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A novel scheme for SMCH equation with two different approaches

Arzu Akbulut[1,*], S. M. Rayhanul Islam[2], S. M. Yasir Arafat[2] and Filiz Taşcan[3]

[1]Department of Mathematics, Faculty of Arts and Science, Bursa Uludag University, Bursa, Turkey.

[2]Department of Mathematics, Pabna University of Science and Technology, Pabna-6600, Bangladesh.

[3]Faculty of Arts and Science, Department of Mathematics and Computer, Eskişehir Osmangazi University, Eskisehir, Turkey.

Abstract

In this study, the unified and improved F-expansion methods are applied to derive exact traveling wave solutions of the simplified modified Camassa-Holm (SMCH) equation. The current methods can calculate all branches of solutions at the same time, even if several solutions are quite near and therefore impossible to identify via numerical methods. All obtained solutions are given by hyperbolic, trigonometric, and rational function solutions which obtained solutions are useful for real-life problems in fluid dynamics, optical fibers, plasma physics and so on. The two-dimensional (2D) and three-dimensional (3D) graphs of the obtained solutions are plotted. Finally, we can state that these strategies are extremely successful, dependable, and simple. These ideas might potentially be applied to many nonlinear evolution models in mathematics and physics.

Keywords. SMCH equation, The improved F-expansion method, The unified method, Symbolic computation, Exact solution, Solitary wave.
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1. Introduction

Nonlinear evolution equations (NLEEs) have an important place for analyzing nonlinear wave phenomena. The studies of nonlinear waves have a significant role in our life and in various research areas such as optical fiber, fluid dynamics, plasma physics, solid-state physics, propagation of shallow water wave, electromagnetism, mathematical fluid dynamics, quantum mechanics and numerous other surfaces [5, 14, 19, 25, 26, 36, 40, 43, 52, 53].

Lots of researchers dealt with some methods deriving exact solutions of NLPDEs such as $\exp(-\varphi(\xi))$ expansion, the enhanced (G'/G) expansion, $(G'/G, 1/G)$ expansion, extended homoclinic test, new auxiliary equation, generalized Kudryashov, extended Fan-sub equation, Φ^6-model expansion, generalized exponential rational function, extended Sinh-Gordon equation expansion, extended auxiliary equation, modified direct algebraic, extended rational sinh-cosh, new extended direct algebraic, (G'/G^2)-expansion, Sardar sub-equation, and unified methods [4, 6–11, 20, 21, 24, 27, 28, 32–35, 37–39, 46–51] as well numerous other approaches.

In 1993, Camassa and Holm developed a new integrable dispersive water wave equation. This equation is called the equation CH by holding on to two terms that are neglected in the small amplitude shallow-water limit [13]. CH equation is given as follows:

$$u_t + 2\eta u_x - u_{xxt} + \gamma u u_x = 2u_x u_{xx} + u u_{xxx}, \tag{1.1}$$

where $\gamma > 0, n \in \mathbb{R}$ and u represent the fluid velocity in the x-direction. Many authors have worked on Camassa-Holm (CH), simplified modified Camassa-Holm (SMCH), and modified Camassa-Holm (mCH) equations. Fisher and Schiff studied with the CH equation, and they investigated the conservation laws and initial value problem [16]. The effective method was applied to the CH equation to obtain some new exact soliton solutions in [15], double soliton solutions,

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∗ Corresponding author. Email:ayakut1987@hotmail.com .

convex peaked, and smooth soliton solutions are obtained in [42]. Tian and Song in [41] studied the mCH equation and discovered new peaked solitary wave solutions. The generalized CH or mCH equation is given by:

$$u_t + 2\eta u_x - u_{xxt} + \gamma u^w u_x = 2u_x u_{xx} + u u_{xxx}, \tag{1.2}$$

where $\gamma > 0, n \in \mathbb{R}, w$ is the strength of the nonlinearity. Exact traveling wave solutions through a Hamiltonian system obtained of Eq. (1.2) and peakon weak solutions are obtained by Gao et al. in [17]. When $w = 2$, Li et al. discussed the dynamics behavior, obtained both solitary wave solutions and explicit periodic blow-up solutions of the mCH equation in [29]. Some researchers gave some new peakons and periodic peakons solutions in [31], obtained some exploration to Bäcklund transformation for the mCH in [44]. Also, Boyd [12] examined that if the solitary wave changed slowly, the two additional terms on the right-hand side of Eq. (1.1) would become smaller and the soliton would be given by the solution of the lowest order

$$u_t + 2\eta u_x - u_{xxt} + \gamma u u_x = 0. \tag{1.3}$$

In the view of Eq. (1.3), Wazwaz [45] investigated exact solutions via compactons, solitons, and plane periodic solutions of the mCH equation by using direct analysis which is simplified from the mCH equation and given by:

$$u_t + 2\eta u_x - u_{xxt} + \gamma u^w u_x = 0. \tag{1.4}$$

When w=2, (1.4) can be written as

$$u_t + 2\eta u_x - u_{xxt} + \gamma u^2 u_x = 0, \tag{1.5}$$

where $\gamma > 0$ and η are an arbitrary constants. In this paper, we use Eq. (1.5) and we call it the SMCH equation. In recent years, Ali et al. provided exact traveling wave solutions of the SMCH equation in [3]. Applying different methods to SMCH, exact solutions have been constructed by Islam et al. in [23]. In addition, Liu et al. explored the soliton solutions of the SMCH equation in [30].

The purpose of this article is to examine new, useful wave solutions of the SMCH equation with the help of a unified and improved F-expansion methods. The unified and improved F-expansion approaches have the key benefit over the previous other methods because it delivers more accurate standing wave solutions. Aside from its physical significance, the closed-form solutions of NLEEs may help numerical solvers assess the correctness of their findings and hence aid in convergence studies. The goal of this book is to propose novel, practical, and more generic standing wave solutions for the SMCH model utilizing unified and improved F-expansion methodologies, as well as simulating the resulting solution from free parameter values using 3D and 2D visualizations. In section 2, we will explain the unified method (UM). Then, in section3, we will apply the UM to given model. In section 4, we will explain the improved F-expansion procedure. In section 5, we will derive exact solutions of the SMCH model with the help of the improved F-expansion approach. The given methods are effective, direct, powerful, and rising methods to explore exact solutions in NLDEs. In section 6, we will plot 3D and 2D figures of some obtained solutions and we will discuss graphs of obtained solutions. Finally, conclusions will be given.

2. Description of the unified method (UM)

Let us consider the NLEEs

$$L(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt} \dots) = 0, \tag{2.1}$$

where $u(x, t)$ is an unidentified function, L is a polynomial.
By applying the wave transformation as follows to Eq. (2.1):

$$u(x, t) = u(\xi), \xi = x - \omega t \tag{2.2}$$

where ω is a wave velocity, we get the following nonlinear ODE:

$$Q(u, u', u'', u''' \dots) = 0, \tag{2.3}$$

Assume that the trail solution of Eq.(2.3) is given by:

$$u(\xi) = A_0 + \sum_{i=1}^{N}[A_i S(\xi)^i + B_i S(\xi)^{-i}], \tag{2.4}$$

here A_0,A_i, and B_i $(i = 0, 1, 2, \dots, N)$ are constants to be investigated later and $A_N \neq 0$ and $B_N \neq 0$, with a different discourse A_N and B_N can not be zero at the same time. S and its derivatives satisfy Riccati equation as follows:

$$S' = (S(\xi))^2 + \sigma \tag{2.5}$$

where $S' = \frac{dS}{d\xi}$ and σ is a constant. The solutions of the Riccati equation is given as follows:

Case 1. If $\sigma < 0$, Eq. (2.5) has following the hyperbolic solutions by:

$$S(\xi) = \frac{\sqrt{-(A^2+B^2)\sigma} - A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi + C))}{Asinh(2\sqrt{-\sigma}(\xi + C)) + B}, \tag{2.6}$$

$$S(\xi) = \frac{-\sqrt{-(A^2+B^2)\sigma} - A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi + C))}{Asinh(2\sqrt{-\sigma}(\xi + C)) + B}, \tag{2.7}$$

$$S(\xi) = \sqrt{-\sigma} + \frac{-2A\sqrt{-\sigma}}{A + cosh(2\sqrt{-\sigma}(\xi + C)) - sinh(2\sqrt{-\sigma}(\xi + C))}, \tag{2.8}$$

$$S(\xi) = -\sqrt{-\sigma} + \frac{2A\sqrt{-\sigma}}{A + cosh(2\sqrt{-\sigma}(\xi + C)) - sinh(2\sqrt{-\sigma}(\xi + C))}. \tag{2.9}$$

Case 2. If $\sigma > 0$, Eq. (2.5) has following the trigonometric solutions by:

$$S(\xi) = \frac{\sqrt{(A^2-B^2)\sigma} - A\sqrt{\sigma}cos(2\sqrt{\sigma}(\xi + C))}{Asin(2\sqrt{\sigma}(\xi + C)) + B}, \tag{2.10}$$

$$S(\xi) = \frac{-\sqrt{(A^2-B^2)\sigma} - A\sqrt{\sigma}cos(2\sqrt{\sigma}(\xi + C))}{Asin(2\sqrt{\sigma}(\xi + C)) + B}, \tag{2.11}$$

$$S(\xi) = i\sqrt{\sigma} + \frac{-2Ai\sqrt{\sigma}}{A + cos(2\sqrt{\sigma}(\xi + C)) - isin(2\sqrt{\sigma}(\xi + C))}, \tag{2.12}$$

$$S(\xi) = -i\sqrt{\sigma} + \frac{2Ai\sqrt{\sigma}}{A + cos(2\sqrt{\sigma}(\xi + C)) + isin(2\sqrt{\sigma}(\xi + C))}. \tag{2.13}$$

Case 3. If $\sigma = 0$, Eq. (2.5) has following the rational solution by:

$$S(\xi) = -\frac{1}{\xi + C}, \tag{2.14}$$

where A and B are real arbitrary constants, C is an arbitrary constant. N is found by the method of homogeneous balance. This means equality between the highest order derivatives and nonlinear terms in Eq. (2.3). On the other hand, the degree of U as $D(U(\xi)) = N$ which gives the order of others expression as follows:

$$D(\frac{d^qU}{d\xi^q}) = N + q, D(U^p(\frac{d^qU}{d\xi^q})^s) = pN + s(N + q). \tag{2.15}$$

Substituting Eq. (2.4) and Eq. (2.5) in Eq. (2.3) and then equating the coefficients of each power of $S(\xi)$ to zero, we get determining equation system. By solving the obtained system, we get different values of here ω, A_i and B_i $(i = 0, 1, 2, \dots, N)$.

Finally, putting obtained values of A_i, B_i $(i = 0, 1, 2, \ldots, N)$, and ω into Eq. (2.4), we obtain trail solution and then the putting obtained trail solution in Eqs. (2.6) to (2.14), we get several sets of solutions [2, 18].

3. Applying the unified method to the SMCH equation:

If we apply the wave transform (2.2) to Eq. (1.5), we get as follows:

$$-\omega u' + 2\eta u' + \omega u''' + \eta u^2 u' = 0. \tag{3.1}$$

By integrating and simplifying Eq. (3.1), we get:

$$\omega u - 2\eta u - \omega u'' - \frac{\gamma}{3} u^3 = 0, \tag{3.2}$$

where we assume the integration constants are zero. Balancing between u'' and u^3, gives $N = 1$, then Eq. (2.4) yields:

$$u(\xi) = A_0 + A_1 S(\xi) + B_1 S(\xi)^{-1}. \tag{3.3}$$

here A_0, A_1, and B_1 are constants and they determined later. Inserting Eq. (3.3) and Eq. (2.5), in Eq. (3.2), equating each coefficient of all powers of $S^i(\xi)$ to zero, obtained determining equation system is given by:

$$\begin{aligned}
&S^3 : -2\omega A_1 - \tfrac{1}{3}\gamma A_1^3 = 0,\\
&S^2 : -\gamma A_0 A_1^2 = 0,\\
&S^1 : -2\eta A_1 - 2\omega A_1 \sigma - \gamma A_1^2 B_1 + \omega A_1 - \gamma A_0^2 A_1 = 0,\\
&S^0 : \omega A_0 - 2\gamma A_0 A_1 B_1 - \tfrac{1}{3}\gamma A_0^3 - 2\eta A_0 = 0,\\
&S^{-1} : -\gamma A_0^2 B_1 - 2\eta B_1 + \omega B_1 - \gamma A_1 B_1^2 - 2\omega B_1 \sigma = 0,\\
&S^{-2} : -\gamma A_0 B_1^2 = 0,\\
&S^{-3} : -\tfrac{1}{3}\gamma B_1^3 - 2\omega B_1 \sigma^2 = 0.
\end{aligned}$$

If we solve the above relationships, we get the following values.
Set 1:

$$\omega = \frac{-2\eta}{2\sigma - 1}, A_0 = 0, A_1 = 0, B_1 = \pm 2\sigma\sqrt{\frac{3\eta}{\gamma(2\sigma - 1)}}.$$

Set 2:

$$\omega = \frac{-2\eta}{2\sigma - 1}, A_0 = 0, A_1 = \pm 2\sqrt{\frac{3\eta}{\gamma(2\sigma - 1)}}, B_1 = 0.$$

Set 3:

$$\omega = \frac{2\eta}{4\sigma + 1}, A_0 = 0, A_1 = \pm 2i\sqrt{\frac{3\eta}{\gamma(4\sigma + 1)}}, B_1 = \pm 6\sigma i\sqrt{\frac{\eta}{3\gamma(4\sigma + 1)}}.$$

Set 4:

$$\omega = -\frac{2\eta}{8\sigma - 1}, A_0 = 0, A_1 = \pm 2\sqrt{\frac{3\eta}{\gamma(8\sigma - 1)}}, B_1 = \pm 6\sigma\sqrt{\frac{\eta}{3\gamma(8\sigma - 1)}}.$$

By replacing Set 1- Set 4 together with Eqs. (2.6) - (2.14) in Eq. (3.3), we are drastically reducing the traveling wave solution of Eq. (1.5) as below.

If $\sigma < 0$, hyperbolic solutions are given by:

$$u_{1,2}(\xi) = \pm 2\sqrt{\frac{3\eta}{\gamma(2\sigma - 1)}} \times \frac{\sigma(Asinh(2\sqrt{-\sigma}(\xi + C)) + B)}{\sqrt{-(A^2 + B^2)\sigma} - A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi + C))},$$

$$u_{3,4}(\xi) = \pm 2\sqrt{\frac{-3\eta\sigma}{\gamma(2\sigma-1)}} \times \frac{\sigma(Asinh(2\sqrt{-\sigma}(\xi+C))+B)}{\sqrt{-(A^2+B^2)\sigma}+A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi+C))},$$

$$u_{5,6}(\xi) = \pm 2\sqrt{\frac{-3\eta\sigma}{\gamma(2\sigma-1)}} \times \frac{(A+cosh(2\sqrt{-\sigma}(\xi+C))-sinh(2\sqrt{-\sigma}(\xi+C))}{cosh(2\sqrt{-\sigma}(\xi+C))-sinh(2\sqrt{-\sigma}(\xi+C))-A},$$

$$u_{7,8}(\xi) = \pm 2\sqrt{\frac{-3\eta\sigma}{\gamma(1-2\sigma)}} \times \frac{A+cosh(2\sqrt{-\sigma}(\xi+C))-sinh(2\sqrt{-\sigma}(\xi+C))}{A-cosh(2\sqrt{-\sigma}(\xi+C))+sinh(2\sqrt{-\sigma}(\xi+C))},$$

$$u_{9,10}(\xi) = \pm 2\sqrt{\frac{3\eta}{\gamma(2\sigma-1)}} \times \frac{\sqrt{-(A^2+B^2)\sigma}-A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi+C))}{Asinh(2\sqrt{-\sigma}(\xi+C))+B},$$

$$u_{11,12}(\xi) = \pm 2\sqrt{\frac{3\eta}{\gamma(2\sigma-1)}} \times \frac{\sqrt{-(A^2+B^2)\sigma}+A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi+C))}{Asinh(2\sqrt{-\sigma}(\xi+C))+B},$$

$$u_{13,14}(\xi) = \pm 2\sqrt{\frac{-3\eta\sigma}{\gamma(1-2\sigma)}} \times \frac{(A+cosh(2\sqrt{-\sigma}(\xi+C))-sinh(2\sqrt{-\sigma}(\xi+C))}{cosh(2\sqrt{-\sigma}(\xi+C))-sinh(2\sqrt{-\sigma}(\xi+C))-A},$$

$$u_{15,16}(\xi) = \pm 2\sqrt{\frac{-3\eta\sigma}{\gamma(1-2\sigma)}} \times \frac{A+cosh(2\sqrt{-\sigma}(\xi+C))-sinh(2\sqrt{-\sigma}(\xi+C))}{A-cosh(2\sqrt{-\sigma}(\xi+C))+sinh(2\sqrt{-\sigma}(\xi+C))},$$

where $\xi = x - \frac{2\eta}{2\sigma-1}t$

$$u_{17,18}(\xi) = \pm 2i\sqrt{\frac{3\eta}{\gamma(4\sigma+1)}} \times [\frac{\sqrt{-(A^2+B^2)\sigma}-A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi+C))}{Asinh(2\sqrt{-\sigma}(\xi+C))+B} + \frac{\sigma(Asinh(2\sqrt{-\sigma}(\xi+C))+B)}{\sqrt{-(A^2+B^2)\sigma}-A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi+C))}],$$

$$u_{19,20}(\xi) = \pm 2i\sqrt{\frac{3\eta}{\gamma(4\sigma+1)}} \times [\frac{\sqrt{-(A^2+B^2)\sigma}+A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi+C))}{Asinh(2\sqrt{-\sigma}(\xi+C))+B} + \frac{\sigma(Asinh(2\sqrt{-\sigma}(\xi+C))+B)}{\sqrt{-(A^2+B^2)\sigma}+A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi+C))}],$$

$$u_{21,22}(\xi) = \pm 2i\sqrt{\frac{-3\eta\sigma}{\gamma(4\sigma+1)}} \times [\frac{cosh(2\sqrt{-\sigma}(\xi+C))-sinh(2\sqrt{-\sigma}(\xi+C))-A}{A+cosh(2\sqrt{-\sigma}(\xi+C))-sinh(2\sqrt{-\sigma}(\xi+C))} - \frac{A+cosh(2\sqrt{-\sigma}(\xi+C))-sinh(2\sqrt{-\sigma}(\xi+C))}{cosh(2\sqrt{-\sigma}(\xi+C))-sinh(2\sqrt{-\sigma}(\xi+C))-A}],$$

$$u_{23,24}(\xi) = \pm 2i\sqrt{\frac{-3\eta\sigma}{\gamma(4\sigma+1)}} \times [\frac{A-cosh(2\sqrt{-\sigma}(\xi+C))+sinh(2\sqrt{-\sigma}(\xi+C))}{A+cosh(2\sqrt{-\sigma}(\xi+C))-sinh(2\sqrt{-\sigma}(\xi+C))} - \frac{A+cosh(2\sqrt{-\sigma}(\xi+C))-sinh(2\sqrt{-\sigma}(\xi+C))}{A-cosh(2\sqrt{-\sigma}(\xi+C))+sinh(2\sqrt{-\sigma}(\xi+C))}],$$

where $\xi = x - \frac{2\eta}{4\sigma+1}t$.

$$u_{25,26}(\xi) = \pm 2\sqrt{\frac{3\eta}{\gamma(8\sigma-1)}} \times [\frac{\sqrt{-(A^2+B^2)\sigma}-A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi+C))}{Asinh(2\sqrt{-\sigma}(\xi+C))+B} - \frac{\sigma(Asinh(2\sqrt{-\sigma}(\xi+C))+B)}{\sqrt{-(A^2+B^2)\sigma}-A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi+C))}],$$

$$u_{27,28}(\xi) = \pm 2\sqrt{\frac{3\eta}{\gamma(8\sigma-1)}} \times [\frac{-\sqrt{-(A^2+B^2)\sigma} - A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi+C))}{Asinh(2\sqrt{-\sigma}(\xi+C))+B}$$
$$-\frac{\sigma(Asinh(2\sqrt{-\sigma}(\xi+C))+B)}{-\sqrt{-(A^2+B^2)\sigma} - A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi+C))}],$$

$$u_{29,30}(\xi) = \pm 2\sqrt{\frac{-3\eta\sigma}{\gamma(8\sigma-1)}} \times [\frac{cosh(2\sqrt{-\sigma}(\xi+C)) - sinh(2\sqrt{-\sigma}(\xi+C)) - A}{A + cosh(2\sqrt{-\sigma}(\xi+C)) - sinh(2\sqrt{-\sigma}(\xi+C))}$$
$$+\frac{A + cosh(2\sqrt{-\sigma}(\xi+C)) - sinh(2\sqrt{-\sigma}(\xi+C))}{cosh(2\sqrt{-\sigma}(\xi+C)) - sinh(2\sqrt{-\sigma}(\xi+C)) - A}],$$

$$u_{31,32}(\xi) = \pm 2\sqrt{\frac{-3\eta\sigma}{\gamma(8\sigma-1)}} \times [\frac{A - cosh(2\sqrt{-\sigma}(\xi+C)) + sinh(2\sqrt{-\sigma}(\xi+C))}{A + cosh(2\sqrt{-\sigma}(\xi+C)) - sinh(2\sqrt{-\sigma}(\xi+C))}$$
$$+\frac{A + cosh(2\sqrt{-\sigma}(\xi+C)) - sinh(2\sqrt{-\sigma}(\xi+C))}{A - cosh(2\sqrt{-\sigma}(\xi+C)) + sinh(2\sqrt{-\sigma}(\xi+C))}],$$

where $\xi = x + \frac{2\eta}{8\sigma-1}t$.

If $\sigma > 0$, trigonometric solutions are given by:

$$u_{33,34}(\xi) = \pm 2\sqrt{\frac{3\eta}{\gamma(2\sigma-1)}} \times \frac{\sigma(Asin(2\sqrt{\sigma}(\xi+C))+B)}{\sqrt{(A^2-B^2)\sigma} - A\sqrt{\sigma}cos(2\sqrt{\sigma}(\xi+C))},$$

$$u_{35,36}(\xi) = \pm 2\sqrt{\frac{3\eta}{\gamma(2\sigma-1)}} \times \frac{\sigma(Asin(2\sqrt{\sigma}(\xi+C))+B)}{\sqrt{(A^2-B^2)\sigma} + A\sqrt{\sigma}cos(2\sqrt{\sigma}(\xi+C))},$$

$$u_{37,38}(\xi) = \pm 2i\sqrt{\frac{3\eta\sigma}{\gamma(2\sigma-1)}} \times \frac{A + cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C))}{A - cos(2\sqrt{\sigma}(\xi+C)) + isin(2\sqrt{\sigma}(\xi+C))},$$

$$u_{39,40}(\xi) = \pm 2i\sqrt{\frac{3\eta\sigma}{\gamma(2\sigma-1)}} \times \frac{A + cos(2\sqrt{\sigma}(\xi+C)) + isin(2\sqrt{\sigma}(\xi+C))}{A - cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C))},$$

$$u_{41,42}(\xi) = \pm 2\sqrt{\frac{3\eta}{\gamma(2\sigma-1)}} \times \frac{\sqrt{(A^2-B^2)\sigma} - A\sqrt{\sigma}cos(2\sqrt{\sigma}(\xi+C))}{Asin(2\sqrt{\sigma}(\xi+C))+B},$$

$$u_{43,44}(\xi) = \pm 2\sqrt{\frac{3\eta}{\gamma(2\sigma-1)}} \times \frac{\sqrt{(A^2-B^2)\sigma} + A\sqrt{\sigma}cos(2\sqrt{\sigma}(\xi+C))}{Asin(2\sqrt{\sigma}(\xi+C))+B},$$

$$u_{45,46}(\xi) = \pm 2i\sqrt{\frac{3\eta\sigma}{\gamma(2\sigma-1)}} \times \frac{(cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C)) - A}{A + cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C))},$$

$$u_{47,48}(\xi) = \pm 2i\sqrt{\frac{3\eta\sigma}{\gamma(2\sigma-1)}} \times \frac{A - cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C))}{A + cos(2\sqrt{\sigma}(\xi+C)) + isin(2\sqrt{\sigma}(\xi+C))},$$

where $\xi = x + \frac{2\eta}{2\sigma-1}t$.

$$u_{49,50}(\xi) = \pm 2i\sqrt{\frac{3\eta}{\gamma(4\sigma+1)}} \times [\frac{\sqrt{(A^2-B^2)\sigma} - A\sqrt{\sigma}cos(2\sqrt{\sigma}(\xi+C))}{Asin(2\sqrt{\sigma}(\xi+C))+B}$$
$$+\frac{\sigma(Asin(2\sqrt{\sigma}(\xi+C))+B)}{\sqrt{(A^2-B^2)\sigma} - A\sqrt{\sigma}cos(2\sqrt{\sigma}(\xi+C))}],$$

$$u_{51,52}(\xi) = \pm 2i\sqrt{\frac{3\eta}{\gamma(4\sigma+1)}} \times [\frac{\sqrt{(A^2-B^2)\sigma} + A\sqrt{\sigma}cos(2\sqrt{\sigma}(\xi+C))}{Asin(2\sqrt{\sigma}(\xi+C)) + B} + \frac{\sigma(Asin(2\sqrt{\sigma}(\xi+C)) + B)}{\sqrt{(A^2-B^2)\sigma} + A\sqrt{\sigma}cos(2\sqrt{\sigma}(\xi+C))}],$$

$$u_{53,54}(\xi) = \pm 2\sqrt{\frac{3\eta\sigma}{\gamma(4\sigma+1)}} \times [\frac{cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C)) - A}{A + cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C))} - \frac{A + cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C))}{cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C)) - A}],$$

$$u_{55,56}(\xi) = \pm 2\sqrt{\frac{3\eta\sigma}{\gamma(4\sigma+1)}} \times [\frac{A - cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C))}{A + cos(2\sqrt{\sigma}(\xi+C)) + isin(2\sqrt{\sigma}(\xi+C))} - \frac{A + cos(2\sqrt{\sigma}(\xi+C)) + isin(2\sqrt{\sigma}(\xi+C))}{A - cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C))}],$$

where $\xi = x - \frac{2\eta}{4\sigma+1}t$.

$$u_{57,58}(\xi) = \pm 2\sqrt{\frac{3\eta}{\gamma(8\sigma-1)}} \times [\frac{\sqrt{-(A^2+B^2)\sigma} - A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi+C))}{Asinh(2\sqrt{-\sigma}(\xi+C)) + B} - \frac{\sigma(Asinh(2\sqrt{-\sigma}(\xi+C)) + B)}{\sqrt{-(A^2+B^2)\sigma} - A\sqrt{-\sigma}cosh(2\sqrt{-\sigma}(\xi+C))}],$$

$$u_{59,60}(\xi) = \pm 2\sqrt{\frac{3\eta}{\gamma(8\sigma-1)}} \times [\frac{\sqrt{(A^2-B^2)\sigma} + A\sqrt{\sigma}cos(2\sqrt{\sigma}(\xi+C))}{Asin(2\sqrt{\sigma}(\xi+C)) + B} - \frac{\sigma(Asin(2\sqrt{\sigma}(\xi+C)) + B)}{\sqrt{(A^2-B^2)\sigma} + A\sqrt{\sigma}cos(2\sqrt{\sigma}(\xi+C))}],$$

$$u_{61,62}(\xi) = \pm 2i\sqrt{\frac{3\eta\sigma}{\gamma(8\sigma-1)}} \times [\frac{cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C)) - A}{A + cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C))} + \frac{A + cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C))}{cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C)) - A}],$$

$$u_{63,64}(\xi) = \pm 2i\sqrt{\frac{3\eta\sigma}{\gamma(8\sigma-1)}} \times [\frac{A - cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C))}{A + cos(2\sqrt{\sigma}(\xi+C)) + isin(2\sqrt{\sigma}(\xi+C))} + \frac{A + cos(2\sqrt{\sigma}(\xi+C)) + isin(2\sqrt{\sigma}(\xi+C))}{A - cos(2\sqrt{\sigma}(\xi+C)) - isin(2\sqrt{\sigma}(\xi+C))}],$$

where $\xi = x + \frac{2\eta}{8\sigma-1}t$.

If $\sigma = 0$, rational solutions are given by:

$$u_{65,66}(\xi) = \pm 2i\sqrt{\frac{3\eta}{\gamma}} \times \frac{1}{\xi + C},$$

where $\xi = x - 2\eta t$.

4. Description of the improved F-expansion method

According to the method, we can give an exact solutions of Eq.(2.3) as follows:

$$u(\xi) = \sum_{i=0}^{N} \alpha_i \left(m + \mathcal{F}(\xi)\right)^i + \sum_{i=1}^{N} \beta_i \left(m + \mathcal{F}(\xi)\right)^{-i}, \tag{4.1}$$

where α_N or β_M are constants. They can be zero at different times. N is balancing term. $\mathcal{F}$ and its first-order derivatives satisfy Riccati equation as follows:

$$\mathcal{F}'(\xi) = k + \mathcal{F}^2(\xi), \tag{4.2}$$

where k is the real parameter and the solutions of Eq. (4.2) is given as follows:
Case 1: If $k < 0$, hyperbolic solutions are given by:

$$\mathcal{F}_1 = -\sqrt{-k}\tanh\left(\sqrt{-k}\xi\right),$$

or

$$\mathcal{F}_2 = -\sqrt{-k}\coth\left(\sqrt{-k}\xi\right),$$

Case 2: If $k > 0$, trigonometric solutions are given by:

$$\mathcal{F}_3 = \sqrt{k}\tan\left(\sqrt{k}\xi\right),$$

or

$$\mathcal{F}_4 = -\sqrt{k}\cot\left(\sqrt{k}\xi\right),$$

Case 3: If $k = 0$, rational solution is given by:

$$\mathcal{F}_5 = -\frac{1}{\xi}.$$

If we substitute Eq. (4.1) in Eq. (2.3) along with Eq. (4.2), we get polynomials in $\left(m + \mathcal{F}(\xi)\right)^i$ and $\left(m + \mathcal{F}(\xi)\right)^{-i}$. Then, equating each coefficient of the resulted polynomial to zero, we obtain a determining equation system for β_N, α_N, w, and m. If we solve the obtained system, we get various values of α_N, β_N, m, and w. Putting the value of α_N, β_N, m, and w into Eq. (4.1) with solutions of Riccati equation, exact solutions of Eq. (2.1) are obtained [1, 22].

5. Applying the improved F-expansion method to SMCH equation:

Now, we will apply an improved F-expansion procedure to the given equation. According to the method, the exact solution of the given equation is given by:

$$u(\xi) = \alpha_0 + \alpha_1\left(m + \mathcal{F}(\xi)\right) + \beta_1\left(m + \mathcal{F}(\xi)\right), \tag{5.1}$$

If we substitute Eq. (5.1) into Eq. (3.2), we obtain the determining equation system. If we solve the obtained determining equation system, we obtain the following solution families.
Family 1: If we use following constants:

$$\alpha_0 = \mp 2\sqrt{3}m\sqrt{\frac{\eta}{2\gamma k - \gamma}}, \alpha_1 = \pm 2\sqrt{3}\sqrt{\frac{\eta}{2\gamma k - \gamma}}, \beta_1 = 0, w = -\frac{2\eta\gamma}{2\gamma k - \gamma}, m = m,$$

exact solutions are given by:
1) If $k < 0$, hyperbolic solutions are given by:

$$u_{65,66}(\xi) = \mp 2\sqrt{3}\tanh\left(\sqrt{-k}\xi\right)\sqrt{-k}\sqrt{\frac{\eta}{2\gamma k - \gamma}}$$

2) If $k > 0$, trigonometric solutions are given by:

$$u_{67,68}(\xi) = \pm 2\sqrt{3}\tan\left(\sqrt{k}\xi\right)\sqrt{k}\sqrt{\frac{\eta}{2\gamma k - \gamma}}$$

3) If $k=0$, rational solutions are given by:

$$u_{69,70}(\xi) = \mp\frac{2\sqrt{3}\sqrt{-\eta}}{\sqrt{\gamma}\xi}$$

where $\xi = x + (\frac{2\eta\gamma}{2\gamma k-\gamma})t$.

Family 2:

$$\alpha_0 = \pm 2\sqrt{3}m\sqrt{\frac{\eta}{2\gamma k-\gamma}}, \alpha_1 = 0, w = -\frac{2\left(\frac{6\eta\gamma m^2}{2\gamma k-\gamma}+\eta\right)}{6m^2+2k-1}, m = m,$$

$$\beta_1 = \frac{2\sqrt{3}\sqrt{\frac{\eta}{2\gamma k-\gamma}}\left(\frac{18\eta m^4\gamma}{2\gamma k-\gamma}+\frac{6\eta m^2\gamma k}{2\gamma k-\gamma}+\frac{6\gamma\eta m^2}{2\gamma k-\gamma}+9m^2\eta+3\eta k\right)}{3\left(\frac{6\eta\gamma m^2}{2\gamma k-\gamma}+\eta\right)},$$

1) If $k<0$,

$$u_{71,72}(\xi) = \pm\frac{2\sqrt{3}\sqrt{\frac{\eta}{2\gamma k-\gamma}}\left(\tanh\left(\sqrt{-k}\xi\right)\sqrt{-k}m+k\right)}{\left(\tanh\left(\sqrt{-k}\xi\right)\sqrt{-k}-m\right)},$$

2) If $k>0$,

$$u_{73,74}(\xi) = \mp\frac{2\sqrt{3}\sqrt{\frac{\eta}{2\gamma k-\gamma}}\left(-\tan\left(\sqrt{k}\xi\right)\sqrt{k}m+k\right)}{\left(\tan\left(\sqrt{k}\xi\right)\sqrt{k}+m\right)},$$

3) If $k=0$,

$$u_{75,76}(\xi) = \mp\frac{2\sqrt{3}m\sqrt{-\eta}}{\sqrt{\gamma}(m\xi-1)},$$

where $\xi = x + \left(\frac{2\left(\frac{6\eta\gamma m^2}{2\gamma k-\gamma}+\eta\right)}{6m^2+2k-1}\right)t$.

Family 3:

$$\alpha_1 = \pm 2\sqrt{3}\sqrt{\frac{\eta}{\gamma(8k-1)}}, \beta_1 = \mp\frac{2\sqrt{3}k\eta}{\gamma(8k-1)\sqrt{\frac{\eta}{\gamma(8k-1)}}}, w = -\frac{2\eta\gamma}{\gamma(8k-1)}, m = \alpha_0 = 0,$$

1) When $k<0$,

$$u_{77,78}(\xi) = \mp\frac{2\sqrt{3}\eta\sqrt{-k}\left(\tanh\left(\sqrt{-k}\xi\right)^2+1\right)}{\sqrt{\gamma}\sqrt{\frac{\eta}{(8k-1)}}(8k-1)\tanh\left(\sqrt{-k}\xi\right)},$$

2) When $k>0$,

$$u_{79,80}(\xi) = \pm\frac{2\sqrt{3}\eta\sqrt{k}\left(\tan\left(\sqrt{k}\xi\right)^2-1\right)}{\sqrt{\gamma}\sqrt{\frac{\eta}{(8k-1)}}(8k-1)\tan\left(\sqrt{k}\xi\right)},$$

3) When $k=0$,

$$u_{81,82}(\xi) = \mp\frac{2\sqrt{3}\sqrt{-\eta}}{\sqrt{\gamma}\xi},$$

where $\xi = x + \left(\frac{2\eta\gamma}{\gamma(8k-1)}\right) t$.

Family 4: If the following values are used:

$$\alpha_1 = \pm 2\sqrt{-\frac{3\eta}{\gamma(4k+1)}}, \beta_1 = \mp \frac{6k\eta}{\gamma(4k+1)\sqrt{-\frac{3\eta}{\gamma(4k+1)}}}, w = \frac{2\eta\gamma}{\gamma(4k+1)}, m = \alpha_0 = 0,$$

The following exact solutions are obtained.

1) When $k < 0$,

$$u_{83,84}(\xi) = \pm \frac{2\sqrt{3}\eta\sqrt{-k}\left(\tanh\left(\sqrt{-k}\xi\right)^2 - 1\right)}{\sqrt{\gamma}\sqrt{-\frac{\eta}{(4k+1)}}(4k+1)\tanh\left(\sqrt{-k}\xi\right)},$$

2) When $k > 0$,

$$u_{85,86}(\xi) = \mp \frac{2\sqrt{3}\eta\sqrt{k}\left(\tan\left(\sqrt{k}\xi\right)^2 + 1\right)}{\sqrt{\gamma}\sqrt{-\frac{\eta}{(4k+1)}}(4k+1)\tan\left(\sqrt{k}\xi\right)},$$

3) When $k = 0$,

$$u_{87,88}(\xi) = \mp \frac{2\sqrt{3}\sqrt{-\eta}}{\sqrt{\gamma}\xi}$$

where $\xi = x - \left(\frac{2\eta\gamma}{\gamma(4k+1)}\right) t$.

Family 5:

$$\beta_1 = \pm 2\sqrt{3}k\sqrt{\frac{\eta}{\gamma(2k-1)}}, w = -\frac{2\eta}{2k-1}, \alpha_0 = \alpha_1 = m = 0,$$

1) When $k < 0$,

$$u_{89,90}(\xi) = \pm \frac{2\sqrt{3}\sqrt{-k}\sqrt{\frac{\eta}{2k-1}}}{\sqrt{\gamma}\tanh\left(\sqrt{-k}\xi\right)},$$

2) When $k > 0$,

$$u_{91,92}(\xi) = \pm \frac{2\sqrt{3}\sqrt{k}\sqrt{\frac{\eta}{2k-1}}}{\sqrt{\gamma}\tan\left(\sqrt{k}\xi\right)},$$

3) When $k = 0$, we obtain the zero solution,

where $\xi = x + \left(\frac{2\eta}{2k-1}\right) t$.

Remark 5.1. If we examine solutions of Reference [23], Islam et al obtained complex solutions. So, our solutions are different from their solutions. Also, our obtained solutions are different from the solutions of [3].

6. Graphical Representation

In this section, we will discuss the obtained solutions of the SMCH equation and plot the physical behavior of these solutions for free parameters with the help of Mathematica. The graphical illustrations of the solutions are shown in Figures 1–6. Figure 1 represents 3D and 2D wave profile of $u_{1,2}(x,t)$. The 3D wave profile depicts the kink form for selecting additional parameters $\sigma = -1$, $C = 0.5$, $A = -1$, $B = 1$, $\eta = -1.5$, $\gamma = 8$ inside the displacement $-5 \leq x, t \leq 5$ the 2D wave profile for various values of η within the displacement$-10 \leq t \leq 10$ shows that getting higher the negative value of η the arbitrary constant causes the asymptotic curve to move from left to right.

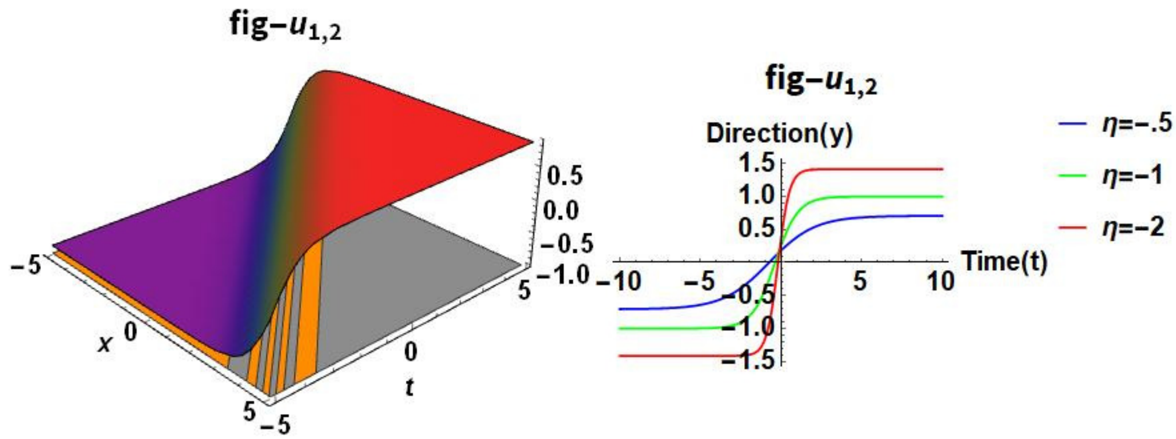

FIGURE 1. 3D and 2D surface of kink wave profile.

3D and 2D wave profiles in Figure 2 represent the solution $u_{11,12}(x,t)$. In Figure 2, the 3D wave profile shows the singular kink shape for choosing free parameters of $\sigma = -3$, $C = 0.5$, $A = -0.5$, $B = 1$, $\eta = -0.75$, $\gamma = 6$ within the displacement$-5 \leq x, t \leq 5$. The singular kink shape arises from one asymptotic state to another asymptotic state. These solitons are referred to as topological solitons. Mathematically, the transition of the wave from the asymptotic state at $\xi = -\infty$ to the other asymptotic state is $\xi = \infty$ localized in ξ, where $\xi = x - \omega t$, and ω is the wave speed. Otherwise, for different values of η, we drew the 2D wave chart at the limit $-10 \leq t \leq 10$.

The 3D and 2D wave profiles are given in Figure 3 for solution $u_{37,38}(x,t)$. For the fixed values of $\sigma = 0.7$, $C = 1$, $A = 0.2$, $B = 1$, $\eta = 0.4$, $\gamma = 1$ within the interval $-5 \leq x, t \leq 5$, the graph of the exact solutions $u_{37,38}(x,t)$ describe the solitary wave solution of periodic peakon wave type solution is depicted. 2D plot also shows the wave is very smooth for the values of $\eta = 0.3, 0.33$, and $\eta = 0.36$.

The common solution profile of $u_{45,46}(x,t)$ is anti compacton, which is a strong wave similar to an attic with infinite free wing. It is depicted in Figure 4 for the fixed values of $\sigma = 0.01$, $C = 0.1$, $A = 1.1$, $\eta = 0.2$, $\gamma = 0.5$ within the limit $-5 \leq x, t \leq 5$. The values of the parameters meet the conditions of the stability and the solutions remain stable according to the analysis. Here, the combined 2D graph of this solution has also been provided. Combined 2D graphics are sets of waves that describe the physical meaning of the various parameter's values.

The wave profile of the solutions $u_{49,50}(x,t)$ and $u_{53,54}(x,t)$ are the bell-shape soliton compressed on both sides. Therefore, the 3D plot of solutions $u_{49,50}(x,t)$ and $u_{53,54}(x,t)$ are portrayed for the values $\sigma = .02, C = 3, A = 2, B = 2.01, \eta = 0.1, \gamma = 0.01$ and $\sigma = .02, C = 0.3, A = 0.9, \eta = 0.3, \gamma = 0.1$ of the parameters within the limit $-5 \leq x, t \leq 5$ in Figs. 5 and 6. From these solutions, we deduce that the wave is zero when the wave is tilted toward the variable ξ tends to infinity. The bell-shape soliton has infinite wings. The stability analysis of this solution is stable. We understand that not all waves have the same amplitude. If we look closely, we can see that multiple waves of very low amplitude are preceded by waves of quite large amplitude. As a result, the wave's amplitude fluctuates gradually

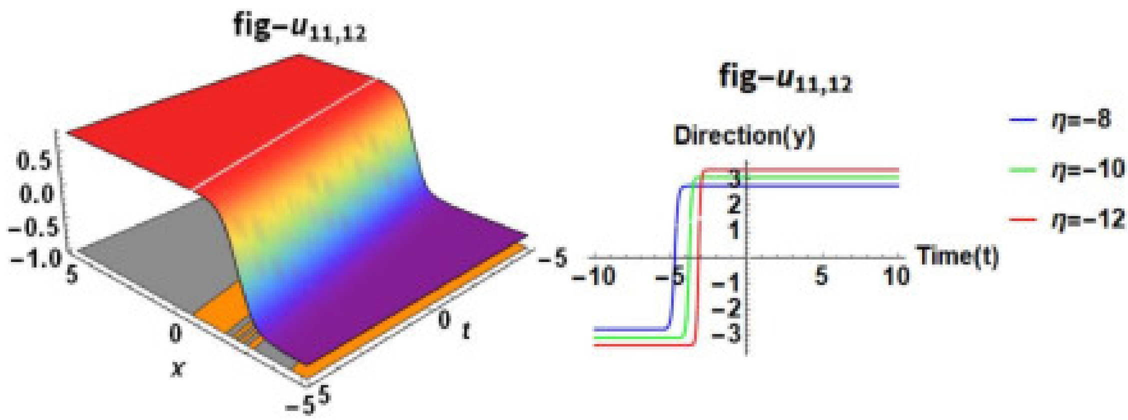

FIGURE 2. 3D and 2D surface of singular kink wave profile.

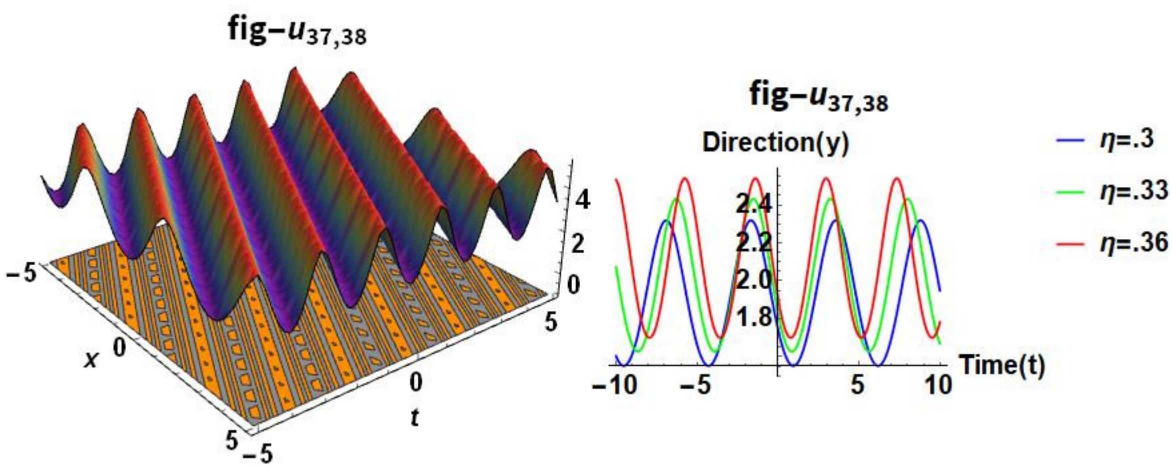

FIGURE 3. 3D and 2D surface of periodic peakon wave profile.

across time and location.

The 3D and 2D wave profiles are involved in Figure 7. For the fixed values of $k = -0.9$, $\eta = 0.75$, $\gamma = 0.001$ within the interval $-5 \leq x, t \leq 5$, the graph of the exact solutions $u_{65,66}(x,t)$ describes the solitary wave solution of kink type wave solution. We take $k = -9.5$, $\gamma = 0.75$ in 2D figure. The stability analysis of this solution is stable because of has infinite wings.

The 3D and 2D wave profiles are involved in Figure 8. For the fixed values of $k = 5$, $eta = -0.05$, $\gamma = 1.1$ within the interval $-5 \leq x, t \leq 5$, the graph of the exact solutions $u_{67,68}(x,t)$ describes the solitary wave solution of singular

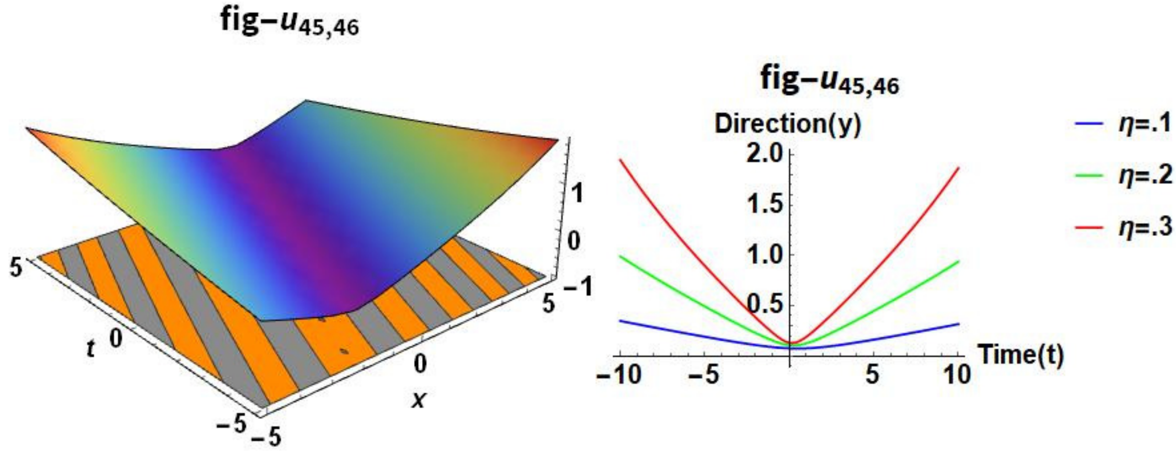

FIGURE 4. 3D and 2D surface of anti compacton wave profile.

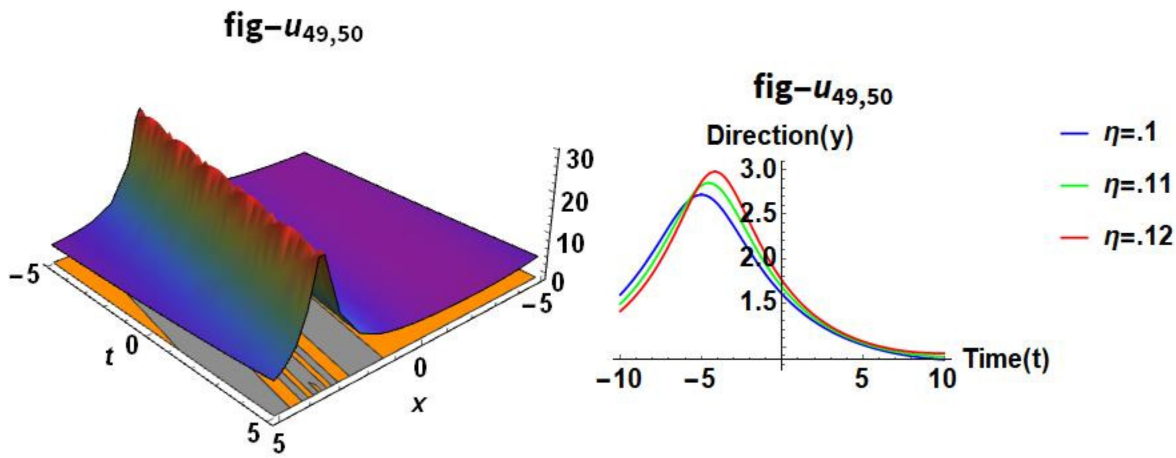

FIGURE 5. 3D and 2D surface of Bell type wave profile.

periodic peakon type wave. For the free parameters $k = 3$, $\gamma = -0.5$, the following 2D figure is obtained.

The 3D and 2D wave profiles are involved in Figure 9. For the fixed values of $k = 0$, $\eta = -0.01$, $\gamma = -1.2$ within the interval $-5 \leq x, t \leq 5$, the graph of the exact solutions $u_{69,70}(x, t)$ describes the solitary wave solution of singular bell shape wave. Rogue waves have a distinct structure, which is seen in this figure. The 2D combined line plots for the free parameters $k = 0, \gamma = -0.89$ confirm the relevant sorts of phenomena.

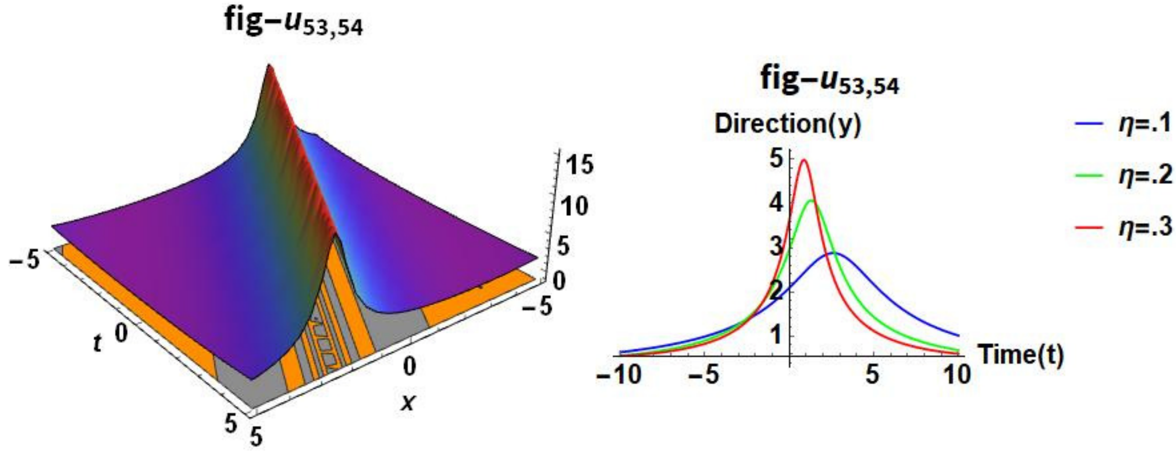

FIGURE 6. 3D and 2D surface of Bell type wave profile.

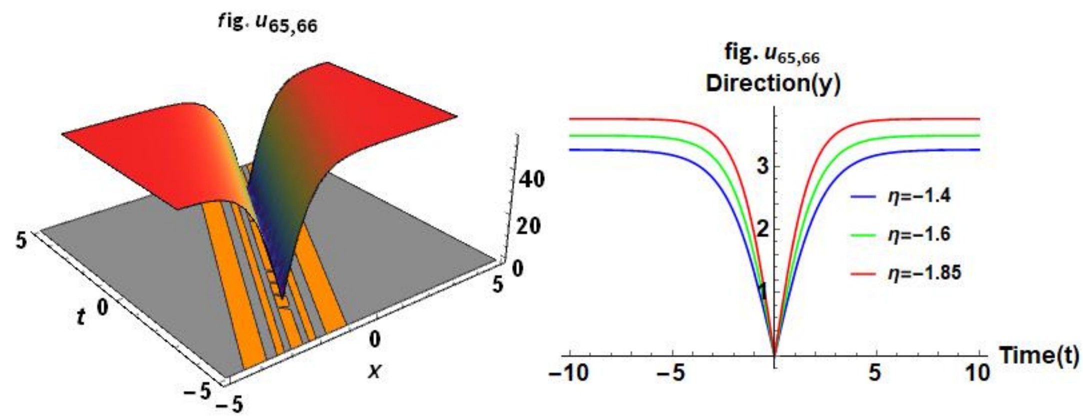

FIGURE 7. 3D and 2D surface of Bell type wave profile.

7. Conclusion

We considered the SMCH model in this paper and obtained some exact solutions by the unified and improved F-expansion procedures. These discussed methods are very effective and powerful for deriving the more and new exact solutions. The solutions interactively and graphically the interactions of waves for example singular kink wave, kink wave, periodic peakon wave, anti compacton, and bell shape wave solutions are presented. As a result, using the aforementioned approaches to solve NLPDEs will be simpler, more efficient, more powerful, fruitful, and aesthetically pleasing. Finally, this technique will provide NLPDEs with a more complex mathematical tool for finding soliton solutions, making waveform phenomena more useful in research and engineering.

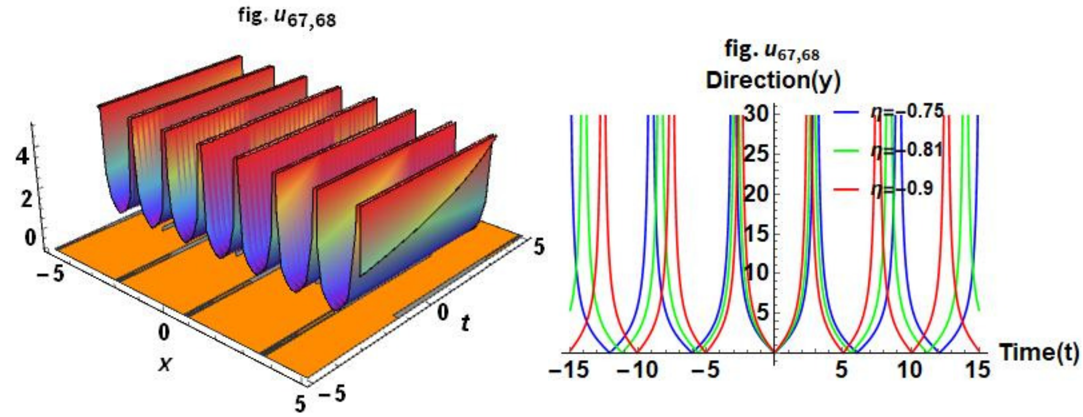

FIGURE 8. 3D and 2D surface of singular periodic peakon type wave profile.

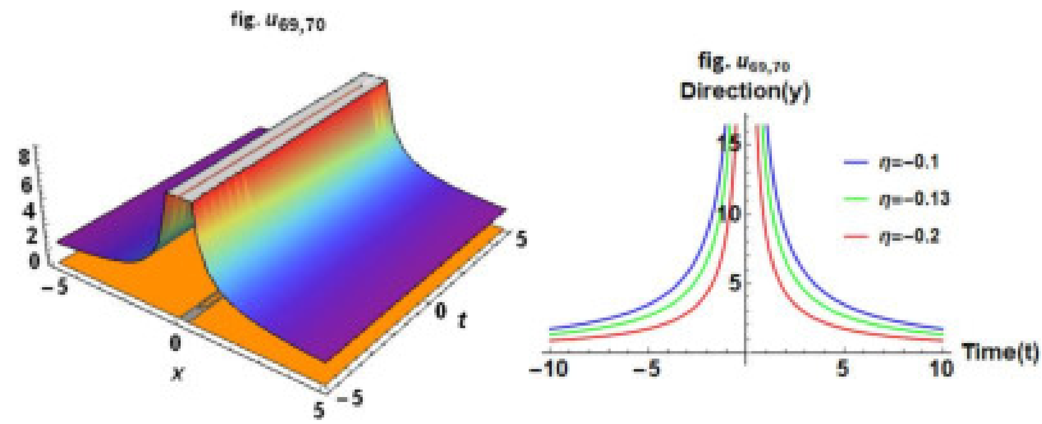

FIGURE 9. 3D and 2D surface of singular periodic peakon type wave profile.

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