3\tau^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{h^3\tau^{3\alpha}}{\Gamma(1+3\alpha)}-\frac{h^3\tau^\alpha}{\Gamma(1+\alpha)}+\frac{2h^2\tau^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{2h^2\tau^\alpha}{\Gamma(1+\alpha)}-\frac{h\tau^\alpha}{\Gamma(1+\alpha)}\right),\\
\varphi_4(\sigma,\tau) &= \sigma^2\left(\frac{3h^4\tau^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{3h^4\tau^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{h^4\tau^{4\alpha}}{\Gamma(1+4\alpha)}-\frac{h^4\tau^\alpha}{\Gamma(1+\alpha)}+\frac{6h^3\tau^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{3h^3\tau^{3\alpha}}{\Gamma(1+3\alpha)}-\frac{3h^3\tau^\alpha}{\Gamma(1+\alpha)}\right.\\
&\quad\left.+\frac{3h^2\tau^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{3h^2\tau^\alpha}{\Gamma(1+\alpha)}-\frac{h\tau^\alpha}{\Gamma(1+\alpha)}\right).
\end{aligned}$$

The term $\varphi_\lambda(\sigma,\tau),\ \lambda\geq 4$ can be computed in similar way. $5th$-order solution of equation (6.9) is obtained as

$$\begin{aligned}
S_5(\sigma,\tau) &= \sum_{\lambda=0}^{5}\varphi_\lambda(\sigma,\tau)=\sigma^2\left(\frac{4h^5\tau^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{6h^5\tau^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{4h^5\tau^{4\alpha}}{\Gamma(1+4\alpha)}-\frac{h^5\tau^{5\alpha}}{\Gamma(1+5\alpha)}-\frac{h^5\tau^\alpha}{\Gamma(1+\alpha)}+\frac{15h^4\tau^{2\alpha}}{\Gamma(1+2\alpha)}\right. \qquad (6.14)\\
&\quad-\frac{15h^4\tau^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{5h^4\tau^{4\alpha}}{\Gamma(1+4\alpha)}-\frac{5h^4\tau^\alpha}{\Gamma(1+\alpha)}+\frac{20h^3\tau^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{10h^3\tau^{3\alpha}}{\Gamma(1+3\alpha)}-\frac{10h^3\tau^\alpha}{\Gamma(1+\alpha)}\\
&\quad\left.+\frac{10h^2\tau^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{10h^2\tau^\alpha}{\Gamma(1+\alpha)}-\frac{5h\tau^\alpha}{\Gamma(1+\alpha)}+1\right),
\end{aligned}$$

and in special case for $h=-1$, (6.14) reduces to

$$S_5(\sigma,\tau)=\sigma^2\sum_{j=0}^{5}\frac{\tau^{j\alpha}}{\Gamma(j\alpha+1)}, \qquad (6.15)$$

which is the closed form of solution (6.10) of TF-FPE (6.9).

TABLE 1. Optimal value of $\hbar$ in ${}_O$HAJTM results $\wp_5(\sigma,\tau)$ and associated minimum residual errors $(\mathcal{R}_5(\sigma,\tau))$ in example 6.1 for different values of α,β.

α	β	Optimal- $\hbar$	$\mathcal{R}_5(\sigma,\tau)$
0.50	0.50	-1.1805874502	1.0910000000E-10
0.75	0.75	-1.0851530113	7.7954899577E-12
0.85	0.85	-1.0631350530	2.0088862040E-12
0.95	0.95	-1.0467549989	4.5574604183E-13

6.3. **EXAMPLE 6.3.**

Example 6.3. The last test example deals with nonlinear TF-FPE of the form

$${}_\tau\mathcal{D}_\alpha^C\varphi(\sigma,\tau)=-\frac{\partial}{\partial x}\left[\left(3\varphi(\sigma,\tau)-\frac{\sigma}{2}\right)\varphi(\sigma,\tau)\right]+\frac{\partial^2}{\partial x^2}\left[\sigma\varphi^2(\sigma,\tau)\right],\ \varphi(\sigma,0)=\sigma,\ 0<\alpha,\leq 1,\ \sigma,\ \tau>0. \qquad (6.16)$$

and the exact solution of equation (6.16) is

$$\varphi(\sigma,\tau)=\sigma\sum_{r=0}^{\infty}\frac{\tau^{r\alpha}}{\Gamma(1+r\alpha)}=E_{\alpha,1}(\tau^\alpha). \qquad (6.17)$$

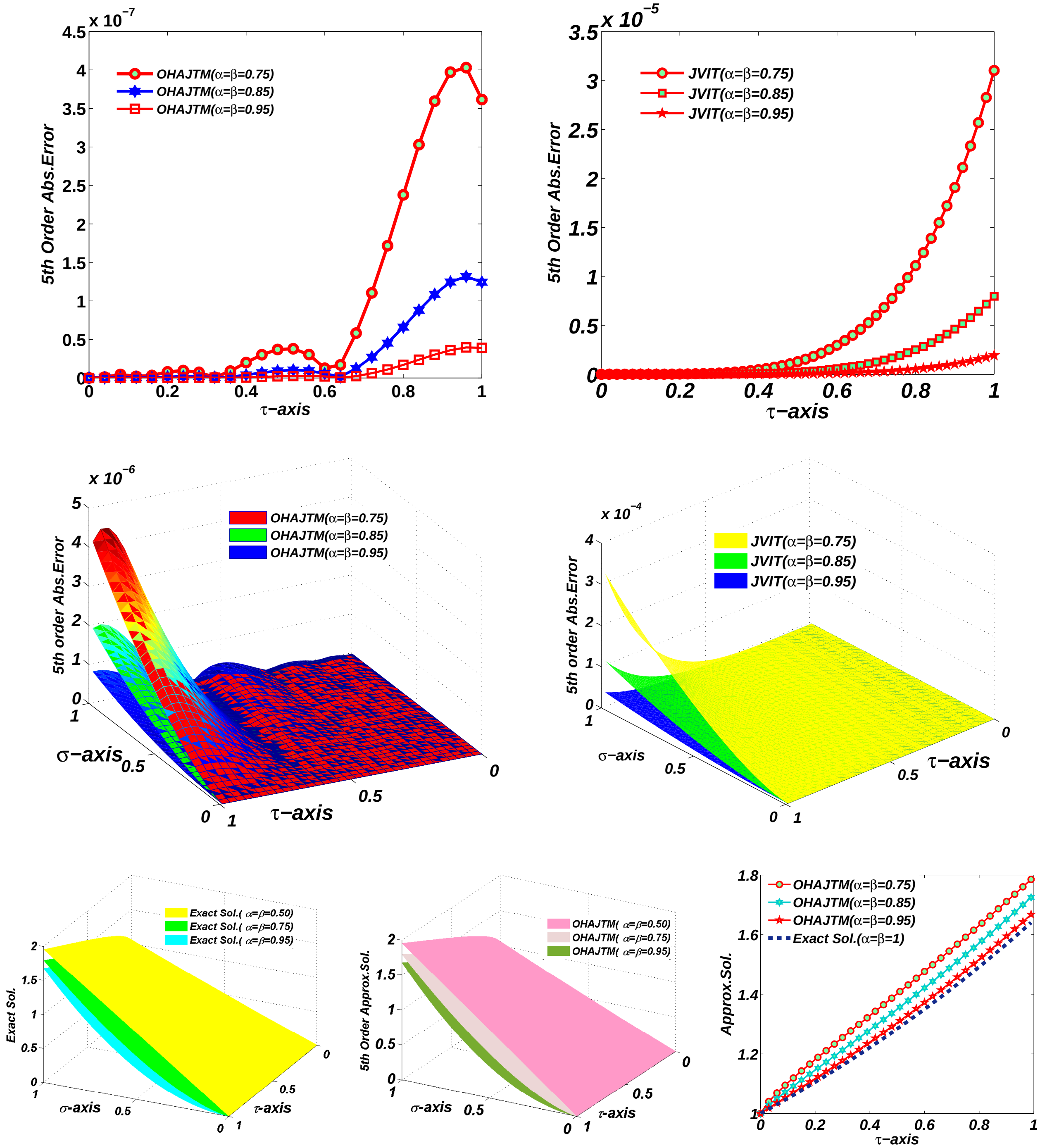

Figure 1. Two dimensional plots and surface behaviour of absolute errors in 5th-order evaluated results for STF-FPE (6.1) with different $\alpha = \beta$, and the comparison of computed results with exact results (6.2).

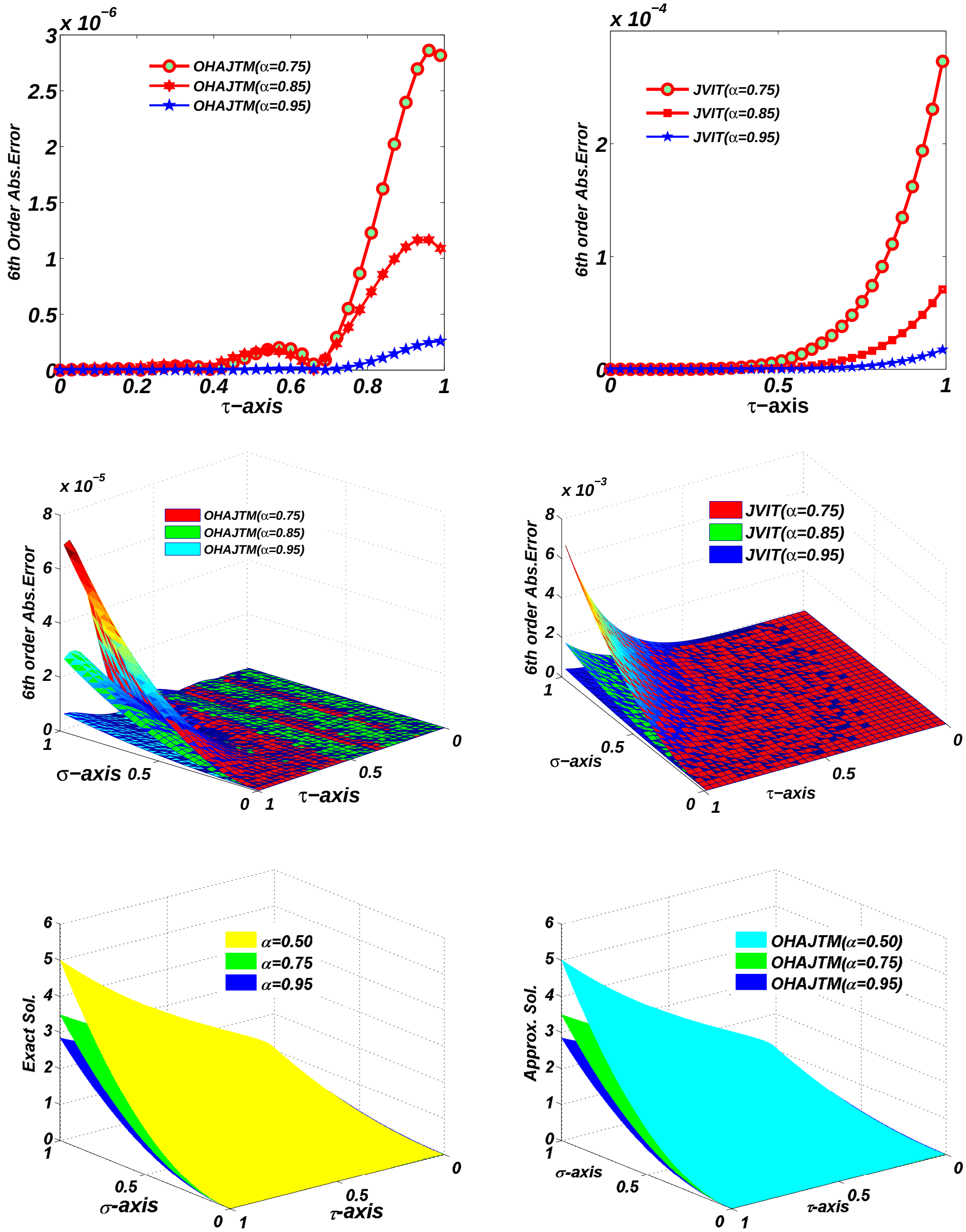

FIGURE 2. Two dimensional plots and surface behaviour of absolute errors in 5th-order evaluated results for TF-FPE (6.9) with different α, and the comparison of the computed results with exact result (6.10).

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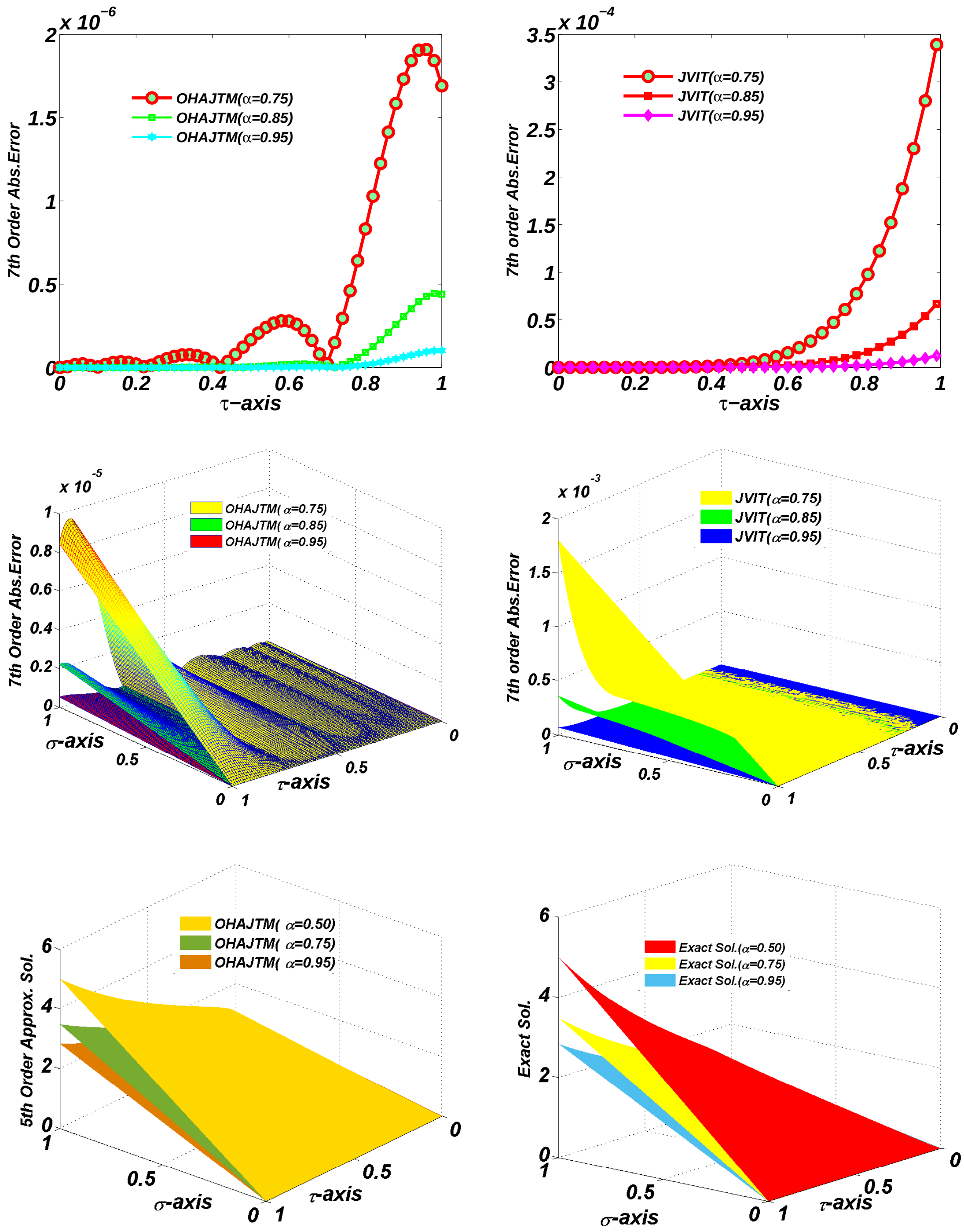

FIGURE 3. Two Dimensional plots and surface behaviour of absolute errors in 7th-order evaluated results for TF-FPE (6.16) with different α, and the comparison of computed 5th-order results with exact result (6.17).

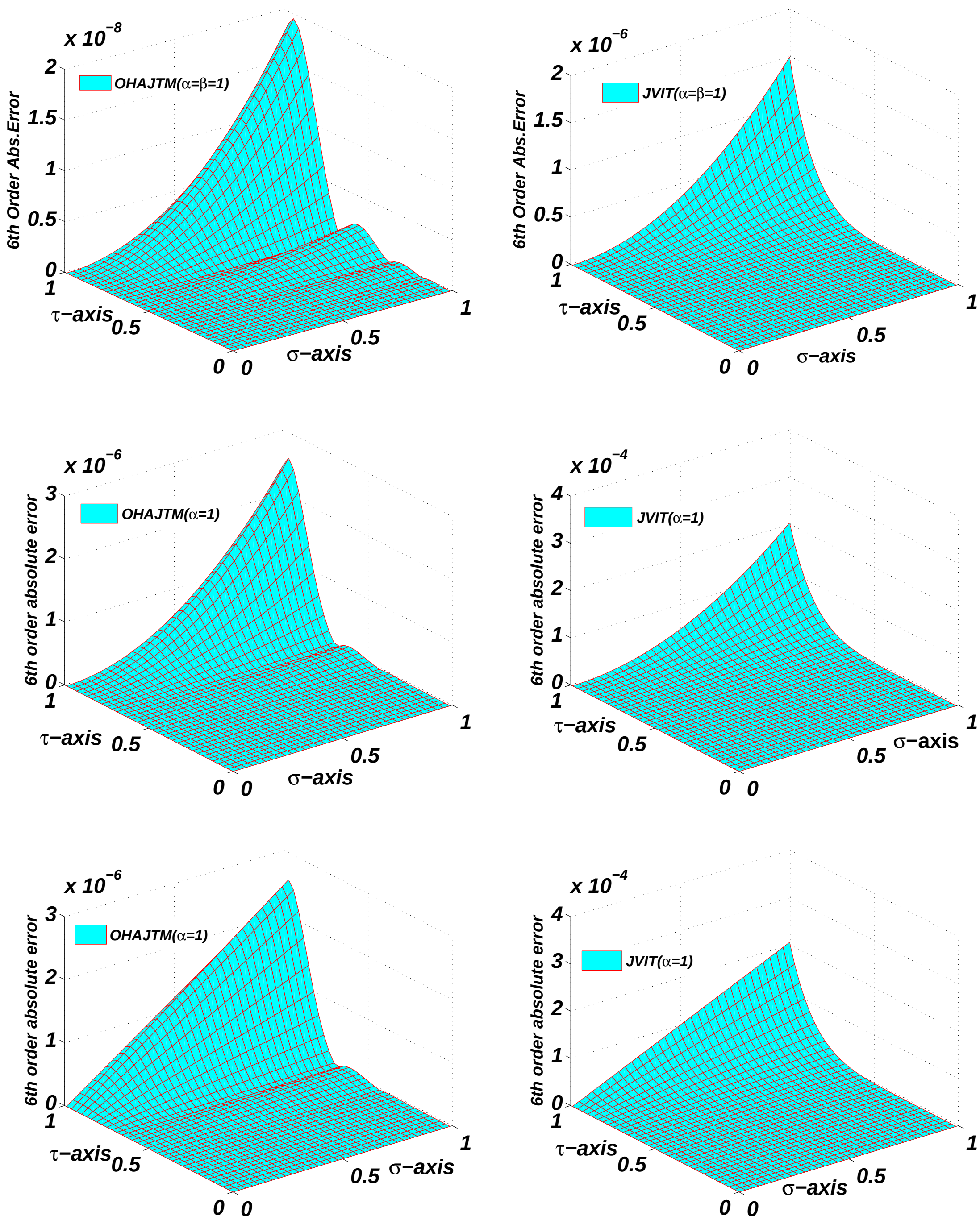

Figure 4. Comparison of 6th-order absolute-errors in computed results for STF-FPE (6.1) with $\alpha = \beta = 1$ (top), TF-FPE (6.9) with $\alpha = 1$ (mid), and TF-FPE (6.16) for $\alpha = 1$ (bottom).

Table 2. Comparison of evaluated 3rd-order results via $\mathbb{J}$-VIT and ${}_O$HA$\mathbb{J}$TM with optimal $\hbar = -1.00039449313177$ with the results from existing techniques: ADM [34], RPSM [70], q-HATM[37], FRDTM and FVIM [41], and exact result $\varphi^e(\sigma,\tau)$ for $\alpha=\beta=1$ in example 6.1.

σ	τ	[34]	[41]	[70]	[37]	$\varphi^e(\sigma,\tau)$	${}_O$HA$\mathbb{J}$TM	$\mathbb{J}$-VIT
0.25	0.20	0.06906	0.0691	0.0691	0.0691	0.06907	0.06907	0.06907
0.50		0.27626	0.2763	0.2763	0.2763	0.27629	0.27629	0.27629
0.75		0.62156	0.6216	0.6216	0.6216	0.62166	0.62165	0.62166
0.25	0.40	0.07625	0.0762	0.0762	0.0762	0.07634	0.07634	0.07633
0.50		0.30500	0.3050	0.3050	0.3050	0.30535	0.30535	0.30533
0.75		0.68625	0.6863	0.6863	0.6863	0.68704	0.68703	0.68700
0.25	0.60	0.08406	0.0841	0.0841	0.0841	0.08437	0.08437	0.08434
0.50		0.33625	0.3362	0.3362	0.3362	0.33746	0.33748	0.33738
0.75		0.75656	0.7566	0.7566	0.7566	0.75930	0.75933	0.75909
					absolute-errors			
0.25	0.20	0.00038	0.0000	0.0000	0.0000		0.00000027	0.00000025
0.50	0.40	0.00040	0.0004	0.0004	0.0004		0.00001736	0.00001696
0.75	0.60	0.00274	0.0027	0.0027	0.0027		0.00020183	0.00019884

Table 3. Evaluation of absolute-errors in Mth-order ($M=3,6$) $\mathbb{J}$-VIT results and ${}_O$HA$\mathbb{J}$TM results for optimal $\hbar$ with existing techniques: HATM[23], RPSM [70], q-HATM[37], FRDTM [41] for $\alpha = 1, \beta = 1$.

example 6.1		$M=3$			$M=6$	
σ	τ	[23, 37, 41, 70]	$\mathbb{J}$-VIT	${}_O$HA$\mathbb{J}$TM	$\mathbb{J}$-VIT	${}_O$HA$\mathbb{J}$TM
-5	0.5	4.28125E-03	4.28125E-03	2.74821E-04	3.12484E-07	5.18385E-08
-10	0.6	3.58808E-02	3.58808E-02	6.04504E-03	4.50760E-06	1.85280E-07
15	0.9	4.21804E-01	4.21804E-01	4.58949E-02	1.76691E-04	4.35378E-06
				$\hbar = -1.0752638316811443$	$\hbar = -1.0493389713885495$	
example 6.2		[23, 37, 41, 70]	$\mathbb{J}$-VIT	${}_O$HA$\mathbb{J}$TM	${}_O$HA$\mathbb{J}$TM	$\mathbb{J}$-VIT
-5	0.5	0.07219845	0.07219845	0.004716389	1.47786E-06	4.131611E-05
-10	0.6	0.61188	0.61188	0.121402721	1.16291E-05	6.000391E-04
5	0.9	0.82757778	0.82757778	0.112903672	5.54490E-05	2.668716E-03
				$\hbar = -1.16192032020698$	$\hbar = -1.06343438388067$	

6.3.1. *$\mathbb{J}$-VIT SOLUTIONS OF EXAMPLE 6.3.* Taking into account the equation (3.7), the desired recurrence function is written down as

$$\varphi_{\lambda+1}(\sigma,\tau) = \varphi_\lambda(\sigma,0) + \mathbb{J}^{-1}\left\{\left(\frac{\vartheta}{s}\right)^\alpha \mathbb{J}\left\{-\frac{\partial}{\partial x}\left[\left(3\varphi(\sigma,\tau)-\frac{\sigma}{2}\right)\varphi(\sigma,\tau)\right] + \frac{\partial^2}{\partial x^2}\left[\sigma\varphi^2(\sigma,\tau)\right]\right\}\right\}. \tag{6.18}$$

Solving recurrence (6.18) with aid of Mathematica 10.0, we get

$$\begin{aligned}\varphi_1(\sigma,\tau) &= \sigma\left(1+\frac{\tau^\alpha}{\Gamma(1+\alpha)}\right),\\ \varphi_2(\sigma,\tau) &= \sigma\left(1+\frac{\tau^\alpha}{\Gamma(1+\alpha)}+\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)}\right),\end{aligned}$$

TABLE 4. Comparison of exact result $\varphi(\sigma,\tau)$ with 5th-order ${}_O$HAJTM results $\wp_5(\sigma,\tau)$, $\mathbb{J}$-VIT results $\varphi_5(\sigma,\tau)$ and their associated absolute-errors in example 6.1 with different values of $\alpha=\beta$ for different grids σ at different time levels.

α	β	τ	σ	$\wp_5(\sigma,\tau)$	$\varphi_5(\sigma,\tau)$	$\varphi(\sigma,\tau)$	$\lvert\varphi(\sigma,\tau)-\varphi_5(\sigma,\tau)\rvert$	$\lvert\varphi(\sigma,\tau)-\wp_5(\sigma,\tau)\rvert$
0.75	0.75	0.3	0.25	0.1568750468	0.1568748651	0.1568750406	1.755660E-07	6.171210E-09
			0.50	0.4437096376	0.4437091235	0.4437096201	4.965740E-07	1.745480E-08
			0.75	0.8151466545	0.8151457102	0.8151466224	9.122650E-07	3.206660E-08
0.75	0.75	0.6	0.25	0.1845480580	0.1845439336	0.1845480754	4.141800E-06	1.745530E-08
			0.50	0.5219807330	0.5219690676	0.5219807824	1.171477E-05	4.937100E-08
			0.75	0.9589398386	0.9589184079	0.9589399293	2.152141E-05	9.070030E-08
0.75	0.75	0.9	0.25	0.2138792805	0.2138520797	0.2138787474	2.666772E-05	5.330920E-07
			0.50	0.6049419583	0.6048650228	0.6049404505	7.542770E-05	1.507810E-06
			0.75	1.1113493414	1.1112080019	1.1113465714	1.385695E-04	2.770030E-06
0.85	0.85	0.3	0.25	0.1148806705	0.1148806450	0.1148806683	2.329080E-08	2.153480E-09
			0.50	0.3732483995	0.3732483168	0.3732483925	7.567220E-08	6.996670E-09
			0.75	0.7436234805	0.7436233158	0.7436234665	1.507620E-07	1.393950E-08
0.85	0.85	0.6	0.25	0.1346430526	0.1346422355	0.1346430608	8.253720E-07	8.229530E-09
			0.50	0.4374565685	0.4374539136	0.4374565952	2.681600E-06	2.673780E-08
			0.75	0.8715455351	0.8715402457	0.8715455883	5.342640E-06	5.326980E-08
0.85	0.85	0.9	0.25	0.1563948811	0.1563879771	0.1563947096	6.732490E-06	1.715370E-07
			0.50	0.5081284678	0.5081060366	0.5081279105	2.187392E-05	5.573240E-07
			0.75	1.0123452915	1.0123006017	1.0123441811	4.357946E-05	1.110360E-06
0.95	0.95	0.3	0.25	0.0845237171	0.0845237136	0.0845237165	3.846290E-09	5.778230E-10
			0.50	0.3154536663	0.3154536533	0.3154536642	1.249660E-08	2.156510E-09
			0.75	0.6815676542	0.6815676261	0.6815676496	2.489710E-08	4.659350E-09
0.95	0.95	0.6	0.25	0.0985224046	0.0985222520	0.0985224074	2.050460E-07	2.858700E-09
			0.50	0.3676986155	0.3676980462	0.3676986261	6.661970E-07	1.066910E-08
			0.75	0.7944478367	0.7944466067	0.7944478597	1.327260E-06	2.305150E-08

TABLE 5. Evaluation of Mth-order $\mathbb{J}$-VIT, ${}_O$HAJTM results for optimal $\hbar$ ($\hbar = -1.00157984223391, -1.06344639936478$ for $M=3,6$, resp.) and their comparison with HATM [23], RPSM [70], q-HATM [37], FRDTM [41] and exact results for $\alpha=1$ in example 6.2.

		$M=3$				$M=6$	
σ	τ	[23, 37, 41, 70]	${}_O$HAJTM	$\varphi^e(\sigma,\tau)$	$\mathbb{J}$-VIT	${}_O$HAJTM	$\mathbb{J}$-VIT
0.25	0.01	0.06312814	0.06312814	0.06312814	0.06312814	0.06312814	0.06312814
0.50		0.25251254	0.25251254	0.25251254	0.25251254	0.25251254	0.25251254
0.75		0.56815322	0.56815322	0.56815322	0.56815322	0.56815322	0.56815322
1.00		1.01005017	1.01005017	1.01005017	1.01005017	1.01005017	1.01005017
0.05	0.6	0.00454000	0.00454042	0.00455530	0.00454000	0.00455530	0.00455528
0.50		0.45400000	0.45404239	0.45552970	0.45404239	0.45552967	0.45552820
0.75		1.02150000	1.02159537	1.02494183	1.02159537	1.02494176	1.02493845
1.00		1.81600000	1.81616955	1.82211880	1.81616955	1.82211868	1.82211280

$$\varphi_3(\sigma,\tau) = \sigma\left(1+\frac{\tau^{\alpha}}{\Gamma(1+\alpha)}+\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\tau^{3\alpha}}{\Gamma(1+3\alpha)}\right),$$

$$\varphi_4(\sigma,\tau) = \sigma\left(1+\frac{\tau^{\alpha}}{\Gamma(1+\alpha)}+\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\tau^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{\tau^{4\alpha}}{\Gamma(1+4\alpha)}\right),$$

TABLE 6. Optimal value of $\hbar$ in ${}_O$HA𝕁TM results $\wp_M(\sigma,\tau)$ $(M=5,6,7)$ and associated minimum residual error $(\mathcal{R}_M(\sigma,\tau))$ for different values of α.

Example 6.2	$M=5$			$M=6$	
α	Optimal-$\hbar$	$\mathcal{R}_5(\sigma,\tau)$	α	Optimal - $\hbar$	$\mathcal{R}_6(\sigma,\tau)$
0.50	-1.4389057859	2.1463891002E-06	0.75	-1.1528124727	1.2763196681E-09
0.75	-1.1857821539	6.5058687767E-08	0.85	-1.1700413321	4.8254748800E-10
0.95	-1.0979845100	2.9183721298E-09	0.95	-1.0777543792	2.3591830740E-11
Example 6.3	$M=5$		$M=7$		
α	Optimal- $\hbar$	$\mathcal{R}_5(\sigma,\tau)$	α	Optimal- $\hbar$	$\mathcal{R}_7(\sigma,\tau)$
0.50	-1.4389057947	3.2619895140E-06	0.75	-1.1871210669	5.0403615504E-11
0.75	-1.1857821539	9.8873385661E-08	0.85	-1.0912891910	2.9581728919E-12
0.95	-1.0979845100	4.4352160033E-09	0.95	-1.0642513232	2.3397249741E-13

TABLE 7. Comparison of exact result $\varphi(\sigma,\tau)$ with 7th-order ${}_O$HA𝕁TM results $\wp_7(\sigma,\tau)$, 𝕁-VIT results $\varphi_7(\sigma,\tau)$ and their associated absolute-errors in example 6.2 with different values of $\alpha=\beta$ for different grids σ at different time levels.

α	β	τ	σ	$\wp_7(\sigma,\tau)$	$\varphi_7(\sigma,\tau)$	$\varphi(\sigma,\tau)$	$\lvert\varphi(\sigma,\tau)-\varphi_7(\sigma,\tau)\rvert$	$\lvert\varphi(\sigma,\tau)-\wp_7(\sigma,\tau)\rvert$
0.75	0.75	0.3	0.25	0.0997530783	0.0997530291	0.0997530990	6.990230E-08	2.073200E-08
			0.50	0.3990123131	0.3990121164	0.3990123960	2.796090E-07	8.292800E-08
			0.75	0.8977777044	0.8977772619	0.8977778910	6.291200E-07	1.865880E-07
0.75	0.75	0.6	0.25	0.1414322410	0.1414273420	0.1414321540	4.812020E-06	8.695540E-08
			0.50	0.5657289639	0.5657093680	0.5657286161	1.924809E-05	3.478220E-07
			0.75	1.2728901688	1.2728460780	1.2728893862	4.330821E-05	7.825990E-07
0.75	0.75	0.9	0.25	0.1959502897	0.1958921818	0.1959508306	5.864879E-05	5.408500E-07
			0.50	0.7838011589	0.7835687271	0.7838033223	2.345952E-04	2.163400E-06
			0.75	1.7635526075	1.7630296361	1.7635574752	5.278391E-04	4.867650E-06
0.85	0.85	0.3	0.25	0.0924190621	0.0924190565	0.0924190620	5.495110E-09	9.074270E-11
			0.50	0.3696762484	0.3696762260	0.3696762480	2.198040E-08	3.629710E-10
			0.75	0.8317715588	0.8317715086	0.8317715580	4.945590E-08	8.166870E-10
0.85	0.85	0.6	0.25	0.1284548920	0.1284542510	0.1284548977	6.467500E-07	5.663340E-09
			0.50	0.5138195682	0.5138170038	0.5138195908	2.587000E-06	2.265340E-08
			0.75	1.1560940284	1.1560882586	1.1560940794	5.820750E-06	5.097010E-08
0.85	0.85	0.9	0.25	0.1761194441	0.1761086068	0.1761193487	1.074188E-05	9.542670E-08
			0.50	0.7044777764	0.7044344271	0.7044773947	4.296750E-05	3.817070E-07
			0.75	1.5850749968	1.5849774611	1.5850741380	9.667689E-05	8.588400E-07
0.95	0.95	0.3	0.25	0.0867386606	0.0867386602	0.0867386606	4.003810E-10	9.172800E-13
			0.50	0.3469546426	0.3469546410	0.3469546426	1.601520E-09	3.669120E-12
			0.75	0.7806479458	0.7806479422	0.7806479458	3.603420E-09	8.255620E-12
0.95	0.95	0.6	0.25	0.1182237316	0.1182236517	0.1182237326	8.089100E-08	9.559210E-10
			0.50	0.4728949265	0.4728946068	0.4728949304	3.235640E-07	3.823680E-09
			0.75	1.0640135847	1.0640128653	1.0640135933	7.280190E-07	8.603290E-09

$$\begin{aligned}
\varphi_5(\sigma,\tau) &= \sigma\left(1+\frac{\tau^{\alpha}}{\Gamma(1+\alpha)}+\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\tau^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{\tau^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{\tau^{5\alpha}}{\Gamma(1+5\alpha)}\right),\\
&\vdots\\
\varphi_M(\sigma,\tau) &= \sigma\sum_{j=0}^{M}\frac{\tau^{j\alpha}}{\Gamma(1+j\alpha)}.
\end{aligned}$$

Table 8. The relative errors in consecutive order solutions for example 6.2 via $\mathbb{J}$-VIT and ${}_O$HA$\mathbb{J}$TM for $\alpha = 0.95$.

$\mathbb{J}$-VIT | σ	τ	E_5	E_6	E_7	E_8	E_9
0.25	0.30	3.002986793E-05	1.823390000E-06	9.569280000E-08	4.425760000E-09	1.830870000E-10
0.50		3.002986793E-05	1.823390000E-06	9.569280000E-08	4.425760000E-09	1.830870000E-10
0.75		3.002986793E-05	1.823390000E-06	9.569280000E-08	4.425760000E-09	1.830870000E-10
0.25	0.50	2.763001990E-04	2.725543856E-05	2.323850000E-06	1.746110000E-07	1.173540000E-08
0.50		2.763001990E-04	2.725543856E-05	2.323850000E-06	1.746110000E-07	1.173540000E-08
0.75		2.763001990E-04	2.725543856E-05	2.323850000E-06	1.746110000E-07	1.173540000E-08
0.25	1.00	4.473888162E-03	8.518778178E-04	1.402975671E-04	2.036501171E-05	2.644160000E-06
0.50		4.473888162E-03	8.518778178E-04	1.402975671E-04	2.036501171E-05	2.644160000E-06
0.75		4.473888162E-03	8.518778178E-04	1.402975671E-04	2.036501171E-05	2.644160000E-06
${}_O$HA$\mathbb{J}$TM						
0.25	0.30	1.272247784E-05	-2.473010000E-06	-2.261070000E-08	3.262460000E-09	-3.986860000E-11
0.50		1.272247784E-05	-2.473010000E-06	-2.261070000E-08	3.262460000E-09	-3.986860000E-11
0.75		1.272247784E-05	-2.473010000E-06	-2.261070000E-08	3.262470000E-09	-3.987800000E-11
0.25	0.50	3.347027446E-05	1.160000000E-06	8.958570000E-08	-1.074469234E-08	4.155450000E-11
0.50		3.347027446E-05	1.160000000E-06	8.958570000E-08	-1.074469234E-08	4.155450000E-11
0.75		3.347027446E-05	1.160000000E-06	8.958570000E-08	-1.074469620E-08	4.155700000E-11
0.25	1.00	-2.374232953E-04	-3.103536238E-05	-2.328220000E-06	4.402400000E-08	1.027810000E-08
0.50		-2.374232953E-04	-3.103536238E-05	-2.328220000E-06	4.402400000E-08	1.027810000E-08
0.75		-2.374232953E-04	-3.103536238E-05	-2.328220000E-06	4.402400000E-08	1.027810000E-08

As $M \to \infty$, we get exact solution $\varphi(\sigma,\tau) = \lim_{M\to\infty} \varphi_M(\sigma,\tau)$ as

$$\varphi(\sigma,\tau) = \sigma \sum_{j=0}^{\infty} \frac{\tau^{j\alpha}}{\Gamma(1+j\alpha)} = \sigma\ E_{\alpha,1}(\tau^{\alpha}), \tag{6.19}$$

which is the exact solution of equation (6.3).

6.3.2. *${}_O$HA$\mathbb{J}$TM SOLUTION OF EXAMPLE 6.3.* Consider the equation (6.16) and impose $\mathbb{J}$-transform to obtain

$$\mathbb{J}\left[\varphi(\sigma,\tau)\right](s,\vartheta) = \frac{\vartheta^2}{s}\varphi(\sigma,0) + \left(\frac{\vartheta}{s}\right)^{\alpha} \mathbb{J}\left[\frac{\partial^2\varphi^2(\sigma,\tau)}{\partial\sigma^2} - \frac{\partial}{\partial\sigma}\left(3\ \varphi^2(\sigma,\tau) - \frac{\sigma\ \varphi(\sigma,\tau)}{2}\right)\right](s,\vartheta), \tag{6.20}$$

and consequently the nonlinear operator

$$\begin{aligned}\Xi[\Theta(\sigma,\tau;\omega)] =& \mathbb{J}[\Theta(\sigma,\tau;\omega)](s,\vartheta) - \frac{\vartheta^2\varphi(\sigma,0)}{s} \\ &- \left(\frac{\vartheta}{s}\right)^{\alpha} \mathbb{J}\left[\frac{\partial^2\Theta^2(\sigma,\tau;\omega)}{\partial\sigma^2} - \frac{\partial}{\partial\sigma}\left(3\ \Theta^2(\sigma,\tau;\omega) - \frac{\sigma\ \Theta(\sigma,\tau;\omega)}{2}\right)\right](s,\vartheta).\end{aligned} \tag{6.21}$$

Utilizing (6.21), the recurrence formula (5.5) can be re-written as

$$\varphi_\lambda(\sigma,\tau) = \chi_\lambda\varphi_{\lambda-1}(\sigma,\tau) + \hbar\mathbb{J}^{-1}\left[\mathbb{P}_\lambda(\overrightarrow{\varphi}_{\lambda-1}(\sigma,\tau))\right], \tag{6.22}$$

here

$$\begin{aligned}\mathbb{P}_\lambda(\overrightarrow{\varphi}_{\lambda-1}(\sigma,\tau)) =& \mathbb{J}\left[\varphi_{\lambda-1}(\sigma,\tau)\right](s,\vartheta) - (1-\chi_\lambda)\frac{\vartheta^2}{s}\varphi(\sigma,0) - \left(\frac{\vartheta}{s}\right)^{\alpha}\mathbb{J}\left[\frac{\partial^2}{\partial\sigma^2}\left(\sigma\sum_{\kappa=0}^{\lambda-1}\varphi_\kappa(\sigma,\tau)\varphi_{\lambda-1-\kappa}(\sigma,\tau)\right)\right. \\ &\left.-\frac{\partial}{\partial\sigma}\left(3\sum_{\kappa=0}^{\lambda-1}\varphi_\kappa(\sigma,\tau)\varphi_{\lambda-1-\kappa}(\sigma,\tau)\right) - \frac{\sigma\varphi_{\lambda-1}(\sigma,\tau)}{2}\right](s,\vartheta).\end{aligned}$$

Table 9. The order of convergence of both the concerned method by taking the example 6.2.

n			${}_{O}$HA$\mathbb{J}$TM, $\alpha = 0.5$			$\mathbb{J}$VIT,$\alpha = 0.5$	
	L2 Error	OC(h=-1)	Optimal h	L2-Error	OC	L2 Error	OC
3	0.46188		-1.667380	3.93139E-02		4.61878E-01	
4	0.23958	2.281732	-1.516470	9.13584E-03	5.07287	2.39580E-01	2.28173
5	0.11720	3.204164	-1.425890	1.99149E-03	6.82665	1.17202E-01	3.20416
6	0.05437	4.212943	-1.366030	4.19024E-04	8.54924	5.43687E-02	4.21294
7	0.02403	5.297509	-1.323580	8.57014E-05	10.29550	2.40269E-02	5.29751
8	0.01016	6.449257	-1.291850	1.70636E-05	12.08646	1.01553E-02	6.44926
9	0.00412	7.661008	-1.335910	2.32273E-06	16.93117	4.11920E-03	7.66101
10	0.00161	8.927083	1.304350	3.67967E-07	17.48763	1.60817E-03	8.92708

n			${}_{O}$HA$\mathbb{J}$TM, $\alpha = 0.7$			$\mathbb{J}$VIT, $\alpha = 0.7$	
	L2 Error	OC(h=-1)	Optimal h	L2 Error	OC	L2 Error	OC
3	0.12715		-1.366430	1.30075E-02		1.27150E-01	
4	0.04446	3.652589	-1.267530	2.02634E-03	6.46302	4.44597E-02	3.65259
5	0.01411	5.142021	-1.211700	2.97111E-04	8.60379	1.41141E-02	5.14202
6	0.00411	6.763736	-1.176180	4.15283E-05	10.79264	4.11237E-03	6.76374
7	0.00111	8.499113	-1.151610	5.53782E-06	13.07016	1.10944E-03	8.49911
8	0.00028	10.333718	-1.180690	6.01158E-07	16.62904	2.79146E-04	10.33372
9	0.00007	12.255953	-1.157090	5.99913E-08	19.56698	6.59035E-05	12.25595
10	0.00001	14.256771	-1.139250	5.84866E-09	22.09544	1.46742E-05	14.25677

			${}_{O}$HA$\mathbb{J}$TM, $\alpha = 0.95$			$\mathbb{J}$VIT, $\alpha = 0.95$	
n	L2 Error	OC(h=-1)	Optimal h	L2 Error	OC	L2 Error	OC
3	0.02382		-1.184900	3.07077E-03		2.38177E-02	
4	0.00472	5.623917	-1.125610	2.81746E-04	8.30319	4.72340E-03	5.62392
5	0.00081	7.910237	-1.094170	2.37886E-05	11.07717	8.08488E-04	7.91024
6	0.00012	10.385317	-1.075000	1.85353E-06	13.99788	1.21717E-04	10.38532
7	0.00002	13.020204	-1.062200	1.33621E-07	17.06019	1.63565E-05	13.02020
8	0.00000	15.793220	-1.076170	7.99796E-09	21.08733	1.98523E-06	15.79322
9	0.00000	18.687670	-1.064050	4.08998E-10	25.24329	2.19730E-07	18.68767
10	0.00000	21.690122	-1.055130	1.99718E-11	28.65769	2.23565E-08	21.69012

Solution of (6.22) is evaluated as

$$
\begin{aligned}
\varphi_1(\sigma,\tau) &= \sigma\left(-\frac{h\tau^{\alpha}}{\Gamma(1+\alpha)}\right), \qquad (6.23)\\
\varphi_2(\sigma,\tau) &= \sigma\left(\frac{h^2\tau^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{h^2\tau^{\alpha}}{\Gamma(1+\alpha)} - \frac{h\tau^{\alpha}}{\Gamma(1+\alpha)}\right),\\
\varphi_3(\sigma,\tau) &= \sigma\left(\frac{2h^3\tau^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{h^3\tau^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{h^3\tau^{\alpha}}{\Gamma(1+\alpha)} + \frac{2h^2\tau^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{2h^2\tau^{\alpha}}{\Gamma(1+\alpha)} - \frac{h\tau^{\alpha}}{\Gamma(1+\alpha)}\right),\\
\vdots \quad & \quad \vdots
\end{aligned}
$$

TABLE 10. Comparison of exact result $\varphi(\sigma,\tau)$ with 7th-order $_{O}$HA$\mathbb{J}$TM results $\wp_7(\sigma,\tau)$, $\mathbb{J}$-VIT results $\varphi_7(\sigma,\tau)$ and their associated absolute-errors in example 6.3 with different values of $\alpha=\beta$ for different grids σ at different time levels.

α	β	τ	σ	$\wp_7(\sigma,\tau)$	$\varphi_7(\sigma,\tau)$	$\varphi(\sigma,\tau)$	$\lvert\varphi(\sigma,\tau)-\varphi_7(\sigma,\tau)\rvert$	$\lvert\varphi(\sigma,\tau)-\wp_7(\sigma,\tau)\rvert$
0.75	0.75	0.3	0.25	0.3990123131	0.3990121164	0.3990123960	2.796090E-07	8.292800E-08
			0.50	0.7980246262	0.7980242328	0.7980247920	5.592180E-07	1.658560E-07
			0.75	1.1970369392	1.1970363492	1.1970371880	8.388270E-07	2.487840E-07
0.75	0.75	0.6	0.25	0.5657289639	0.5657093680	0.5657286161	1.924809E-05	3.478220E-07
			0.50	1.1314579278	1.1314187360	1.1314572322	3.849618E-05	6.956440E-07
			0.75	1.6971868918	1.6971281040	1.6971858483	5.774428E-05	1.043470E-06
0.75	0.75	0.9	0.25	0.7838011589	0.7835687271	0.7838033223	2.345952E-04	2.163400E-06
			0.50	1.5676023178	1.5671374543	1.5676066446	4.691903E-04	4.326800E-06
			0.75	2.3514034767	2.3507061814	2.3514099669	7.037855E-04	6.490200E-06
0.85	0.85	0.3	0.25	0.3696762484	0.3696762260	0.3696762480	2.198040E-08	3.629700E-10
			0.50	0.7393524967	0.7393524521	0.7393524960	4.396080E-08	7.259410E-10
			0.75	1.1090287451	1.1090286781	1.1090287440	6.594130E-08	1.088910E-09
0.85	0.85	0.6	0.25	0.5138195682	0.5138170038	0.5138195908	2.587000E-06	2.265340E-08
			0.50	1.0276391363	1.0276340077	1.0276391817	5.174000E-06	4.530670E-08
			0.75	1.5414587045	1.5414510115	1.5414587725	7.761000E-06	6.796010E-08
0.85	0.85	0.9	0.25	1.0262243049	0.7044344271	0.7044773947	4.296750E-05	3.817070E-07
			0.50	2.0524486099	1.4088688543	1.4089547893	8.593501E-05	7.634130E-07
			0.75	3.0786729148	2.1133032814	2.1134321840	1.289025E-04	1.145120E-06
0.95	0.95	0.3	0.25	0.3469546426	0.3469546410	0.3469546426	1.601520E-09	3.669120E-12
			0.50	0.6939092851	0.6939092819	0.6939092851	3.203040E-09	7.338240E-12
			0.75	1.0408639277	1.0408639229	1.0408639277	4.804570E-09	1.100990E-11
0.95	0.95	0.6	0.25	0.4728949265	0.4728946068	0.4728949304	3.235640E-07	3.823680E-09
			0.50	0.9457898531	0.9457892136	0.9457898607	6.471280E-07	7.647370E-09
			0.75	1.4186847796	1.4186838204	1.4186847911	9.706920E-07	1.147110E-08

the term $\varphi_\lambda(\sigma,\tau),\quad \lambda\geq 4$ can be evaluated is same fashion. In addition, $5th$-order solution of equation (6.16) as (5.6) is evaluated as

$$
\begin{aligned}
S_5(\sigma,\tau) &= \sum_{\lambda=0}^{5}\varphi_\lambda(\sigma,\tau)\\
&= \sigma\Bigg(\frac{4h^5\tau^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{6h^5\tau^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{4h^5\tau^{4\alpha}}{\Gamma(1+4\alpha)}-\frac{h^5\tau^{5\alpha}}{\Gamma(1+5\alpha)}-\frac{h^5\tau^{\alpha}}{\Gamma(1+\alpha)}+\frac{15h^4\tau^{2\alpha}}{\Gamma(1+2\alpha)}\\
&-\frac{15h^4\tau^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{5h^4\tau^{4\alpha}}{\Gamma(1+4\alpha)}-\frac{5h^4\tau^{\alpha}}{\Gamma(1+\alpha)}+\frac{20h^3\tau^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{10h^3\tau^{3\alpha}}{\Gamma(1+3\alpha)}-\frac{10h^3\tau^{\alpha}}{\Gamma(1+\alpha)}\\
&+\frac{10h^2\tau^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{10h^2\tau^{\alpha}}{\Gamma(1+\alpha)}-\frac{5h\tau^{\alpha}}{\Gamma(1+\alpha)}+1\Bigg).
\end{aligned}
$$

In the special case, when $h=-1$ the above equation reduces to

$$
S_5(\sigma,\tau)=\sigma\left(1+\frac{\tau^{\alpha}}{\Gamma(1+\alpha)}+\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\tau^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{\tau^{4\alpha}}{\Gamma(1+4\alpha)}+\frac{\tau^{5\alpha}}{\Gamma(1+5\alpha)}\right), \tag{6.24}
$$

which converges towards the exact solution (6.17) of TF-FPE (6.16) with the increase in number of iterations.

7. RESULTS & DISCUSSIONS

To test how precise the referred methods, we have taken three different examples of Fokker-Plank equation and evaluated the absolute errors, relative errors and order of convergence. The relative error in kth order solution can be evaluated by using formula: $E_k(\sigma,\tau) = \left|\frac{S_k(\sigma,\tau)-S_{k-1}(\sigma,\tau)}{S_k(\sigma,\tau)}\right|$.

For example 6.1, the optimal $\hbar$ for 5ith-order ${}_O$HA$\mathbb{J}$TM results and associated minimum residual errors are computed for different $\alpha = \beta$ as reported in Table 1. Evaluated 3rd-order results and the absolute-errors utilizing $\mathbb{J}$-VIT and ${}_O$HA$\mathbb{J}$TM with optimal $\hbar = -1.00039449313177$ are compared with the results by ADM [34], RPSM [70], q-HATM[37], FRDTM, FVIM [41] and exact result for $\alpha = \beta = 1$ in Table 2. Table 4 records the approximate 5th-order results and their absolute-errors for different values of α, β. The absolute-errors in Mth-order ($M = 3, 6$) $\mathbb{J}$-VIT results and ${}_O$HA$\mathbb{J}$TM results for optimal $\hbar$ are compared with existing results by HATM[23], RPSM [70], q-HATM[37], FRDTM [41] for examples 6.1-6.2 for associated integer order case in Table 3.

Table 5 reports the comparison of fifth-order computed results with HATM [23], RPSM [70], q-HATM [37], FRDTM [41] and exact results for $\alpha = 1$ for example 6.2. Table 6 records optimal value of $\hbar$ in Mth-order ${}_O$HA$\mathbb{J}$TM results ($M = 5, 6, 7$) and the minimum residual error for example 6.2-6.3 with different α. Table 7-10 records the 7th-order approximate results and their absolute-errors for different values of α for example 6.2 and 6.3, respectively. Table 6, 8 and 9 confirms that both $\mathbb{J}$-VIT and ${}_O$HA$\mathbb{J}$TM produces more precise approximations that converging very fast to the exact solution behavior.

Figure 1 depicts two dimension plots and surface behaviour of the absolute-errors in 5th-order evaluated results for STF-FPE (6.1) with different $\alpha = \beta$, and the comparison of the computed results with exact results (6.2). Figure 2 depicts two dimension plots and surface behavior of the absolute-error in 6th-order evaluated approximation for TF-FPE (6.9) with different α, and the comparison of the computed results with exact results (6.10). Figure 3 depicts two dimension plots and surface behavior of the absolute-errors in 7th-order evaluated results for TF-FPE (6.16) with different α, and the comparison of 5th-order results with exact results (6.17). Figure 4 depicts the comparison of the absolute error in 6th-order evaluated approximation for STF-FPE (6.1) with $\alpha = \beta = 1$, TF-FPE (6.9) with $\alpha = 1$, TF-FPE (6.16) with $\alpha = 1$.

8. CONCLUSION

In the present study, two new hybrid methods: ${}_O$HA$\mathbb{J}$TM and $\mathbb{J}$-VIT have been successfully implemented for analytical assessment of space-time fractional Fokker-Planck equations.

Banach approach is utilized to analyze the convergence of these methods. In addition, it is demonstrated that $\mathbb{J}$-VIT is T-stable.

The computed new approximations are presented as the closed expression of Mittag-Leffler function, and in addition, the effectiveness/validity of the proposed new approximations is demonstrated via three test problems of STF-FPE by computing the error norms: L_2, relative errors, and absolute-errors.

From numerical assessment in Section 7, it is demonstrated that the proposed methods perform better for the study of STF-FPE and additionally for a fixed iteration, new approximations obtained via ${}_O$HA$\mathbb{J}$TM shows better accuracy and efficiency in comparison to those obtained via $\mathbb{J}$-VIT and recently developed some rigourous method.

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Authors' contributions

All authors contributed enough in the production and writing of this research.

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