Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 11, No. 2, 2023, pp. 291-302
DOI:10.22034/cmde.2022.50918.2119

Numerical computation of exponential functions in frame of Nabla fractional calculus

Jagan Mohan Jonnalagadda[1], and Jehad Alzabut[2,3,*]

[1]Department of Mathematics, Birla Institute of Technology and Science Pilani, Hyderabad - 500078, Telangana, India.

[2]Department of Mathematics and Sciences, Prince Sultan University, Riyadh 12435, Saudi Arabia.

[3]Department of Industrial Engineering, OSTİM Technical University, Ankara 06374, Türkey.

Abstract

Exponential functions play an essential role in describing the qualitative properties of solutions of nabla fractional difference equations. In this article, we illustrate their asymptotic behavior. We know that these functions involve infinite series of ratios of gamma functions, and it is challenging to compute them. For this purpose, we propose a novel matrix technique to compute the addressed functions numerically. The results are supported by illustrative examples. The proposed method can be extended to obtain numerical solutions for non-homogeneous nabla fractional difference equations.

Keywords. Nabla fractional difference, Exponential function, Triangular strip matrix, General solution, Asymptotic behaviour.
2010 Mathematics Subject Classification. 39A12.

1. Introduction

Fractional calculus is the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the integer order differentiation and n-fold integration. The combined efforts of a number of mathematicians produced a fairly strong theory of fractional calculus for functions of a real variable. Fractional calculus represents a natural instrument to model nonlocal phenomena either in space or time. The fractional derivative of any function contains information about the function at earlier points, so it possesses a long memory effect. From science to engineering, there are many processes that involve different space/time scales. In many problems of the above context, the dynamics of the system can be formulated by fractional differential equations which include the nonlocal effects. Fractional differential equations characterize many real world dynamical systems better than ordinary differential equations [6, 21–25, 33, 34].

On the other hand, the dynamics of many phenomena in nature, for instance, biological species, change only at discrete times. Generally, one assumes that if a natural system could be modeled by a discrete time system, many qualitative aspects of such a system could be determined. Motivated by the above facts, researchers initiated the study of the theory of fractional differential systems qualitatively and model natural systems using fractional differences instead of integer order differences.

Nabla fractional calculus is an integrated theory of arbitrary order sums and differences. The concept of nabla fractional difference traces back to the works of Miller & Ross [27], Gray & Zhang [19], Atici & Eloe [10], and Anastassiou [9]. During the past decade, there has been an increasing interest in this field. For a detailed introduction to the evolution of nabla fractional calculus, we refer to [18] and the references therein.

Acar et al. [4] and Nagai [28] introduced the exponential functions of nabla fractional calculus as the unique solutions of the following initial value problems associated with the Riemann–Liouville and the Caputo nabla fractional

Received: 29 March 2022 ; Accepted: 15 September 2022.
∗ Corresponding author. Email: jalzabut@psu.edu.sa; jehad.alzabut@ostimteknik.edu.tr.

differences:

$$\begin{cases} \left(\nabla^{\nu}_{\rho(0)} w\right)(t) = \lambda w(t), \quad t \in \mathbb{N}_1, \\ \left(\nabla^{-(1-\nu)}_{\rho(0)} w\right)(0) = w(0) = 1, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \left(\nabla^{\nu}_{0*} x\right)(t) = \lambda x(t), \quad t \in \mathbb{N}_1, \\ x(0) = 1, \end{cases} \tag{1.2}$$

where $0 < \nu < 1$ and $|\lambda| < 1$. The unique solutions of the initial value problems (1.1) and (1.2) are represented by $\hat{e}_{\nu,\nu}(\lambda, t^{\overline{\nu}})$ and $\hat{e}_{\nu}(\lambda, t^{\overline{\nu}})$, respectively, where

$$\hat{e}_{\nu,\nu}(\lambda, t^{\overline{\nu}}) = (1-\lambda) \sum_{k=0}^{\infty} \lambda^k H_{\nu k+\nu-1}(t, \rho(0)), \quad t \in \mathbb{N}_0, \tag{1.3}$$

and

$$\hat{e}_{\nu}(\lambda, t^{\overline{\nu}}) = \sum_{k=0}^{\infty} \lambda^k H_{\nu k}(t, 0), \quad t \in \mathbb{N}_0. \tag{1.4}$$

Atici [11], Čermák [13], Eloe [15], Jia [20], and Wu [35] obtained the following asymptotic results for the discrete exponential functions:

$$\lim_{t\to\infty} \hat{e}_{\nu,\nu}(\lambda, t^{\overline{\nu}}) = 0, \quad \lambda \in (-1, 0], \tag{1.5}$$

$$\lim_{t\to\infty} \hat{e}_{\nu,\nu}(\lambda, t^{\overline{\nu}}) = \infty, \quad \lambda \in (0, 1), \tag{1.6}$$

$$\lim_{t\to\infty} \hat{e}_{\nu}(\lambda, t^{\overline{\nu}}) = 0, \quad \lambda \in (-1, 0), \tag{1.7}$$

$$\lim_{t\to\infty} \hat{e}_{\nu}(\lambda, t^{\overline{\nu}}) = \infty, \quad \lambda \in (0, 1). \tag{1.8}$$

Using triangular strip matrices, Podlubny [30] described a matrix approach to find numerical solutions of fractional differential equations. Motivated by this technique, we present a matrix method to compute the exponential functions (1.3) and (1.4) numerically. The results are confirmed by numerical examples.

2. Preliminaries

We use the following notations, definitions, and known results of nabla fractional calculus throughout the article. The notations and terms are borrowed from several places but we cite here the remarkable monograph of Goodrich [18] as an adequate reference.

Denote by $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}$ for any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$. The backward jump operator $\rho : \mathbb{N}_{a+1} \to \mathbb{N}_a$ is defined by

$$\rho(t) = t - 1, \quad t \in \mathbb{N}_{a+1}.$$

Define the μ^{th}-order nabla fractional Taylor monomial by

$$H_{\mu}(t, a) = \frac{\Gamma(t - a + \mu)}{\Gamma(t - a)\Gamma(\mu + 1)}, \quad \mu \in \mathbb{R} \setminus \{\dots, -2, -1\},$$

provided the right-hand side of this equation is sensible. Here $\Gamma(\cdot)$ denotes the Euler gamma function.

Lemma 2.1. *[18] We observe the following properties of nabla fractional Taylor monomials.*

1. $H_{\mu}(t, a) = 0$ *for all* $\mu \in \{\dots, -2, -1\}$ *and* $t \in \mathbb{N}_a$.
2. $H_{\mu}(t, \rho(t)) = 1$ *for all* $\mu \in \mathbb{R} \setminus \{\dots, -2, -1\}$ *and* $t \in \mathbb{N}_a$.
3. $H_{\mu}(t, t) = 0$ *for all* $\mu \in \mathbb{R} \setminus \{\dots, -2, -1\}$ *and* $t \in \mathbb{N}_a$.

Definition 2.2. [12] Let $u : \mathbb{N}_a \to \mathbb{R}$. The first order backward (nabla) difference of u is defined by

$$\big(\nabla u\big)(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1}.$$

Definition 2.3. [18] Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-\nu} u\big)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s))u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\big(\nabla_a^{-\nu} u\big)(a) = 0$.

Definition 2.4. [18] Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $0 < \nu \leq 1$. The ν^{th}-order *Riemann–Liouville* nabla difference of u based at a is given by

$$\big(\nabla_a^{\nu} u\big)(t) = \Big(\nabla\big(\nabla_a^{-(1-\nu)} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+1}.$$

In [5], Ahrendt et al. showed that the definition of a fractional difference can be rewritten in a form similar to the definition of a fractional sum.

Theorem 2.5. *[5] Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $0 < \nu < 1$. Then,*

$$\big(\nabla_a^{\nu} u\big)(t) = \sum_{s=a+1}^{t} H_{-\nu-1}(t, \rho(s))u(s), \quad t \in \mathbb{N}_{a+1}.$$

Definition 2.6. [9] Let $u : \mathbb{N}_a \to \mathbb{R}$ and $0 < \nu \leq 1$. The ν^{th}-order *Caputo* nabla fractional difference of u based at a is given by

$$\big(\nabla_{a*}^{\nu} u\big)(t) = \Big(\nabla_a^{-(1-\nu)}\big(\nabla u\big)\Big)(t), \quad t \in \mathbb{N}_{a+1}.$$

The following identity is useful in transforming the Caputo nabla fractional difference into the Riemann–Liouville nabla fractional difference.

Theorem 2.7. *[1] Let $u : \mathbb{N}_a \to \mathbb{R}$ and $0 < \nu < 1$. Then,*

$$\big(\nabla_{a*}^{\nu} u\big)(t) = \big(\nabla_a^{\nu} u\big)(t) - H_{-\nu}(t, a)u(a), \quad t \in \mathbb{N}_{a+1}.$$

3. Exponential Functions of Nabla Fractional Calculus

This section is devoted to the main results where we compute the exponential functions numerically. This is presented in two subtitles.

3.1. **Computation of** (1.3)**:** Let $m \in \mathbb{N}_1$ and consider the initial value problem associated with (1.1):

$$\begin{cases} \big(\nabla_{\rho(0)}^{\nu} w\big)(t) = \lambda w(t), \quad t \in \mathbb{N}_1^m, \\ \big(\nabla_{\rho(0)}^{-(1-\nu)} w\big)(0) = w(0) = 1. \end{cases} \tag{3.1}$$

Rewriting the equation in (3.1) using Theorem 2.5, we have

$$\sum_{s=0}^{t} H_{-\nu-1}(t, \rho(s))w(s) = \lambda w(t), \quad t \in \mathbb{N}_1^m. \tag{3.2}$$

Rearranging the terms in (3.2), we obtain

$$(1-\lambda)w(t) + \sum_{s=1}^{t-1} H_{-\nu-1}(t, \rho(s))w(s) = -H_{-\nu-1}(t, \rho(0))w(0), \quad t \in \mathbb{N}_1^m. \tag{3.3}$$

Denote by $\tilde{w} = [w(1), w(2), \cdots, w(m)]^T$. Then, the matrix form of (3.3) is given by

$$\mathcal{L}\tilde{w} = -\mathcal{B},$$

where

$$\mathcal{L} = \begin{pmatrix} 1-\lambda & 0 & \cdots & \cdots & 0 & 0 \\ H_{-\nu-1}(2,\rho(1)) & 1-\lambda & \cdots & \cdots & 0 & 0 \\ H_{-\nu-1}(3,\rho(1)) & H_{-\nu-1}(3,\rho(2)) & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ H_{-\nu-1}(m-1,\rho(1)) & H_{-\nu-1}(m-1,\rho(2)) & \cdots & \cdots & 1-\lambda & 0 \\ H_{-\nu-1}(m,\rho(1)) & H_{-\nu-1}(m,\rho(2)) & \cdots & \cdots & H_{-\nu-1}(m,\rho(m-1)) & 1-\lambda \end{pmatrix}_{m\times m}$$

is a lower triangular strip matrix and

$$\mathcal{B} = \begin{pmatrix} H_{-\nu-1}(1,\rho(0)) \\ H_{-\nu-1}(2,\rho(0)) \\ H_{-\nu-1}(3,\rho(0)) \\ \vdots \\ \vdots \\ H_{-\nu-1}(m-1,\rho(0)) \\ H_{-\nu-1}(m,\rho(0)) \end{pmatrix}_{m\times 1}.$$

Since $\mathcal{L}$ is non-singular, the exponential function (1.3) can be computed by the following numerical algorithm:

$$\hat{e}_{\nu,\nu}(\lambda, t^{\overline{\nu}}) = -\mathcal{L}^{-1}\mathcal{B}, \quad t \in \mathbb{N}_1^m.$$

Here $\mathcal{L} = [\mathcal{L}_{ij}]_{m\times m}$ and $\mathcal{B} = [\mathcal{B}_i]_{m\times 1}$, where

$$\mathcal{L}_{ij} = \begin{cases} 1-\lambda, & i = j, \\ 0, & i < j, \\ H_{-\nu-1}(i,\rho(j)), & i > j, \end{cases}$$

and

$$\mathcal{B}_i = H_{-\nu-1}(i,\rho(0)).$$

Example 1. Computation of $\hat{e}_{0.5,0.5}(-0.5, t^{\overline{0.5}})$ for $t \in \mathbb{N}_1^{10}$:

We have

$$\mathcal{L} = \begin{pmatrix} 1.5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5000 & 1.5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.1250 & -0.5000 & 1.5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0625 & -0.1250 & -0.5000 & 1.5000 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0391 & -0.0625 & -0.1250 & -0.5000 & 1.5000 & 0 & 0 & 0 & 0 & 0 \\ -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 1.5000 & 0 & 0 & 0 & 0 \\ -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 1.5000 & 0 & 0 & 0 \\ -0.0161 & -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 1.5000 & 0 & 0 \\ -0.0131 & -0.0161 & -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 1.5000 & 0 \\ -0.0109 & -0.0131 & -0.0161 & -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 1.5000 \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} -0.5000 \\ -0.1250 \\ -0.0625 \\ -0.0391 \\ -0.0273 \\ -0.0205 \\ -0.0161 \\ -0.0131 \\ -0.0109 \\ -0.0093 \end{pmatrix}.$$

Then, for $t \in \mathbb{N}_1^{10}$,

$$\hat{e}_{0.5,0.5}(-0.5, t^{\overline{0.5}}) = -\mathcal{L}^{-1}\mathcal{B} = \begin{pmatrix} 0.3333 \\ 0.1944 \\ 0.1343 \\ 0.1009 \\ 0.0798 \\ 0.0654 \\ 0.0550 \\ 0.0472 \\ 0.0411 \\ 0.0362 \end{pmatrix}.$$

Example 2. Computation of $\hat{e}_{0.5,0.5}(0.5, t^{\overline{0.5}})$ for $t \in \mathbb{N}_1^{10}$:

We have

$$\mathcal{L} = \begin{pmatrix} 0.5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5000 & 0.5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.1250 & -0.5000 & 0.5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0625 & -0.1250 & -0.5000 & 0.5000 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0391 & -0.0625 & -0.1250 & -0.5000 & 0.5000 & 0 & 0 & 0 & 0 & 0 \\ -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 0.5000 & 0 & 0 & 0 & 0 \\ -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 0.5000 & 0 & 0 & 0 \\ -0.0161 & -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 0.5000 & 0 & 0 \\ -0.0131 & -0.0161 & -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 0.5000 & 0 \\ -0.0109 & -0.0131 & -0.0161 & -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 0.5000 \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} -0.5000 \\ -0.1250 \\ -0.0625 \\ -0.0391 \\ -0.0273 \\ -0.0205 \\ -0.0161 \\ -0.0131 \\ -0.0109 \\ -0.0093 \end{pmatrix}.$$

Then, for $t \in \mathbb{N}_1^{10}$,

$$\hat{e}_{0.5,0.5}(0.5, t^{\overline{0.5}}) = -\mathcal{L}^{-1}\mathcal{B} = \begin{pmatrix} 1 \\ 1.2500 \\ 1.6250 \\ 2.1406 \\ 2.8359 \\ 3.7676 \\ 5.0127 \\ 6.6749 \\ 8.8925 \\ 11.8505 \end{pmatrix}.$$

Example 3. The graphs of $\hat{e}_{0.5,0.5}(-0.5, t^{\overline{0.5}})$ and $\hat{e}_{0.5,0.5}(0.5, t^{\overline{0.5}})$ for $t \in \mathbb{N}_1^{100}$ are shown in Figures 1 and 2, respectively.

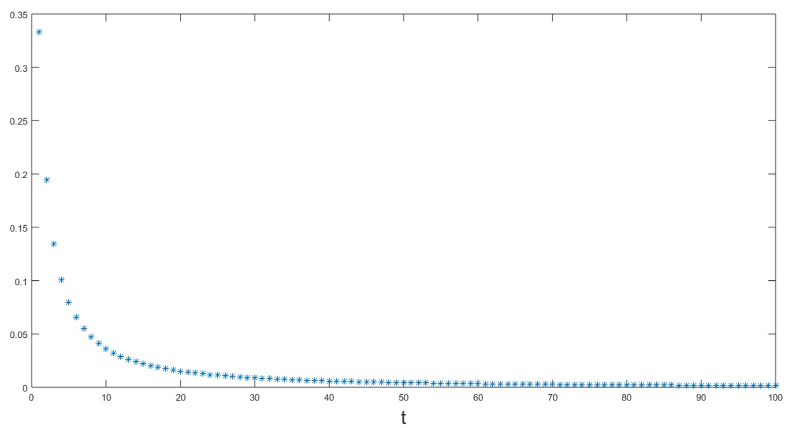

FIGURE 1. $\hat{e}_{0.5,0.5}(-0.5, t^{\overline{0.5}})$

3.2. **Computation of** (1.4)**:** Let $m \in \mathbb{N}_1$ and consider the initial value problem associated with (1.2):

$$\begin{cases} \left(\nabla_{0*}^{\nu} x\right)(t) = \lambda x(t), \quad t \in \mathbb{N}_1^m, \\ x(0) = 1. \end{cases} \tag{3.4}$$

Rewriting the equation in (3.4) using Theorems 2.5 and 2.7, we have

$$\sum_{s=1}^{t} H_{-\nu-1}(t, \rho(s))x(s) - H_{-\nu}(t, 0)x(0) = \lambda x(t), \quad t \in \mathbb{N}_1^m. \tag{3.5}$$

Rearranging the terms in (3.5), we obtain

$$(1-\lambda)x(t) + \sum_{s=1}^{t-1} H_{-\nu-1}(t, \rho(s))x(s) = H_{-\nu}(t, 0)x(0), \quad t \in \mathbb{N}_1^m. \tag{3.6}$$

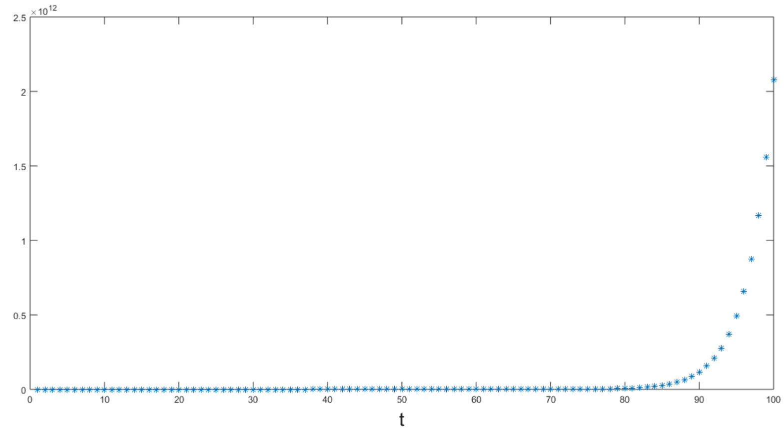

FIGURE 2. $\hat{e}_{0.5,0.5}(0.5, t^{\overline{0.5}})$

Denote by $\tilde{x} = [x(1), x(2), \cdots, x(m)]^T$. Then, the matrix form of (3.6) is given by

$$\mathcal{L}\tilde{x} = \mathcal{C},$$

where

$$\mathcal{C} = \begin{pmatrix} H_{-\nu}(1,0) \\ H_{-\nu}(2,0) \\ H_{-\nu}(3,0) \\ \vdots \\ \vdots \\ H_{-\nu}(m-1,0) \\ H_{-\nu}(m,0) \end{pmatrix}_{m\times 1}.$$

Since $\mathcal{L}$ is non-singular, the exponential function (1.4) can be computed by the following numerical algorithm:

$$\hat{e}_{\nu}(\lambda, t^{\overline{\nu}}) = \mathcal{L}^{-1}\mathcal{C}, \quad t \in \mathbb{N}_1^m.$$

Here $\mathcal{L} = [\mathcal{L}_{ij}]_{m\times m}$ and $\mathcal{C} = [\mathcal{C}_i]_{m\times 1}$, where

$$\mathcal{L}_{ij} = \begin{cases} 1-\lambda, & i = j, \\ 0, & i < j, \\ H_{-\nu-1}(i, \rho(j)), & i > j, \end{cases}$$

and

$$\mathcal{C}_i = H_{-\nu}(i, 0).$$

Example 4. Computation of $\hat{e}_{0.5}(-0.5, t^{\overline{0.5}})$ for $t \in \mathbb{N}_1^{10}$:

We have

$$\mathcal{C} = \begin{pmatrix} 1 \\ 0.5000 \\ 0.3750 \\ 0.3125 \\ 0.2734 \\ 0.2461 \\ 0.2256 \\ 0.2095 \\ 0.1964 \\ 0.1855 \end{pmatrix}.$$

Then, from Example 1, for $t \in \mathbb{N}_1^{10}$,

$$\hat{e}_{0.5}(-0.5, t^{\overline{0.5}}) = \mathcal{L}^{-1}\mathcal{C} = \begin{pmatrix} 0.6667 \\ 0.5556 \\ 0.4907 \\ 0.4460 \\ 0.4124 \\ 0.3857 \\ 0.3639 \\ 0.3456 \\ 0.3299 \\ 0.3162 \end{pmatrix}.$$

Example 5. Computation of $\hat{e}_{0.5}(0.5, t^{\overline{0.5}})$ for $t \in \mathbb{N}_1^{10}$:

We have

$$\mathcal{C} = \begin{pmatrix} 1 \\ 0.5000 \\ 0.3750 \\ 0.3125 \\ 0.2734 \\ 0.2461 \\ 0.2256 \\ 0.2095 \\ 0.1964 \\ 0.1855 \end{pmatrix}.$$

Then, from Example 2, for $t \in \mathbb{N}_1^{10}$,

$$\hat{e}_{0.5}(0.5, t^{\overline{0.5}}) = \mathcal{L}^{-1}\mathcal{C} = \begin{pmatrix} 2 \\ 3 \\ 4.2500 \\ 5.8750 \\ 8.0156 \\ 10.8516 \\ 14.6191 \\ 19.6318 \\ 26.3067 \\ 35.1992 \end{pmatrix}.$$

Example 6. The graphs of $\hat{e}_{0.5}(0.5, t^{\overline{0.5}})$ and $\hat{e}_{0.5}(-0.5, t^{\overline{0.5}})$ for $t \in \mathbb{N}_1^{100}$ are shown in Figures 3 and 4, respectively.

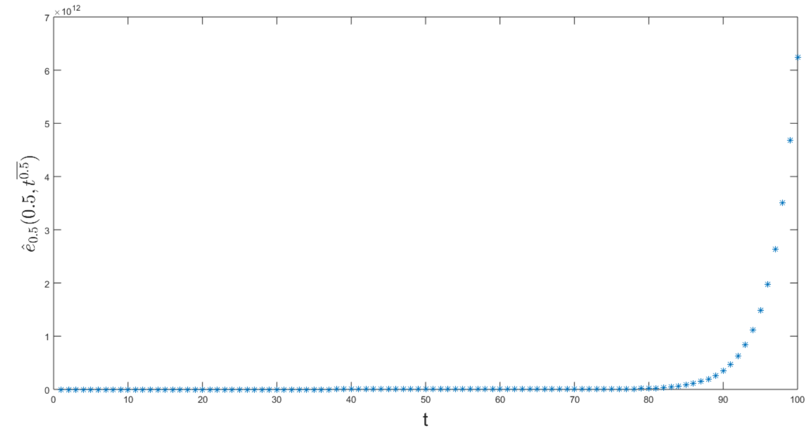

FIGURE 3. $\hat{e}_{0.5}(0.5, t^{\overline{0.5}})$

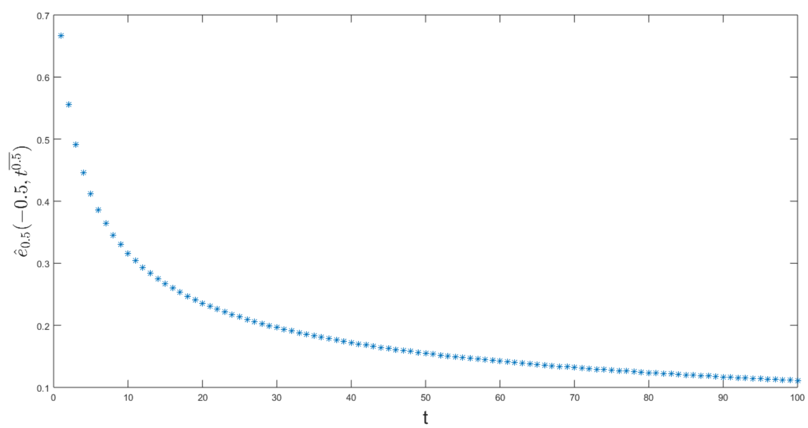

FIGURE 4. $\hat{e}_{0.5}(-0.5, t^{\overline{0.5}})$

4. Extensions

The method described in Section 2 can be extended to obtain numerical solutions of initial value problems involving linear non-homogeneous nabla fractional difference equations.

Let $0 < \nu < 1$ and $m \in \mathbb{N}_1$. Consider the initial value problem

$$\begin{cases} \left(\nabla^{\nu}_{\rho(0)} u\right)(t) = a(t)u(t) + f(t), \quad t \in \mathbb{N}_1^m, \\ \left(\nabla^{-(1-\nu)}_{\rho(0)} u\right)(0) = u(0) = c, \end{cases} \tag{4.1}$$

where $a,\ f : \mathbb{N}_1^m \to \mathbb{R}$ such that

$$a(t) \neq 1, \quad t \in \mathbb{N}_1^m.$$

Denote by $\tilde{u} = [u(1), u(2), \cdots, u(m)]^T$ and $\mathcal{F} = [f(1), f(2), \cdots, f(m)]^T$. Then, the matrix form of (4.1) is given by

$$\mathcal{L}_1 \tilde{u} = \mathcal{F} - c\mathcal{B},$$

where

$$\mathcal{L}_1 = \begin{pmatrix} 1-a(1) & 0 & \cdots & \cdots & 0 & 0 \\ H_{-\nu-1}(2,\rho(1)) & 1-a(2) & \cdots & \cdots & 0 & 0 \\ H_{-\nu-1}(3,\rho(1)) & H_{-\nu-1}(3,\rho(2)) & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ H_{-\nu-1}(m-1,\rho(1)) & H_{-\nu-1}(m-1,\rho(2)) & \cdots & \cdots & 1-a(m-1) & 0 \\ H_{-\nu-1}(m,\rho(1)) & H_{-\nu-1}(m,\rho(2)) & \cdots & \cdots & H_{-\nu-1}(m,\rho(m-1)) & 1-a(m) \end{pmatrix}_{m\times m}$$

is a lower triangular strip matrix. Since $\mathcal{L}_1$ is non-singular, the solution of (4.1) can be computed by the following numerical algorithm:

$$u(t) = \mathcal{L}_1^{-1}[\mathcal{F} - c\mathcal{B}], \quad t \in \mathbb{N}_1^m.$$

Replacing the ν-th order Riemann–Liouville nabla fractional difference operator $\nabla^{\nu}_{\rho(0)}$ in (4.1) with the ν-th order Caputo operator ∇^{ν}_{0*}, the matrix form of the initial value problem

$$\begin{cases} \left(\nabla^{\nu}_{0*}u\right)(t) = a(t)u(t) + f(t), & t \in \mathbb{N}_1^m, \\ u(0) = c, \end{cases} \tag{4.2}$$

is given by

$$\mathcal{L}_1 \tilde{u} = \mathcal{F} + c\mathcal{C}.$$

Since $\mathcal{L}_1$ is non-singular, the solution of (4.2) can be computed by the following numerical algorithm:

$$u(t) = \mathcal{L}_1^{-1}[\mathcal{F} + c\mathcal{C}], \quad t \in \mathbb{N}_1^m.$$

5. Concluding Remark

Unlike the nabla calculus case, the computation of exponential functions of nabla fractional calculus is quite complicated. In this article, we proposed a novel matrix technique to compute the exponential functions of nabla fractional calculus numerically. We also extended the proposed method to obtain numerical solutions for non-homogeneous nabla fractional difference equations.

Declarations Statements

Acknowledgements J. Alzabut is thankful to Prince Sultan University and OSTİM Technical University for their endless support.

Funding Not applicable.

Availability of Data and Materials: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing Interests: The authors declare no conflict of interest.

Authors' Contributions: All the authors have equally contributed to the contents of the manuscript.

References

[1] T. Abdeljawad, *On delta and nabla Caputo fractional differences and dual identities*, Discrete Dyn. Nat. Soc.,*2013* (2013), 1-12.

[2] T. Abdeljawad and J. Alzabut, *On Riemann–Liouville fractional q-difference equations and their application to retarded logistic type model*, Math. Methods Appl. Sci., *41*(18) (2018), 8953–8962.

[3] T. Abdeljawad and F. M. Atici, *On the definitions of nabla fractional operators*, Abstr. Appl. Anal., *2012* (2012), 1-13 .

[4] N. Acar and F. M. Atıcı, *Exponential functions of discrete fractional calculus*, Appl. Anal. Discrete Math., *7*(2) (2013), 343–353.

[5] K. Ahrendt, L. Castle, M. Holm, and K. Yochman, *Laplace transforms for the nabla-difference operator and a fractional variation of parameters formula*, Commun. Appl. Anal., *16*(3) (2012), 317–347.

[6] A. Alkhazzan, P. Jiang, D. Baleanu, H. Khan, and A. Khan, *Stability and existence results for a class of nonlinear fractional differential equations with singularity*, Math. Methods Appl. Sci., *41*(18) (2018), 9321–9334.

[7] J. Alzabut, S. Tyagi, and S. Abbas, *Discrete fractional-order BAM neural networks with leakage delay: existence and stability results*, Asian J. Control, *22*(1) (2020), 143–155.

[8] J. Alzabut and T. Abdeljawad, *A generalized discrete fractional Gronwall inequality and its application on the uniqueness of solutions for nonlinear delay fractional difference system*, Appl. Anal. Discrete Math., *12*(1) (2018), 36–48.

[9] G. A. Anastassiou, *Nabla discrete fractional calculus and nabla inequalities*, Math. Comput. Modelling, *51*(5-6) (2010), 562–571.

[10] F. M. Atıcı and P. W. Eloe, *Discrete fractional calculus with the nabla operator*, Electron. J. Qual. Theory Differ. Equ., Special Edition I, *1*(3) (2009), 1-12.

[11] F. M. Atıcı and P. W. Eloe, *Linear systems of fractional nabla difference equations*, Rocky Mountain J. Math., *41*(2) (2011), 353–370.

[12] M. Bohner and A. Peterson, *Dynamic equations on time scales. An introduction with applications*, Birkhäuser, Boston, MA, 2001.

[13] J. Čermák, T. Kisela, and L. Nechvátal, *Stability and asymptotic properties of a linear fractional difference equation*, Adv. Difference Equ., *122* (2012), 1-14.

[14] S. Elaydi, *An introduction to difference equations*, Third edition, Springer, New York, 2005.

[15] P. Eloe and J. Jonnalagadda, *Mittag–Leffler stability of systems of fractional nabla difference equations*, Bull. Korean Math. Soc., *56*(4) (2019), 977–992.

[16] S. R. Grace, H. Adıguzel, J. Alzabut, and J. M. Jonnalagadda, *Asymptotic behavior of positive solutions for three types of fractional difference equations with forcing term*, Vietnam J. Math., *49*(4) (2021), 1151–1164.

[17] S. R. Grace, J. Alzabut, S. Punitha, V. Muthulakshmi, and H. Adıguzel, *On the nonoscillatory behavior of solutions of three classes of fractional difference equations*, Opuscula Math., *40*(5) (2020), 549–568.

[18] C. Goodrich and A. C. Peterson, *Discrete fractional calculus*, Springer, Cham, 2015.

[19] H. L. Gray and N. F. Zhang, *On a new definition of the fractional difference*, Math. Comp., *50*(182) (1988), 513–529.

[20] B. Jia, L. Erbe, and A. Peterson, *Comparison theorems and asymptotic behavior of solutions of discrete fractional equations*, Electron. J. Qual. Theory Differ. Equ., *89* (2015), 1-18.

[21] A. Khan, H. M. Alshehri, T. Abdeljawad, Q. M. Al-Mdallal, and H. Khan, *Stability analysis of fractional nabla difference COVID-19 model*, Results in Physics, *22* (2021), 1-8.

[22] A. Khan, H. M. Alshehri, J. F. Gomez-Aguilar, Z. A. Khan, and G. A. Fernandez-Anaya, *Predator-prey model involving variable-order fractional differential equations with Mittag–Leffler kernel*, Adv. Difference Equ., *189* (2021), 1-18.

[23] A. Khan, Z. A. Khan, T. Abdeljawad, and H. Khan, *Analytical analysis of fractional-order sequential hybrid system with numerical application*, Adv. Contin. Discrete Models, *12* (2022), 1-19.

[24] Z. A. Khan, A. Khan, T. Abdeljawad, and H. Khan, *Computational analysis of fractional order imperfect testing infection disease model*, Fractals, *30*(5) (2022), 1-17.

[25] H. Khan, C. Tunc, and A. Khan, *Green function's properties and existence theorems for nonlinear singular-delay-fractional differential equations*, Discrete Contin. Dyn. Syst. Ser. S, *13*(9) (2020), 2475–2487.

[26] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Elsevier Science B. V., Amsterdam, 2006.

[27] K. S. Miller and B. Ross, *Fractional difference calculus. Univalent functions, fractional calculus, and their applications*, Kōriyama, 1988, 139–152, Ellis Horwood Ser. Math. Appl., Horwood, Chichester, 1989.

[28] A. Nagai, *An integrable mapping with fractional difference*, J. Phys. Soc. Japan, *72*(9) (2003), 2181–2183.

[29] I. Podlubny, *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Academic Press, San Diego, CA, 1999.

[30] I. Podlubny, *Matrix approach to discrete fractional calculus*, Fract. Calc. Appl. Anal., *3*(4) (2000), 359–386.

[31] A. G. M. Selvam, D. Baleanu, J. Alzabut, D. Vignesh, and S. Abbas, *On Hyers–Ulam Mittag–Leffler stability of discrete fractional Duffing equation with application on inverted pendulum*, Adv. Difference Equ., *456* (2020), 1-16.

[32] A. G. M. Selvam, J. Alzabut, R. Dhineshbabu, S. Rashid and M. Rehman, *Discrete fractional order two-point boundary value problem with some relevant physical applications*, J. Inequal. Appl., (2020), 1-19.

[33] K. Shah, Z. A. Khan, A. Ali, R. Amin, H. Khan, and A. Khan, *Haar wavelet collocation approach for the solution of fractional order COVID-19 model using Caputo derivative*, Alexandria Engineering Journal, *59* (2020), 3221–3231.

[34] I. Ullah, S. Ahmad, Q. Al-Mdallal, Z. A. Khan, H. Khan, and A. Khan, *Stability analysis of a dynamical model of tuberculosis with incomplete treatment*, Adv. Difference Equ., (2020), 1-14.

[35] G. Wu, D. Baleanu, W. Luo, and H. Wei, *Lyapunov functions for Riemann–Liouville-like fractional difference equations*, Appl. Math. Comput., *314* (2017), 228–236.