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Non-polynomial cubic spline method for solution of higher order boundary value problems

Shahna Khan[1] **and Arshad Khan**[2]

[1]Department of Mathematics, Sri Venkateswara College, University of Delhi, New Delhi-21, India.

[2]Department of Mathematics, Jamia Millia Islamia, New Delhi-25, India.

Abstract

In this paper, a new algorithm based on non-polynomial spline is developed for the solution of higher order boundary value problems(BVPs). Employment of the method is done by decomposing the higher order BVP into a system of third order BVPs. Convergence analysis of the developed method is also discussed. The method is tested on higher order linear as well as non-linear BVPs which shows the accuracy and efficiency of the method and also compared our results with some existing fourth order methods.

Keywords. Non-polynomial spline, Higher-order, Non-linear, Convergence analysis, Boundary value problems.

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1. Introduction

Higher order BVPs arise in diversified fields of sciences, particularly in fluid dynamics, astrophysics, induction motors, beam theory, astronomy and applied sciences. In recent years, mostly higher-order BVPs are solved due to their mathematical importance. In [1] author discussed conditions for the existence and uniqueness of the solutions of BVPs. Several techniques have been developed to obtain the numerical solutions of BVPs of higher-order. For example, finite difference scheme [6], spline collocation method [12], modified decomposition method [17], and Petrov-Galerkin Method [7] were developed to solve higher order BVPs. In [3, 10, 11, 18], various splines methods were given to obtain the solution of differential equations. Here, our aim is to give the solution of BVPs of the form:

$$z^{(3N)} = F(t, z, z', z^{(2)}, z^{(3)}..., z^{(3N-1)}),\ r < t < s \tag{1.1}$$

with

$$z(r) = \mu_1, z^{(3)}(r) = \mu_2, z^{(6)}(r) = \mu_3, z^{(9)}(r) = \mu_4, ..., z^{(3N-3)}(r) = \mu_N, \tag{1.2}$$

$$z'(r) = \nu_1, z^{(4)}(r) = \nu_2, z^{(7)}(r) = \nu_3, z^{(10)}(r) = \nu_4, ..., z^{(3N-2)}(r) = \nu_N, \tag{1.3}$$

$$z'(s) = \lambda_1, z^{(4)}(s) = \lambda_2, z^{(7)}(s) = \lambda_3, z^{(10)}(s) = \lambda_4, ..., z^{(3N-2)}(s) = \lambda_N. \tag{1.4}$$

where F is sufficiently smooth function in the interval $[r, s]$, $N = 2, 3, 4$, μ_l, ν_l and λ_l $(l = 1, 2..., n)$ are real constants. We rewrite the equation (1.1)-(1.4) as follows:

$$z_1^{(3)}(t) = z_2(t), \tag{1.5}$$

$$z_2^{(3)}(t) = z_3(t), \tag{1.6}$$

$$\vdots$$

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∗ Corresponding author. Email: khan.shahana5@gmail.com.

$$z_N^{(3)}(t) = f(t, z_1, z_1', z_1^{(2)}, z_2, z_2', z_2^{(2)}, ..., z_N, z_N', z_N^{(2)}), \tag{1.7}$$

with modified conditions;

$$z_1(r) = \mu_1, \; z_1'(r) = \nu_1, \; z_1'(s) = \lambda_1, \tag{1.8}$$
$$z_2(r) = \mu_2, \; z_2'(r) = \nu_2, \; z_2'(s) = \lambda_2, \tag{1.9}$$
$$\vdots$$
$$z_N(r) = \mu_N, \; z_N'(r) = \nu_N, \; z_N'(s) = \lambda_N. \tag{1.10}$$

The above system involves third order BVPs. For example, cubic spline scheme [2], quartic non-polynomial spline method [5] and quintic spline methods [8] were used to solve third order BVPs. Here, we solve higher order BVPs. There are various methods like the collocation method [13], non-polynomial spline method [4, 14, 15] were developed to determine the numerical solutions of these problems. After implementing the linear problem over the developed scheme, we get a system which consists of linear equations and in the case of a nonlinear problem, we get a system of non-linear equations. The linear system is solved by the LU decomposition method and nonlinear system is solved by the Newton Raphson method. The paper is comprised of five sections. Section 2 gives a derivation of method. Application of the method for solving ninth order BVPs is discussed in section 3. Convergence analysis of the fourth order method is discussed in section 4 and in section 5, six numerical examples and their comparison with some existing fourth order methods are presented.

2. Derivation of scheme

Let

$$t_l = r + lh, \; l = 0, 1, ..., n \text{ and } h = (s - r)/(n + 1),$$

and

$$S_l(t) = a_l \sin k(t - t_l) + b_l \cos k(t - t_l) + c_l(t - t_l) + d_l, \tag{2.1}$$

be a non-polynomial spline S_l is defined on [r,s] of class $C^2[r, s]$ which reduces into an ordinary cubic spline in [r,s] as $k \longrightarrow 0$ and $k > 0$. To calculate the coefficients a_l, b_l, c_l *and* d_l, we define

$$S_l(t_l) = z_l, \; S_l(t_{l+1}) = z_{l+1}, \tag{2.2}$$
$$S_l'(t_l) = D_l, \; S_l'(t_{l+1}) = D_{l+1}, \tag{2.3}$$
$$S_l'''(t_l) = \frac{1}{2}(F_l + F_{l+1}), \; l = 0, 1, ..., n. \tag{2.4}$$

using (2.2), (2.3), and (2.4) we calculated the coefficients as

$$\begin{aligned} a_l &= -\frac{F_{l+1} + F_l}{2k^3}, \\ b_l &= \frac{z_{l+1} - z_l}{\eta} + \frac{(F_{l+1} + F_l)}{2k^3}\left(\frac{-\phi + \sin\phi}{\eta}\right) - \frac{hD_l}{\eta}, \\ c_l &= D_l + \frac{F_{l+1} + F_l}{2k^3}, \\ d_l &= z_l - b_l, \end{aligned}$$

where, $\phi = kh$ and $\eta = -1 + \cos\phi$.
Using the continuity conditions, $S_{l-1}^m(t_l) = S_l^m(t_l), m = 0, 1, 2$ the following equations are derived as

$$A_1 D_{l-1} + A_2 D_l = A_3 z_{l-1} + A_4 z_l + A_5(F_{l-1} + F_l), \tag{2.5}$$

$$B_1 D_{l-1} + B_2 D_l = B_3 z_{l-1} + B_4 z_l + B_5 z_{l+1} + B_6 F_{l-1} + B_7 F_l + B_8 F_{l+1}, \tag{2.6}$$

where

$$A_1 = \frac{h(\eta + \sin\phi)}{\eta}, \qquad A_2 = -h, \qquad A_3 = -\frac{\phi \sin\phi}{\eta}, \qquad A_4 = \frac{\phi\sin\phi}{\eta}, \qquad A_5 = \frac{h^3(2 - 2\cos\phi - \phi\sin\phi)}{2\phi^2\eta},$$

$$B_1 = -hB_3, \quad B_2 = h, \quad B_3 = \cos\phi, \quad B_4 = -1 - B_3, \quad B_5 = 1, \quad B_6 = \frac{\phi B_3 - \sin\phi}{2k^3}, \quad B_7 = \frac{\phi\eta}{2k^3}, \quad B_8 = \frac{-\phi + \sin\phi}{2k^3}.$$

Using (2.5) and (2.6), we obtain the following relation in terms of z_l and F_l

$$\tau z_{l-2} + \sigma z_{l-1} + \omega z_l + \rho z_{l+1} = h^3[\psi(F_{l-2} + F_{l+1}) + \tilde{\psi}(F_{l-1} + F_l)], \; l = 2, 3, ..., n-1, \tag{2.7}$$

where,

$$\tau = \cos^2\phi, \qquad \sigma = \frac{B_3 + \tau - 2\cos^3\phi}{\eta},$$
$$\omega = \frac{-2B_3 + \tau + \cos^3\phi}{\eta}, \qquad \rho = -B_3,$$
$$\psi = \frac{-\phi B_3 + B_3 \sin\phi}{2\phi^3}, \qquad \tilde{\psi} = \frac{\tau(2\phi B_3 - 3\phi - \sin\phi) + \phi B_3(1 + \sin\phi)}{2\phi^3\eta}.$$

The above recurrence relation gives $(n-2)$ linear equations in n unknowns $z_l, l = 1, 2, ..., n$. We need two more equations. These two equations are obtained for second and fourth order method respectively by using method of undetermined coefficients given by [2]

$$3z_0 - 4z_1 + z_2 = -2hD_0 + \frac{h^3}{12}[3F_0 + 4F_1 + F_2] + O(h^5), \; l = 1, \tag{2.8}$$

$$-3z_{n-2} + 8z_{n-1} - 5z_n = -2hD_{n+1} + \frac{h^3}{12}[3F_{n-2} + 10F_{n-1} + 31F_n] + O(h^5), \; l = n, \tag{2.9}$$

and

$$3z_0 - 4z_1 + z_2 = -2hD_0 + \frac{h^3}{60}[8F_0 + 35F_1 - 4F_2 + F_3] + O(h^7), \; l = 1, \tag{2.10}$$

$$-3z_{n-2} + 8z_{n-1} - 5z_n = -2hD_{n+1} + \frac{h^3}{60}[-8F_{n-3} + 33F_{n-2} + 38F_{n-1} + 157F_n] + O(h^7), \; l = n. \tag{2.11}$$

Remark: Our method reduces to [2] when

$$(\psi, \tilde{\psi}) = \frac{1}{12}(1, 5). \tag{2.12}$$

Truncation error (TE). After expanding equation (2.7) using Taylor series we obtained TE as follows:

$$\begin{aligned} t_l = {} & (\tau + \sigma + \omega + \rho)z_l + (-2\tau - \sigma + \rho)hz_l' + (4\tau + \sigma + \rho)\frac{h^2}{2!}z_l^{(2)} + \left(\frac{-8\tau - \sigma + \rho}{3!} - (2\psi + 2\tilde{\psi})\right)h^3 z_l^{(3)} \\ & + \left(\frac{16\tau + \sigma + \rho}{4!} + (\psi + \tilde{\psi})\right)h^4 z_l^{(4)} + \left(\frac{-32\tau - \sigma + \rho}{5!} - \frac{5\psi + \tilde{\psi}}{2!}\right)h^5 z_l^{(5)} + \left(\frac{64\tau + \sigma + \rho}{6!} + \frac{7\psi + \tilde{\psi}}{3!}\right)h^6 z_l^{(6)} \\ & + \left(\frac{-128\tau - \sigma + \rho}{7!} - \frac{7\psi + \tilde{\psi}}{4!}\right)h^7 z_l^{(7)} + O(h^8), \; l = 2, 3, ..., n-1. \end{aligned} \tag{2.13}$$

Method of different orders are obtained for various values of ψ and $\tilde{\psi}$.
The TE for the method of second order when $(\tau, \sigma, \omega, \rho, \psi, \tilde{\psi}) = \left(-1, 3, -3, 1, \frac{1}{12}, \frac{5}{12}\right)$ is given as

$$t_l = \begin{cases} -\dfrac{1}{10} h^5 z_0^{(5)} + O(h^6), & l\text{=1}, \\ -\dfrac{1}{6} h^5 z_l^{(5)} + O(h^6), & l\text{=2,3,...,}n\text{-1}, \\ -\dfrac{1}{10} h^5 z_n^{(5)} + O(h^6), & l\text{=}n. \end{cases} \tag{2.14}$$

The TE for the method of fourth order when $(\tau, \sigma, \omega, \rho, \psi, \tilde{\psi}) = \left(-1, 3, -3, 1, 0, \frac{1}{2}\right)$ is given as

$$t_l = \begin{cases} -\dfrac{29}{2520} h^7 z_0^{(7)} + O(h^8), & l\text{=1}, \\ \dfrac{1}{240} h^7 z_l^{(7)} + O(h^8), & l\text{=2,3,...,}n\text{-1}, \\ \dfrac{677}{5040} h^7 z_n^{(7)} + O(h^8), & l\text{=}n. \end{cases} \tag{2.15}$$

3. Application to ninth order BVPs

We consider a ninth order BVP of the form

$$\begin{aligned} z^{(9)}(t) &= a(t)z^{(8)}(t) + b(t)z^{(7)}(t) + c(t)z^{(6)}(t) + d(t)z^{(5)}(t) + p(t)z^{(4)}(t) + w(t)z^{(3)}(t) \\ &\quad + g(t)z^{(2)}(t) + h(t)z'(t) + q(t)z(t) + m(t), \end{aligned} \tag{3.1}$$

with

$$z(r) = \mu_1, \quad z^{(3)}(r) = \mu_2, \quad z^{(6)}(r) = \mu_3, \tag{3.2}$$

$$z'(r) = \nu_1, \quad z^{(4)}(r) = \nu_2, \quad z^{(7)}(r) = \nu_3, \tag{3.3}$$

$$z'(s) = \lambda_1, \quad z^{(4)}(s) = \lambda_2, \quad z^{(7)}(s) = \lambda_3, \tag{3.4}$$

where μ_l , ν_l and λ_l $(l = 1, 2, 3)$ are constants and $a(t)$, $b(t)$, $c(t)$, $d(t)$, $p(t)$, $w(t)$, $g(t)$, $h(t)$, $q(t)$, and $m(t)$ are continuously differentiable functions defined on $[r, s]$. We rewrite the above problem as follows:

$$z^{(3)}(t) = u(t), \tag{3.5}$$

$$u^{(3)}(t) = v(t), \tag{3.6}$$

$$\begin{aligned} v^{(3)}(t) &= a(t)v^{(2)}(t) + b(t)v'(t) + c(t)v(t) + d(t)u^{(2)}(t) + p(t)u'(t) + w(t)u(t) \\ &\quad + g(t)z^{(2)}(t) + h(t)z'(t) + q(t)z(t) + m(t), \end{aligned} \tag{3.7}$$

with

$$z(r) = \mu_1, \quad z'(r) = \nu_1, \quad z'(s) = \lambda_1, \tag{3.8}$$

$$u(r) = \mu_2, \quad u'(r) = \nu_2, \quad u'(s) = \lambda_2, \tag{3.9}$$

$$v(r) = \mu_3, \quad v'(r) = \nu_3, \quad v'(s) = \lambda_3. \tag{3.10}$$

The following are the higher order approximations to derivatives used in (3.7)

$$z_l' = \frac{z_{l+1} - z_{l-1}}{2h}, \quad z_{l-2}' = \frac{-5z_{l-1} + 8z_l - 3z_{l+1}}{2h}, \tag{3.11}$$

$$z_{l-1}' = \frac{-3z_{l-1} + 4z_l - z_{l+1}}{2h}, \quad z_{l+1}' = \frac{z_{l-1} - 4z_l + 3z_{l+1}}{2h}, \tag{3.12}$$

$$z_l'' = \frac{z_{l-1} - 2z_l + z_{l+1}}{h^2}, \quad z_{l-2}'' = \frac{z_{l-1} - 2z_l + z_{l+1}}{h^2} - 2hz_l''', \tag{3.13}$$

$$z_{l-1}'' = \frac{z_{l-1} - 2z_l + z_{l+1}}{h^2} - hz_l''', \quad z_{l+1}'' = \frac{z_{l-1} - 2z_l + z_{l+1}}{h^2} + hz_l''', \tag{3.14}$$

$$\tilde{z}_l' = \frac{z_{l+1} - z_{l-1}}{2h} + \frac{h^2}{6} z_l''' - \frac{h^2}{24}(z_{l+1}''' - z_{l-1}''') \quad \tilde{z}_l'' = \frac{z_{l+1} - 2z_l + z_{l-1}}{h^2} + \frac{h}{3}(z_l''' - z_{l-1}'''). \tag{3.15}$$

Here, we derive the scheme for method of fourth order when $(\tau, \sigma, \omega, \rho) = (-1, 3, -3, 1)$, $\psi = 0$ and $\tilde{\psi} = 1/2$. Therefore by implementing the BVPs (3.5)-(3.7) on the scheme (2.7), we get the following system

$$\tau z_{l-2} + \sigma z_{l-1} + \omega z_l + \rho z_{l+1} = \frac{h^3}{2}[u_{l-1} + u_l], \tag{3.16}$$

$$\tau u_{l-2} + \sigma u_{l-1} + \omega u_l + \rho u_{l+1} = \frac{h^3}{2}[v_{l-1} + v_l], \tag{3.17}$$

$$\tau v_{l-2} + \sigma v_{l-1} + \omega v_l + \rho v_{l+1} = \frac{h^3}{2}[F_{l-1} + \tilde{F}_l], \tag{3.18}$$

where,

$$\begin{aligned}
\tilde{F}_l &= F(t, z_l, u_l, v_l, \tilde{z}_l', \tilde{u}_l', \tilde{v}_l', \tilde{z}_l'', \tilde{u}_l'', \tilde{v}_l''), \\
F_{l-1} &= F(t, z_{l-1}, u_{l-1}, v_{l-1}, z_{l-1}', u_{l-1}', v_{l-1}', z_{l-1}'', u_{l-1}'', v_{l-1}''), \quad l = 2, 3, ..., n-1.
\end{aligned}$$

Finally, we get the vector difference equation for the BVPs (3.1)

$$A_l U_{l-2} + B_l U_{l-1} + C_l U_l + D_l U_{l+1} = H_l, \tag{3.19}$$

which are as follows:

$$\begin{bmatrix} al_{11} & al_{12} & al_{13} \\ al_{21} & al_{22} & al_{23} \\ al_{31} & al_{32} & al_{33} \end{bmatrix} \begin{bmatrix} z_{l-2} \\ u_{l-2} \\ v_{l-2} \end{bmatrix} + \begin{bmatrix} bl_{11} & bl_{12} & bl_{13} \\ bl_{21} & bl_{22} & bl_{23} \\ bl_{31} & bl_{32} & bl_{33} \end{bmatrix} \begin{bmatrix} z_{l-1} \\ u_{l-1} \\ v_{l-1} \end{bmatrix} + \begin{bmatrix} cl_{11} & cl_{12} & cl_{13} \\ cl_{21} & cl_{22} & cl_{23} \\ cl_{31} & cl_{32} & cl_{33} \end{bmatrix} \begin{bmatrix} z_l \\ u_l \\ v_l \end{bmatrix} +$$

$$\begin{bmatrix} dl_{11} & dl_{12} & dl_{13} \\ dl_{21} & dl_{22} & dl_{23} \\ dl_{31} & dl_{32} & dl_{33} \end{bmatrix} \begin{bmatrix} z_{l+1} \\ u_{l+1} \\ v_{l+1} \end{bmatrix} = \begin{bmatrix} h_{l1} \\ h_{l2} \\ h_{l3} \end{bmatrix}, \ l = 2, 3, ..., n-1 \tag{3.20}$$

where,

$$\begin{aligned}
al_{11} &= \tau, al_{12} = 0, al_{13} = 0, \\
al_{21} &= 0, al_{22} = \tau, al_{23} = 0, \\
al_{31} &= 0, al_{32} = 0, al_{33} = \tau, \\
bl_{11} &= \sigma, bl_{12} = -\frac{h^3}{2}, bl_{13} = 0, \\
bl_{21} &= 0, bl_{22} = \sigma, bl_{23} = -\frac{h^3}{2}, \\
bl_{31} &= -\frac{\delta_1 h g_l}{2} + \frac{\delta_1 h^2 h_l}{4} - \frac{\delta_2 h g_{l-1}}{2} + \frac{3\delta_2 h^2 h_{l-1}}{4} - \frac{\delta_3 h g_{l+1}}{2} - \frac{\delta_3 h^2 h_{l+1}}{4} - \frac{\delta_1 h^3 q_l}{2}, \\
bl_{32} &= -\frac{\delta_1 h d_l}{2} + \frac{\delta_1 h^2 p_l}{4} - \frac{\delta_2 h d_{l-1}}{2} + \frac{3\delta_2 h^2 p_{l-1}}{4} - \frac{\delta_3 h d_{l+1}}{2} - \frac{\delta_3 h^2 p_{l+1}}{4} - \frac{\delta_1 h^3 w_l}{2} + \frac{h^3}{2}\left(\frac{g_l h}{3} - \frac{h_l h^2}{24}\right),
\end{aligned}$$

$$
\begin{aligned}
bl_{33} &= \sigma - \frac{\delta_1 h a_l}{2} + \frac{\delta_1 h^2 b_l}{4} - \frac{\delta_2 h a_{l-1}}{2} + \frac{3\delta_2 h^2 b_{l-1}}{4} - \frac{\delta_3 h a_{l+1}}{2} - \frac{\delta_3 h^2 b_{l+1}}{4} - \frac{\delta_1 h^3 c_l}{2} + \frac{h^3}{2}(\frac{d_l h}{3} - \frac{p_l h^2}{24}),\\
cl_{11} &= \omega, cl_{12} = -\frac{h^3}{2}, cl_{13} = 0,\\
cl_{21} &= 0, cl_{22} = \omega, cl_{23} = -\frac{h^3}{2},\\
cl_{31} &= \delta_1 h g_l - \delta_2 h^2 h_{l-1} + \delta_2 h g_{l-1} - \frac{\delta_2 h^3 q_{l-1}}{2} + \delta_3 h g_{l+1} + \delta_3 h^2 h_{l+1}\\
cl_{32} &= \delta_1 h d_l - \delta_2 h^2 p_{l-1} + \delta_2 h d_{l-1} - \frac{\delta_2 h^3 w_{l-1}}{2} + \delta_3 h d_{l+1} + \delta_3 h^2 p_{l+1} + \frac{h^3}{2}(-\frac{g_l h}{3} - \frac{h_l h^2}{6}),\\
cl_{33} &= \omega + \delta_1 h a_l - \delta_2 h^2 b_{l-1} + \delta_2 h a_{l-1} - \frac{\delta_2 h^3 c_{l-1}}{2} + \delta_3 h a_{l+1} + \delta_3 h^2 b_{l+1} + \frac{h^3}{2}(-\frac{d_l h}{3} - \frac{p_l h^2}{6}),\\
dl_{11} &= \rho, dl_{12} = 0, dl_{13} = 0,\\
dl_{21} &= 0, dl_{22} = \rho, dl_{23} = 0,\\
dl_{31} &= -\frac{\delta_1 h g_l}{2} - \delta_1 h^2 h_l - \frac{\delta_2 h g_{l-1}}{2} + \frac{\delta_2 h^2 h_{l-1}}{4} - \frac{\delta_3 h g_{l+1}}{2} - \frac{3\delta_3 h^2 h_{l+1}}{4} - \frac{\delta_3 h^3 q_{l+1}}{2},\\
dl_{32} &= -\frac{\delta_1 h d_l}{2} - \delta_1 h^2 p_l - \frac{\delta_2 h d_{l-1}}{2} + \frac{\delta_2 h^2 p_{l-1}}{4} - \frac{\delta_3 h d_{l+1}}{2} - \frac{3\delta_3 h^2 p_{l+1}}{4} - \frac{\delta_3 h^3 w_{l+1}}{2} + \frac{h_l h^5}{48},\\
dl_{33} &= \rho - \frac{\delta_1 h a_l}{2} - \delta_1 h^2 b_l - \frac{\delta_2 h a_{l-1}}{2} + \frac{\delta_2 h^2 b_{l-1}}{4} - \frac{\delta_3 h a_{l+1}}{2} - \frac{3\delta_3 h^2 b_{l+1}}{4} - \frac{\delta_3 h^3 c_{l+1}}{2} + \frac{p_l h^5}{48},\\
h_{l1} &= 0, h_{l2} = 0,\\
h_{l3} &= h^3(\delta_2 m_{l-1} + \delta_1 m_l + \delta_3 m_{l+1}),\ l = 2, 3, ..., n-1,\\
\textit{where}&\\
\delta_1 &= 1 + \frac{a_l h}{3} + \frac{b_l h^2}{6} - \delta_2 a_{l-1} h + \delta_3 a_{l+1} h,\\
\delta_2 &= 1 - \frac{a_l h}{3} + \frac{b_l h^2}{24},\\
\delta_3 &= -\frac{b_l h^2}{24}.
\end{aligned}
$$

Now for *l=1*, we have

$$
A_1 U_1 + B_1 U_2 + C_1 U_3 + D_1 U_4 + E_1 U_5 = H_1, \tag{3.21}
$$

which can be written as

$$
\begin{bmatrix} a1_{11} & a1_{12} & a1_{13} \\ a1_{21} & a1_{22} & a1_{23} \\ a1_{31} & a1_{32} & a1_{33} \end{bmatrix}
\begin{bmatrix} z_1 \\ u_1 \\ v_1 \end{bmatrix} +
\begin{bmatrix} b1_{11} & b1_{12} & b1_{13} \\ b1_{21} & b1_{22} & b1_{23} \\ b1_{31} & b1_{32} & b1_{33} \end{bmatrix}
\begin{bmatrix} z_2 \\ u_2 \\ v_2 \end{bmatrix} +
\begin{bmatrix} c1_{11} & c1_{12} & c1_{13} \\ c1_{21} & c1_{22} & c1_{23} \\ c1_{31} & c1_{32} & c1_{33} \end{bmatrix}
\begin{bmatrix} z_3 \\ u_3 \\ v_3 \end{bmatrix} +
\begin{bmatrix} d1_{11} & d1_{12} & d1_{13} \\ d1_{21} & d1_{22} & d1_{23} \\ d1_{31} & d1_{32} & d1_{33} \end{bmatrix}
\begin{bmatrix} z_4 \\ u_4 \\ v_4 \end{bmatrix} +
\begin{bmatrix} e1_{11} & e1_{12} & e1_{13} \\ e1_{21} & e1_{22} & e1_{23} \\ e1_{31} & e1_{32} & e1_{33} \end{bmatrix}
\begin{bmatrix} z_5 \\ u_5 \\ v_5 \end{bmatrix} =
\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \end{bmatrix}, \tag{3.22}
$$

where,

$$
\begin{aligned}
&a1_{11}=-4, a1_{12}=-\frac{7h^3}{12}, a1_{13}=0,\\
&a1_{21}=a1_{13}, a1_{22}=a1_{11}, a1_{23}=a1_{12},\\
&a1_{31}=\frac{77}{45}hg_0-\frac{1}{120}h^2h_3+\frac{1}{720}hg_3+\frac{35}{48}hg_1+\frac{1}{90}h^2h_2+\frac{1}{72}h^2h_1+\frac{4}{45}hg_2+\frac{7}{12}h^3q_1,\\
&a1_{32}=\frac{77}{45}hd_0+\frac{35}{48}hd_1+\frac{1}{72}h^2p_1+\frac{4}{45}hd_2+\frac{1}{90}h^2p_2+\frac{1}{720}hd_3-\frac{1}{120}h^2p_3+\frac{7}{12}h^3w_1,\\
&a1_{33}=-4+\frac{77}{45}ha_0+\frac{35}{48}ha_1+\frac{1}{72}h^2b_1+\frac{4}{45}ha_2+\frac{1}{90}h^2b_2+\frac{1}{720}ha_3-\frac{1}{120}h^2b_3+\frac{7}{12}h^3c_1,
\end{aligned}
$$

$$
\begin{aligned}
&b1_{11}=1, b1_{12}=\frac{h^3}{15}, b1_{13}=0,\\
&b1_{21}=0, b1_{22}=1, b1_{23}=\frac{h^3}{15},\\
&b1_{31}=-\frac{107}{45}hg_0+\frac{35}{180}hg_1-\frac{35}{40}h^2h_1-\frac{1}{6}hg_2-\frac{1}{45}hg_3+\frac{1}{40}h^2h_3+\frac{1}{15}h^3q_2,\\
&b1_{32}=-\frac{107}{45}hd_0+\frac{35}{180}hd_1-\frac{35}{40}h^2p_1-\frac{1}{6}hd_2-\frac{1}{45}hd_3+\frac{1}{40}h^2p_3+\frac{1}{15}h^3w_2,\\
&b1_{33}=1-\frac{107}{45}ha_0+\frac{35}{180}ha_1-\frac{35}{40}h^2b_1-\frac{1}{6}ha_2-\frac{1}{45}ha_3+\frac{1}{40}h^2b_3+\frac{1}{15}h^3c_2,
\end{aligned}
$$

$$
\begin{aligned}
&c1_{11}=0, c1_{12}=-\frac{h^3}{60}, c1_{13}=0,\\
&c1_{21}=c1_{13}, c1_{22}=c1_{11}, c1_{23}=c1_{12},\\
&c1_{31}=\frac{104}{60}hg_0-\frac{1}{72}h^2h_3+\frac{35}{120}h^2h_1-\frac{49}{72}hg_1+\frac{2}{45}h^2h_2+\frac{4}{45}hg_2+\frac{1}{24}hg_3-\frac{1}{60}h^3q_3,\\
&c1_{32}=\frac{104}{60}hd_0-\frac{49}{72}hd_1+\frac{35}{120}h^2p_1+\frac{4}{45}hd_2+\frac{2}{45}h^2p_2+\frac{1}{24}hd_3-\frac{1}{72}h^2p_3-\frac{1}{60}h^3w_3,\\
&c1_{33}=\frac{104}{60}ha_0-\frac{49}{72}ha_1+\frac{35}{120}h^2b_1+\frac{4}{45}ha_2+\frac{2}{45}h^2b_2+\frac{1}{24}ha_3-\frac{1}{72}h^2b_3-\frac{1}{60}h^3c_3,
\end{aligned}
$$

$$
\begin{aligned}
&d1_{11}=0, d1_{12}=0, d1_{13}=0,\\
&d1_{21}=0, d1_{22}=0, d1_{23}=0,\\
&d1_{31}=-\frac{61}{90}hg_0+\frac{35}{120}hg_1-\frac{35}{720}h^2h_1-\frac{1}{180}hg_2-\frac{1}{180}h^2h_2-\frac{1}{45}hg_3-\frac{1}{240}h^2h_3,\\
&d1_{32}=-\frac{61}{90}hd_0-\frac{35}{720}h^2p_1-\frac{1}{180}hd_2-\frac{1}{180}h^2p_2+\frac{35}{120}hd_1-\frac{1}{240}h^2p_3-\frac{1}{45}hd_3,\\
&d1_{33}=-\frac{61}{90}ha_0-\frac{1}{180}ha_2+\frac{35}{120}ha_1-\frac{1}{180}h^2b_2-\frac{1}{45}ha_3-\frac{35}{720}h^2b_1-\frac{1}{240}h^2b_3,
\end{aligned}
$$

$$\begin{aligned}
e1_{11} &= 0, e1_{12} = 0, e1_{13} = 0,\\
e1_{21} &= 0, e1_{22} = 0, e1_{23} = 0,\\
e1_{31} &= \frac{1}{9}hg_0 - \frac{35}{720}hg_1 + \frac{1}{720}hg_3,\\
e1_{32} &= \frac{1}{9}hd_0 - \frac{35}{720}hd_1 + \frac{1}{720}hd_3,\\
e1_{33} &= \frac{1}{9}ha_0 - \frac{35}{720}ha_1 + \frac{1}{720}ha_3,
\end{aligned}$$

$$\begin{aligned}
h_{11} &= -2hz_0' + \frac{2}{15}h^3u_0, h_{12} = -2hu_0' - 3u_0 + \frac{2}{15}h^3v_0,\\
h_{13} &= -2hv_0' + \frac{h^3}{60}\Big(8\big(b_0v_0' + p_0u_0' + h_0z_0'\big) + v_0\big(8c_0 + \frac{1}{2}a_0h + \frac{35}{72}a_1h - \frac{7}{48}h^2b_1\\
&+ \frac{1}{180}ha_2 - \frac{1}{180}h^2b_2 - \frac{1}{720}h^2b_3\big) + u_0\big(8f_0 + \frac{1}{2}d_0h + \frac{35}{72}d_1h - \frac{7}{48}h^2p_1 + \frac{1}{180}hd_2 - \frac{1}{180}h^2p_2 - \frac{1}{720}h^2p_3\big)\\
&+ z_0\big(8l_0 + \frac{1}{2}g_0h + \frac{35}{72}g_1h - \frac{7}{48}h^2h_1 + \frac{1}{180}hg_2 - \frac{1}{180}h^2h_2 - \frac{1}{720}h^2h_3\big)\Big) + \frac{h^3}{60}\Big(8m_0 + 35m_1 - 4m_2 + m_3\Big).
\end{aligned}$$

Now for $l{=}n$, we have

$$A_nU_{n-5} + B_nU_{n-4} + C_nU_{n-3} + D_nU_{n-2} + E_nU_{n-1} + F_nU_n \quad = \quad H_n, \tag{3.23}$$

which can be written as

$$\begin{aligned}
&\begin{bmatrix} an_{11} & an_{12} & an_{13}\\ an_{21} & an_{22} & an_{23}\\ an_{31} & an_{32} & an_{33}\end{bmatrix}\begin{bmatrix} z_{n-5}\\ u_{n-5}\\ v_{n-5}\end{bmatrix} + \begin{bmatrix} bn_{11} & bn_{12} & bn_{13}\\ bn_{21} & bn_{22} & bn_{23}\\ bn_{31} & bn_{32} & bn_{33}\end{bmatrix}\begin{bmatrix} z_{n-4}\\ u_{n-4}\\ v_{n-4}\end{bmatrix} + \begin{bmatrix} cn_{11} & cn_{12} & cn_{13}\\ cn_{21} & cn_{22} & cn_{23}\\ cn_{31} & cn_{32} & cn_{33}\end{bmatrix}\begin{bmatrix} z_{n-3}\\ u_{n-3}\\ v_{n-3}\end{bmatrix} +\\
&\begin{bmatrix} dn_{11} & dn_{12} & dn_{13}\\ dn_{21} & dn_{22} & dn_{23}\\ dn_{31} & dn_{32} & dn_{33}\end{bmatrix}\begin{bmatrix} z_{n-2}\\ u_{n-2}\\ v_{n-2}\end{bmatrix} + \begin{bmatrix} en_{11} & en_{12} & en_{13}\\ en_{21} & en_{22} & en_{23}\\ en_{31} & en_{32} & en_{33}\end{bmatrix}\begin{bmatrix} z_{n-1}\\ u_{n-1}\\ v_{n-1}\end{bmatrix} +\\
&\begin{bmatrix} fn_{11} & fn_{12} & fn_{13}\\ fn_{21} & fn_{22} & fn_{23}\\ fn_{31} & fn_{32} & fn_{33}\end{bmatrix}\begin{bmatrix} z_n\\ u_n\\ v_n\end{bmatrix} = \begin{bmatrix} h_{n1}\\ h_{n2}\\ h_{n3}\end{bmatrix},
\end{aligned} \tag{3.24}$$

where,

$$\begin{aligned}
an_{11} &= 0, an_{12} = 0, an_{13} = 0,\\
an_{21} &= 0, an_{22} = 0, an_{23} = 0,\\
an_{31} &= \frac{1}{90}hg_{n-3} + \frac{19}{360}hg_{n-1} - \frac{157}{72}hg_n,\\
an_{32} &= \frac{1}{90}hd_{n-3} + \frac{19}{360}hd_{n-1} - \frac{157}{72}hd_n,\\
an_{33} &= \frac{1}{90}ha_{n-3} + \frac{19}{360}ha_{n-1} - \frac{157}{72}ha_n,
\end{aligned}$$

$$
\begin{aligned}
&bn_{11}=0, bn_{12}=0, bn_{13}=0,\\
&bn_{21}=0, bn_{22}=0, bn_{23}=0,\\
&bn_{31}=-\frac{8}{45}hg_{n-3}-\frac{1}{30}h^2h_{n-3}-\frac{11}{240}hg_{n-2}-\frac{19}{360}h^2h_{n-2}+\frac{19}{60}hg_{n-1}+\frac{11}{240}h^2h_{n-1}+\frac{9577}{720}hg_n-\frac{157}{240}h^2h_n,\\
&bn_{32}=-\frac{8}{45}hd_{n-3}-\frac{1}{30}h^2p_{n-3}-\frac{11}{240}hd_{n-2}-\frac{19}{360}h^2p_{n-2}+\frac{19}{60}hd_{n-1}+\frac{11}{240}h^2p_{n-1}+\frac{9577}{720}hd_n-\frac{157}{240}h^2p_n,\\
&bn_{33}=-\frac{1}{30}h^2b_{n-3}+\frac{19}{60}ha_{n-1}-\frac{11}{240}ha_{n-2}+\frac{11}{240}h^2b_{n-1}-\frac{19}{360}h^2b_{n-2}+\frac{9577}{720}ha_n-\frac{8}{45}ha_{n-3}-\frac{157}{240}h^2b_n,
\end{aligned}
$$

$$
\begin{aligned}
&cn_{11}=0, cn_{12}=\frac{2h^3}{15}, cn_{13}=0,\\
&cn_{21}=0, cn_{22}=0, cn_{23}=\frac{2h^3}{15},\\
&cn_{31}=\frac{1}{3}hg_{n-3}+\frac{11}{15}hg_{n-2}-\frac{1}{9}h^2h_{n-3}+\frac{133}{180}hg_{n-1}+\frac{11}{30}h^2h_{n-2}-\frac{19}{60}h^2h_{n-1}+\frac{157}{45}h^2h_n-\frac{2041}{60}hg_n+\frac{2h^3q_{n-3}}{15},\\
&cn_{32}=\frac{1}{3}hd_{n-3}+\frac{11}{15}hd_{n-2}+\frac{157}{45}h^2p_n-\frac{1}{9}h^2p_{n-3}+\frac{133}{180}hd_{n-1}+\frac{11}{30}h^2p_{n-2}-\frac{2041}{60}hd_n-\frac{19}{60}h^2p_{n-1}+\frac{2h^3w_{n-3}}{15},\\
&cn_{33}=\frac{1}{3}ha_{n-3}+\frac{157}{45}h^2b_n+\frac{11}{30}h^2b_{n-2}+\frac{2h^3c_{n-3}}{15}+\frac{133}{180}ha_{n-1}-\frac{19}{60}h^2b_{n-1}-\frac{1}{9}h^2b_{n-3}-\frac{2041}{60}ha_n+\frac{11}{15}ha_{n-2},
\end{aligned}
$$

$$
\begin{aligned}
&dn_{11}=-3, dn_{12}=-\frac{33h^3}{60}, dn_{13}=0,\\
&dn_{21}=dn_{13}, dn_{22}=dn_{11}, dn_{23}=dn_{12},\\
&dn_{31}=-\frac{8}{45}hg_{n-3}+\frac{16799}{360}hg_n+\frac{1}{5}h^2h_{n-3}-\frac{38}{180}hg_{n-1}-\frac{11}{8}hg_{n-2}+\frac{19}{20}h^2h_{n-1}-\frac{157}{20}h^2h_n-\frac{33h^3q_{n-2}}{60},\\
&dn_{32}=-\frac{8}{45}hd_{n-3}-\frac{11}{8}hd_{n-2}+\frac{19}{20}h^2p_{n-1}-\frac{157}{20}h^2p_n+\frac{1}{5}h^2p_{n-3}+\frac{16799}{360}hd_n-\frac{38}{180}hd_{n-1}-\frac{33h^3w_{n-2}}{60},\\
&dn_{33}=-3-\frac{8}{45}ha_{n-3}+\frac{16799}{360}ha_n-\frac{11}{8}ha_{n-2}-\frac{38}{180}ha_{n-1}+\frac{1}{5}h^2b_{n-3}+\frac{19}{20}h^2b_{n-1}-\frac{157}{20}h^2b_n-\frac{33h^3c_{n-2}}{60},
\end{aligned}
$$

$$
\begin{aligned}
&en_{11}=8, en_{12}=-\frac{38h^3}{60}, en_{13}=0,\\
&en_{21}=en_{13}, en_{22}=en_{11}, en_{23}=en_{12},\\
&en_{31}=\frac{1}{90}hg_{n-3}-\frac{11}{30}h^2h_{n-2}-\frac{19}{24}hg_{n-1}-\frac{19}{36}h^2h_{n-1}-\frac{1}{15}h^2h_{n-3}-\frac{12089}{360}hg_n+\frac{11}{15}hg_{n-2}+\frac{157}{15}h^2h_n-\frac{38h^3q_{n-1}}{60},\\
&en_{32}=\frac{1}{90}hd_{n-3}-\frac{1}{15}h^2p_{n-3}+\frac{11}{15}hd_{n-2}-\frac{11}{30}h^2p_{n-2}-\frac{19}{24}hd_{n-1}-\frac{19}{36}h^2p_{n-1}-\frac{12089}{360}hd_n+\frac{157}{15}h^2p_n-\frac{38h^3w_{n-1}}{60},\\
&en_{33}=8+\frac{1}{90}ha_{n-3}-\frac{1}{15}h^2b_{n-3}+\frac{11}{15}ha_{n-2}-\frac{11}{30}h^2b_{n-2}-\frac{19}{24}ha_{n-1}-\frac{19}{36}h^2b_{n-1}-\frac{12089}{360}ha_n+\frac{157}{15}h^2b_n\\
&-\frac{38h^3c_{n-1}}{60},
\end{aligned}
$$

$$
\begin{aligned}
&fn_{11} = -5, fn_{12} = -\frac{157h^3}{60}, fn_{13} = 0,\\
&fn_{21} = fn_{13}, fn_{22} = fn_{11}, fn_{23} = fn_{12},\\
&fn_{31} = \frac{1}{90}h^2h_{n-3} + \frac{11}{240}h^2h_{n-2} - \frac{785}{144}h^2h_n - \frac{11}{240}hg_{n-2} - \frac{38}{240}h^2h_{n-1} + \frac{19}{36}hg_{n-1} + \frac{157}{16}hg_n - \frac{157h^3q_n}{60},\\
&fn_{32} = \frac{1}{90}h^2p_{n-3} + \frac{11}{240}h^2p_{n-2} + \frac{19}{36}hd_{n-1} - \frac{11}{240}hd_{n-2} - \frac{38}{240}h^2p_{n-1} - \frac{785}{144}h^2p_n + \frac{157}{16}hd_n - \frac{157h^3w_n}{60},\\
&fn_{33} = -5 + \frac{1}{90}h^2b_{n-3} - \frac{785}{144}h^2b_n + \frac{11}{240}h^2b_{n-2} - \frac{11}{240}ha_{n-2} + \frac{157}{16}ha_n - \frac{38}{240}h^2b_{n-1} + \frac{19}{36}ha_{n-1} - \frac{157h^3c_n}{60},\\
&h_{n1} = -2hz'_{n+1},\\
&h_{n2} = -2hu'_{n+1},\\
&h_{n3} = -2hv'_{n+1} + \frac{h^3}{60}\Big(157m_n + 38m_{n-1} + 33m_{n-2} - 8m_{n-3}\Big).
\end{aligned}
$$

4. Convergence analysis

Here, we discuss the convergence analysis of the method. We rewrite our method in the form

$$WX = H, \tag{4.1}$$

where,

$$
W = \begin{bmatrix}
A_1 & B_1 & C_1 & D_1 & & & & & \\
B_2 & C_2 & D_2 & & & & & & \\
A_3 & B_3 & C_3 & D_3 & & & & & \\
 & & \ddots & \ddots & \ddots & & & & \\
 & & & \ddots & \ddots & \ddots & & & \\
 & & & & \ddots & \ddots & \ddots & & \\
 & & & & & A_{n-1} & B_{n-1} & C_{n-1} & D_{n-1} \\
 & & & & A_n & B_n & C_n & D_n & E_n
\end{bmatrix} \tag{4.2}
$$

where, $A_l, B_l, ..., E_l (l = 1, 2, ..., n)$ are matrices of order 3×3 , $X = [x_1, x_2, ..., x_{n-1}]^T$, where $x_l = [z_l, u_l, v_l]^T$ and the right side column vector $H = [h_1, h_2, ..., h_{n-1}]^T$, where $h_l = [h_{l1}, h_{l2}]^T$.
Also,

$$W\tilde{X} = H + T, \tag{4.3}$$

where $T = [t_1, t_2, ..., t_{n-1}]^T$, where $t_l = [\tilde{z}_l - z_l, \tilde{u}_l - u_l, \tilde{v}_l - v_l]^T$ be the truncation error and $\tilde{X} = [\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_{n-1}]^T$, where $\tilde{x}_l = [\tilde{z}_l, \tilde{u}_l, \tilde{v}_l]^T$ be the exact solution. From (4.1) and (4.3) we get,

$$W(\tilde{X} - X) = T, \tag{4.4}$$
$$WE = T, \tag{4.5}$$
$$E = \tilde{X} - X. \tag{4.6}$$

Now by calculating the sum of entries of each row of the matrix W, we get

$$S_{1j} = \begin{cases} -3 - \dfrac{8}{15}h^3, & \text{j=1} \\ -3 - \dfrac{8}{15}h^3, & \text{j=2} \\ -3 + h\dfrac{1}{2}(a_0 + d_0 + g_0) - \dfrac{7}{48}h^2(b_1 + p_1 + h_1) + \dfrac{35}{72}h(a_1 + d_1 + g_1) & \\ +\dfrac{1}{180}h(a_2 + d_2 + g_2) - \dfrac{1}{180}h^2(b_2 + p_2 + h_2) - \dfrac{1}{720}h^2(b_3 + p_3 + h_3) & \\ -\dfrac{35}{60}h^3(q_1 + w_1 + c_1) + \dfrac{4}{60}h^3(q_2 + w_2 + c_2) - \dfrac{1}{60}h^3(q_3 + w_3 + c_3), & \text{j=3} \end{cases} \tag{4.7}$$

$$S_{lj} = \begin{cases} \tau + \sigma + \omega + \rho - h^3,\ l = 1, 4, 7, ..., n-3 & \text{j=1} \\ \tau + \sigma + \omega + \rho - h^3,\ l = 2, 5, 8, ..., n-2 & \text{j=2} \\ \tau + \sigma + \omega + \rho + h^3\Big(- \dfrac{(c_{l-1} + w_{l-1} + q_{l-1})}{2} & \\ -\dfrac{(c_l + w_l + q_l)}{2}\Big) - \dfrac{h^5}{12}(h_l + p_l),\ l = 3, 6, 9, ..., n-1 & \text{j=3} \end{cases} \tag{4.8}$$

$$S_{n\mathbf{j}} = \begin{cases} -\dfrac{11}{3}h^3, & \text{j=1} \\ -\dfrac{11}{3}h^3, & \text{j=2} \\ \dfrac{8}{60}h^3(q_{n-3} + w_{n-3} + c_{n-3}) - \dfrac{33}{60}h^3(q_{n-2} + w_{n-2} + c_{n-2}) & \\ -\dfrac{38}{60}h^3(q_{n-1} + w_{n-1} + c_{n-1}) - \dfrac{157}{60}h^3(q_n + w_n + c_n), & \text{j=3.} \end{cases} \tag{4.9}$$

Let $0 < M \in Z^+$ is the minimum of $\mid a_l \mid, \mid b_l \mid, \mid c_l \mid, \mid d_l \mid, \mid p_l \mid, \mid w_l \mid, \mid g_l \mid, \mid h_l \mid, \mid q_l \mid$ and $\mid m_l \mid$. For sufficiently small h, we can say that

$$S_{1j} \geq \begin{cases} \dfrac{8}{15}h^3, & \text{j=1} \\ \dfrac{8}{15}h^3, & \text{j=2} \\ \dfrac{8}{15}h^3 M, & \text{j=3} \end{cases} \tag{4.10}$$

$$S_{lj} \geq \begin{cases} h^3,\ l = 1, 4, ..., n-3 & \text{j=1} \\ h^3,\ l = 2, 5, ..., n-2 & \text{j=2} \\ h^3 M,\ l = 3, 6, ..., n-1 & \text{j=3} \end{cases} \tag{4.11}$$

$$S_{n\mathbf{j}} \geq \begin{cases} \frac{11}{3}h^3, & \text{j=1} \\ \frac{11}{3}h^3, & \text{j=2} \\ \frac{11}{3}h^3 M, & \text{j=3} \end{cases} \tag{4.12}$$

$$S_1 \geq \frac{8}{15}Mh^3,\ l = 1 \tag{4.13}$$

$$S_l \geq Mh^3,\ l = 2, 3, ..., n-1 \tag{4.14}$$

$$S_n \geq \frac{11}{3}Mh^3,\ l = n. \tag{4.15}$$

Therefore,

$$\frac{1}{S_l} \leq \begin{cases} \frac{15}{8h^3M},\ l = 1 \\ \frac{1}{Mh^3},\ l = 2, 3, ..., n-1 \\ \frac{3}{11h^3M},\ l = n. \end{cases} \tag{4.16}$$

We can easily show the irreducibility and monotonicity of matrix W for sufficiently small value of $\boldsymbol{h}$. Then, W^{-1} exist and $W^{-1} \geq 0$.
Hence,

$$\|W\|\|E\| \quad = \quad \|T\|. \tag{4.17}$$

Let $W^{-1} = (w^*_{l,j})$, then by [16], we get

$$\sum_{l=1}^{n-1} w^*_{l,j} S_l \quad = \quad 1,\ 1 \leq j \leq n-1. \tag{4.18}$$

Therefore,

$$w^*_{l,j} \quad \leq \quad \frac{1}{S_l}, \tag{4.19}$$

$$\|W^{-1}\| \quad = \quad \max_{1\leq l\leq n-1} \sum_{j=1}^{n-1} |w^*_{l,j}| \leq \sum_{l=1}^{n-1} \frac{1}{S_l} = \frac{1}{h^3M}\left(\frac{15}{8} + 1 + \frac{3}{11}\right), \quad 1 \leq l \leq n-1 \ and \tag{4.20}$$

$$\|\mathbf{T}_l\| \quad = \quad \max_{1\leq l\leq n-1} \sum_{l=1}^{n-1} |T_l|. \tag{4.21}$$

The error is obtained as

$$\|E\| \quad = \quad \|W^{-1}\|\|T\| \leq \frac{277}{88h^3M}\|T\|. \tag{4.22}$$

For fourth order method $\|T\| = O(h^7)$ by (2.13). Hence error is of order four. The above analysis shows that the developed method is fourth order convergent. Similarly, we can prove the second order convergence of the method.

5. Numerical Illustrations

To verify the applicability of our developed method on existing problems, we solve the following six BVPs of the form (1.1)-(1.4). The maximum absolute errors (MAE) are given in the Tables I-VI. Each problem has been decomposed into system of third order BVPs. It is clearly shown from MAE that our method gives better accuracy to BVPs solved by numerical methods such as [4, 7, 9, 13–15, 17].

Example 5.1. Sixth order BVP is considered as

$$z^{(6)}(t) + tz(t) = -(24 + 11t + t^3)\exp(t),\ 0 \le t \le 1 \tag{5.1}$$

with

$$\begin{aligned} z(0) &= 0,\ z^{(3)}(0) = -3, &(5.2)\\ z'(0) &= 1,\ z^{(4)}(0) = -8, &(5.3)\\ z'(1) &= \exp(1),\ z^{(4)}(1) = -16z'(1). &(5.4)\end{aligned}$$

The exact solution is $z(t) = t(1-t)\exp(t)$. The MAE of the problem (5.1) are given in Table 1 and results are compared with [4].

Example 5.2. Ninth order BVP with variable coefficients is considered as

$$\begin{aligned} z^{(9)}(t) + z^{(7)}(t) + tz^{(4)}(t) + z^{(3)}(t) + \sin tz'(t) + z(t) &= 5t\sin(t) - \cos(t) + t^2\cos(t) - t\sin^2(t)\\ &\quad + \sin(t)\cos(t) + t\cos(t),\ 0 \le t \le 1 \qquad (5.5)\end{aligned}$$

where,

$$\begin{aligned} z(0) &= 0,\ z^{(3)}(0) = -3,\ z^{(6)}(0) = 0, &(5.6)\\ z'(0) &= 1,\ z^{(4)}(0) = 0,\ z^{(7)}(0) = -7, &(5.7)\\ z'(1) &= \cos(1) - \sin(1),\ z^{(4)}(1) = 4\sin(1) + \cos(1),\ z^{(7)}(1) = \sin(1) - 7\cos(1). &(5.8)\end{aligned}$$

The exact solution is $z(t) = t\cos(t)$. The MAE of the problem (5.5) are given in Table 2 and results are compared with [7].

Example 5.3. Ninth order BVP is considered as

$$z^{(9)}(t) - z(t) = -9\exp(t),\ 0 \le t \le 1 \tag{5.9}$$

with

$$\begin{aligned} z(0) &= 1,\ z^{(3)}(0) = -2,\ z^{(6)}(0) = -5, &(5.10)\\ z'(0) &= 0,\ z^{(4)}(0) = -3,\ z^{(7)}(0) = -6, &(5.11)\\ z'(1) &= -\exp(1),\ z^{(4)}(1) = 4z'(1),\ z^{(7)}(1) = 7z'(1). &(5.12)\end{aligned}$$

The exact solution is $z(t) = (1-t)\exp(t)$. The MAE of the problem (5.9) are given in Table 3 and results are compared with [13].

Example 5.4. Twelfth order BVP is considered as

$$z^{(12)}(t) - z(t) = -12(2t\cos(t) + 11\sin(t)),\ -1 \le t \le 1 \tag{5.13}$$

with

$$\begin{aligned} z(-1) &= 0,\ z^{(3)}(-1) = 6\cos(1) - 6\sin(1),\ z^{(6)}(-1) = -12\cos(1) - 30\sin(1), &(5.14)\\ z^{(9)}(-1) &= -72\cos(1) + 18\sin(1),\ z'(-1) = 2\sin(1),\ z^{(4)}(-1) = 8\cos(1) + 6z'(-1), &(5.15)\\ z^{(7)}(-1) &= 42\cos(1) - 7z'(-1),\ z^{(10)}(-1) = -20\cos(1) - 45z'(-1),\ z'(1) = 2\sin(1), &(5.16)\\ z^{(4)}(1) &= -8\cos(1) - 6z'(1),\ z^{(7)}(1) = 42\cos(1) - 7z'(1),\ z^{(10)}(1) = 20\cos(1) + 45z'(1). &(5.17)\end{aligned}$$

The exact solution is $z(t) = (t^2-1)\sin(t)$. The MAE of the problem (5.13) are given in Table 4 and results are compared with [14, 15].

TABLE 1. MAE for Example 5.1

Our method	$n=8$	$n=16$	$n=32$
Fourth order method for $(\psi,\tilde{\psi})=\left(0,\frac{1}{2}\right)$	7.1632×10^{-6}	3.8149×10^{-7}	2.1596×10^{-8}
[4]	2.39×10^{-4}	3.43×10^{-6}	7.34×10^{-8}
Second order method for $(\psi,\tilde{\psi})=\left(\frac{1}{12},\frac{5}{12}\right)$	4.190×10^{-3}	1.369×10^{-3}	3.817×10^{-4}
[4]	2.99×10^{-2}	7.00×10^{-3}	1.80×10^{-3}

TABLE 2. MAE for Example 5.2

Our method	$n=10$	$n=20$	$n=40$
Fourth order method for$(\psi,\tilde{\psi})=\left(0,\frac{1}{2}\right)$	2.5256×10^{-7}	2.3357×10^{-8}	2.4397×10^{-9}
Second order method for$(\psi,\tilde{\psi})=\left(\frac{1}{12},\frac{5}{12}\right)$	1.13×10^{-3}	4.512×10^{-4}	3.505×10^{-4}
[7]	2.324×10^{-6}	–	–

Example 5.5. Sixth order BVP is considered as

$$z^{(6)}(t)=z^2(t)\exp(t),\ 0\le t\le 1 \tag{5.18}$$

with

$$\begin{aligned} z(0) &= 1,\ z^{(3)}(0)=z(0), && (5.19)\\ z'(0) &= 1,\ z^{(4)}(0)=z(0), && (5.20)\\ z'(1) &= \exp(1),\ z^{(4)}(1)=z'(1). && (5.21)\end{aligned}$$

The exact solution is $z(t)=\exp(t)$. The MAE of the problem (5.18) are given in Table 5 and results are compared with [9].

Example 5.6. Twelfth order non-linear BVP is considered as

$$z^{(12)}(t)+z^{(3)}(t) = 2z^2(t)\exp(t)\ ,0\le t\le 1 \tag{5.22}$$

with

$$\begin{aligned} z(0) &= 1,\ z^{(3)}(0)=1,\ z^{(6)}(0)=1,\ z^{(9)}(0)=1 && (5.23)\\ z'(0) &= -1,\ z^{(4)}(0)=1,\ z^{(7)}(0)=-1,\ z^{(10)}(0)=1, && (5.24)\\ z'(1) &= -\exp(-1),\ z^{(4)}(1)=\exp(-1),\ z^{(7)}(1)=z'(1),\ z^{(10)}(1)=z^{(4)}(1) && (5.25)\end{aligned}$$

The analytical solution is $z(t)=\exp(-t)$. The MAE of the problem (5.22) are given in Table 6 and results are compared with [17].

6. CONCLUSION

The numerical solution of higher order BVPs is given by non-polynomial spline. In literature, higher even order BVPs are solved by decomposing into the system of second order BVPs but here we decomposed the problem into system of third order BVPs. Then the developed new algorithm was applied on higher order like ninth order BVPs. Computationally our method is more viable due to use the lower degree splines rather than the higher degree splines

Table 3. MAE for Example 5.3

Our method	$n=10$	$n=20$	$n=40$
Fourth order method for$(\psi,\tilde{\psi})=\left(0,\frac{1}{2}\right)$	4.3270×10^{-7}	2.3426×10^{-8}	1.3816×10^{-9}
Second order method for$(\psi,\tilde{\psi})=\left(\frac{1}{12},\frac{5}{12}\right)$	8.484×10^{-4}	2.393×10^{-4}	6.184×10^{-5}
[13]	1.232×10^{-5}	–	–

Table 4. MAE for Example 5.4

Our method	$n=8$	$n=16$	$n=32$
Fourth order method for$(\psi,\tilde{\psi})=\left(0,\frac{1}{2}\right)$	2.1347×10^{-4}	1.3210×10^{-5}	9.2074×10^{-7}
Second order method for$(\psi,\tilde{\psi})=\left(\frac{1}{24},\frac{11}{24}\right)$	2.627×10^{-2}	7.013×10^{-3}	1.827×10^{-3}
[14]	–	4.69×10^{-5}	–
[15]	–	2.07×10^{-3}	–

Table 5. MAE for Example 5.5

Our method	$n=8$	$n=16$	$n=32$
Fourth order method for$(\psi,\tilde{\psi})=\left(0,\frac{1}{2}\right)$	1.6824×10^{-7}	9.0944×10^{-9}	5.2628×10^{-10}
[9]	7.02×10^{-6}	4.35×10^{-6}	7.87×10^{-7}
Second order method for$(\psi,\tilde{\psi})=\left(\frac{1}{240},\frac{119}{240}\right)$	2.3033×10^{-5}	4.5168×10^{-6}	1.0856×10^{-6}
[9]	2.19×10^{-4}	3.88×10^{-5}	1.59×10^{-6}

Table 6. MAE for Example 5.6

t	Our method	[17]
0.1	2.4840×10^{-10}	1.41×10^{-6}
0.2	2.0083×10^{-9}	2.69×10^{-6}
0.3	4.9198×10^{-9}	3.70×10^{-6}
0.4	8.6549×10^{-9}	4.35×10^{-6}
0.5	1.2912×10^{-8}	4.58×10^{-6}
0.6	1.7417×10^{-8}	4.36×10^{-6}
0.7	2.1912×10^{-8}	3.71×10^{-6}
0.8	2.6160×10^{-8}	2.69×10^{-6}

used by other authors. Error analysis of the developed algorithm is discussed in Section 4 which proved the fourth order accuracy of the scheme (2.7). In this paper, sixth, ninth, and twelfth order BVPs have been solved by lower degree non-polynomial spline. Six numerical illustrations of linear as well as non-linear BVPs are discussed. MAE shows that our results are better in accuracy and effectiveness than some existing fourth order methods.

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References

[1] R. P. Agarwal, *Boundary Value Problems for Higher-Order Differential Equations*, World Scientific Singapore, (1986).
[2] E. A. Al-Said and M. A. Noor, *Cubic splines method for a system of third-order boundary value problems*, Appl Math and Comput, *142*(2003), 195–204.
[3] K. Farajeyan, J. Rashidinia, R. Jalilian, and N. R. Maleki, *Application of spline to approximate the solution of singularly perturbed boundary-value problems*, Computational Methods for Differential Equations, *8* (2020), 373–388.
[4] S. U. Islam, I. A. Tirmizi, F. Haq, and M. A. Khan, *Non-polynomial splines approach to the solution of sixth-order boundary-value problems*, Appl Math and Comput, *195* (2008), 270–284.
[5] S. U. Islam, I. A. Tirmizi, and M. A. Khan, *Quartic non-polynomial spline approach to the solution of a system of third-order boundary-value problems*, J Math Anal Appl *335* (2007), 1095–1104.
[6] N. Jha, R. K. Mohanty, and V. Chauhan, *The Convergence of Geometric Mesh Cubic Spline Finite Difference Scheme for Nonlinear Higher Order Two-Point Boundary Value Problems*, Intern J Comput Math, (2014), Article ID 527924, 12 pages.
[7] K. N. S. Kasi Viswanadham and S. M. Reddy, *Numerical Solution of Ninth Order Boundary Value Problems by Petrov-Galerkin Method with Quintic B-splines as Basis Functions and Septic B-splines as Weight Functions*, International Conference on Computational Heat and Mass Transfer-2015 Procedia Engineering, *127* (2015), 1227–1234.
[8] A. Khan and T. Aziz, *The numerical solution of third-order boundary-value problems using quintic splines*, Appl Math Comput, *137* (2003), 253–260.
[9] A. Khan and P. Khandelwal, *Solution of Non-linear Sixth-order Two Point Boundary-value Problems Using Parametric Septic Splines*, Intern J Nonlin Science, *12* (2011), 184–195.
[10] K. Maleknejad, J. Rashidinia, and H. Jalilian, *Quintic Spline functions and Fredholm integral equation*, Computational Methods for Differential Equations, *9* (2021), 211–224.
[11] M. Kumar and P. K. Srivastava, *Computational Techniques for Solving Differential Equations by Cubic, Quintic, and Sextic Spline*, Intern J Comput Methods in Engg Science and Mechanics, *10* (2009), 108–115.
[12] J. Rashidinia, M. Khazaei, and H. Nikmarvani, *Spline collocation method for solution of higher order linear boundary value problems*, J Pure Appl Math, *6* (2015), 38–47.
[13] S. M. Reddy, *Collocation Method for Ninth Order Boundary Value Problems by Using Sextic B-splines*, Intern Research Journal of Engg and Tech *3* (2016), 56–72.
[14] S. S. Siddiqi and G. Akram, *Solutions of 12th order boundary value problems using non-polynomial spline technique*, Appl Math and Comput, *199* (2008), 559–571.
[15] S. S. Siddiqi and E. H. Twizell, *Spline solutions of linear twelfth-order boundary value problems*, Appl Math and Comput, *78* (1997), 371–390.
[16] R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall Englewood Cliffs NJ, (1962).
[17] A. M. Wazwaz, *Approximate solutions to boundary value problems of higher order by the modified decomposition method*, Comput Math Appl, *40* (2000), 679–691.
[18] H. Zadvan and J. Rashidina, *Development of non polynomial spline and New B-spline with application to solution of Klein-Gordon equation*, Computational Methods for Differential Equations, *8* (2020), 794–814.