Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 11, No. 2, 2023, pp. 241-253
DOI:10.22034/cmde.2022.50054.2082

A numerical method for solving the Duffing equation involving both integral and non-integral forcing terms with separated and integral boundary conditions

Mohammad Reza Doostdar[1,*], **Manochehr Kazemi**[2], **and Alireza Vahidi**[3]

[1]Department of Mathematics, Zarandieh Branch, Islamic Azad University, Zarandieh, Iran.
[2]Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran.
[3]Department of Mathematics, Yadegar-e-Imam Khomeini (RAH) Shahre Rey Branch, Islamic Azad University, Tehran, Iran.

Abstract

This paper presents an efficient numerical method to solve two versions of the Duffing equation by the hybrid functions based on the combination of Block-pulse functions and Legendre polynomials. This method reduces the solution of the considered problem to the solution of a system of algebraic equations. Moreover, the convergence of the method is studied. Some examples are given to demonstrate the applicability and effectiveness of the proposed method. Also, the obtained results are compared with some other results.

Keywords. Integral boundary conditions, Boundary value problem, Duffing equation, Hybrid functions, Legendre polynomials.
2010 Mathematics Subject Classification. 34B15, 65L10.

1. Introduction

The Duffing equation or Duffing oscillator as a powerful instrument to discuss some phenomena in various fields of science and engineering has been considered by scientists. This well-known dynamical system with exhibits chaotic behavior and bifurcations, which is described by a nonlinear differential equation, was presented by George Duffing in 1918 [38]. Although the Duffing equation was first introduced in studying electronics, its applications were developed in various fields such as brain modeling, orbit extraction, nonlinear mechanical oscillator, disease prediction, fuzzy modeling, nonuniformity caused by an infinite domain, biological systems, and so on [6, 7, 11, 14–16, 36, 37].

Integral boundary value problems have received a great deal of attention due to their widespread applications in different areas of applied sciences and engineering such as chemical engineering, underground water flow, heat conduction, thermo-elasticity, and population dynamics [4, 18, 21, 24, 39]. It is worth mentioning that integral boundary conditions can cover other kinds of non-local boundary conditions. For instance, three-point and multi-point boundary conditions are considered as special cases [28, 29]. Here, as boundary value problems, we consider two following versions of the Duffing equation, which we call them Version 1 and Version 2, respectively:

(1) The Duffing equation involving both integral and non-integral forcing terms is

$$v''(t) + \mu v'(t) + f(t, v(t), v'(t)) + \int_0^t h(t, x, v(x))\,\mathrm{d}x = 0,\ \ 0 < t < 1,\ \ \mu \in \mathbb{R} - \{0\}, \tag{1.1}$$

with the separated boundary conditions

$$r_0 v(0) - s_0 v'(0) = a,\ \ r_1 v(1) - s_1 v'(1) = b, \tag{1.2}$$

where functions $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$, $h : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous and $r_0, r_1, s_0, s_1, a, b \in \mathbb{R}$.

Received: 24 January 2022 ; Accepted: 28 July 2022.
∗ Corresponding author. Email: mrdoostdar@yahoo.com.

(2) The forced Duffing equation is

$$v''(t) + \mu v'(t) + f(t, v(t), v'(t)) = 0, \ 0 < t < 1, \ \mu \in \mathbb{R} - \{0\}, \tag{1.3}$$

with the integral boundary conditions

$$v(0) - \lambda_0 v'(0) = \int_0^1 k_1(x) v(x)\, \mathrm{d}x, \ v(1) + \lambda_1 v'(1) = \int_0^1 k_2(x) v(x)\, \mathrm{d}x, \tag{1.4}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$, and constants λ_0, λ_1 are non-negative.

The existence and uniqueness of the solution of these two versions of the Duffing equation have been studied by some researchers. Ahmad and Alghamdi [1] presented a generalized quasilinearization technique for studying the existence and uniqueness of the solution of the Duffing equation Version 1. Also, the authors in [2, 3] using the quasilinearization technique and in [12] using a constructive method have investigated these aspects for the Duffing equation Version 2. Furthermore, since the Duffing equation Version 1 is a Volterra integro-differential equation of second order, some other aspects of such equations can be studied in [40–42].

Although many researchers have analytically and numerically solved the Duffing equation with initial value conditions and two-point boundary conditions [8, 25, 27, 32, 33, 43, 44], there are few references for solving the above-mentioned versions of the Duffing equation. Yao in [45] presented an iteration algorithm to solve the Version 1 of the Duffing equation in the reproducing kernel space. Also, Geng [19] used an improved variational iteration method, Balaji [9] used Legendre wavelet operational matrix, and Saadatmandi and Mashhadi-Fini [34] used the Legendre pseudospectral method have solved the Version 1. For solving the Version 2, Geng and Cui [20] developed a method based on the homotopy perturbation method (HPM) and the reproducing kernel Hilbert space method (RKHSM).

In recent years, the use of hybrid functions consisting of the combination of polynomials such as Legendre, Chebyshev, Lagrange, Bernoulli, Bernstein, and Taylor or Fourier series with the Block-pulse functions (BPFs), has been considered by researchers and various numerical methods have been proposed for solving selected mathematical models [5, 10, 17, 22, 23, 30].

In this work, an attempt is made to solve the two above-mentioned Duffing equations using the hybrid Legendre Block-pulse functions. Hence, we organize the remainder of the paper as follows: In section 2, some definitions and properties of hybrid Legendre Block-pulse functions are reviewed. In section 3, we apply the hybrid Legendre Block-pulse functions method to construct approximate solutions for the selected Duffing equations. In section 4, the error and convergence analysis of the method is studied by obtaining an upper bound of the error. Furthermore, the applicability and effectiveness of the method are illustrated by means of some examples in section 5.

2. Preliminaries and definitions

In the current section, some definitions and properties of hybrid BPFs and Legendre polynomials are reviewed. Also, required operational matrices are presented.

2.1. Hybrid functions.

Definition 2.1. A P-set of BPFs $b_p(t)$ on the interval $[0, 1)$ is defined as [17]

$$b_p(t) = \begin{cases} 1, & \dfrac{p-1}{P} \le t < \dfrac{p}{P}, \\ 0, & \text{o.w}, \end{cases}$$

where $p = 1, 2, \ldots, P$ is the order of BPFs. The set $\{b_p(t)\}$ has orthogonality and disjointness properties on $[0, 1)$.

Definition 2.2. A PQ-set of hybrid Legendre Block-pulse functions $\varphi_{pq}(t)$ on the interval $[0, 1)$ is defined as [17]

$$\varphi_{pq}(t) = \begin{cases} L_q(2Pt - 2p + 1), & \dfrac{p-1}{P} \le t < \dfrac{p}{P}, \\ 0, & \text{o.w}, \end{cases}$$

where L_q is the well-known Legendre polynomial of order $q = 0, 1, 2, \ldots, Q-1$, which is obtained with the following formulas:

$$L_0(t) = 1, \quad L_1(t) = t,$$
$$(q+1)L_{q+1}(t) = (2q+1)tL_q(t) - qL_{q-1}(t), \quad t \in [-1, 1].$$

Note that since the BPFs and Legendre polynomials are both orthogonal and complete, then the set $\{\varphi_{pq}(t)\}$ forms an orthogonal complete system in $L^2[0,1)$. Using the basis functions $\{\varphi_{pq}(t)\}$, a function $g \in L^2[0,1)$ may be expanded as [22]

$$g(t) = \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} g_{pq}\,\varphi_{pq}(t).$$

Theorem 2.3. [26] *Let Θ be a strictly convex normed space and Υ be a finite dimensional subspace of Θ. For each $g \in \Theta$ there exists a unique best approximation $\bar{g} \in \Upsilon$.*

We let $\Theta = L^2[0,1)$ and consider the normed space Θ with the following norm:

$$||g||_2 = \langle g, g\rangle^{\frac{1}{2}} = \left(\int_0^1 |g(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}},$$

where $\langle .,.\rangle$ denotes the inner product. Since Θ is a Hilbert space, it is also strictly convex. Let

$$\Upsilon = \mathrm{span}\{\varphi_{10}(t), \ldots, \varphi_{1(Q-1)}(t), \varphi_{20}(t), \ldots, \varphi_{2(Q-1)}(t), \ldots, \varphi_{P0}(t), \ldots, \varphi_{P(Q-1)}(t)\}.$$

Since Υ is a finite dimensional subspace of Θ, using Theorem 2.3 we have

$$g(t) \simeq \bar{g}(t) = g_{PQ}(t) = \sum_{p=1}^{P}\sum_{q=0}^{Q-1} g_{pq}\,\varphi_{pq}(t) = \mathbf{G}^T\mathbf{\Phi}(t) = \mathbf{\Phi}^T(t)\mathbf{G}, \tag{2.1}$$

where

$$\mathbf{\Phi}(t) = [\varphi_{10}(t), \ldots, \varphi_{1(Q-1)}(t), \varphi_{20}(t), \ldots, \varphi_{2(Q-1)}(t), \ldots, \varphi_{P0}(t), \ldots, \varphi_{P(Q-1)}(t)]^T,$$

and

$$\mathbf{G} = [g_{10}, g_{11}, \ldots, g_{1(Q-1)}, g_{20}, g_{21}, \ldots, g_{2(Q-1)}, \ldots, g_{P0}, g_{P1}, \ldots, g_{P(Q-1)}]^T.$$

The hybrid coefficients g_{pq} are obtained by

$$g_{pq} = \frac{\langle g(t), \varphi_{pq}(t)\rangle}{\langle \varphi_{pq}(t), \varphi_{pq}(t)\rangle}, \qquad p = 1, 2, \ldots, P, \quad q = 0, 1, \ldots, Q-1. \tag{2.2}$$

2.2. **Operational matrices.** To convert the integral term of the Duffing equation to an algebraic equation, we consider the approximation of the integration of the vector $\mathbf{\Phi}(t)$ as [17]

$$\int_0^t \mathbf{\Phi}(\varsigma)\,\mathrm{d}\varsigma \simeq \Upsilon\,\mathbf{\Phi}(t), \tag{2.3}$$

where Υ, i.e. the operational matrix of integration is defined as

$$\Upsilon = \begin{bmatrix} \mathrm{M} & \mathrm{S} & \mathrm{S} & \ldots & \mathrm{S} \\ \mathbf{0} & \mathrm{M} & \mathrm{S} & \ldots & \mathrm{S} \\ \mathbf{0} & \mathbf{0} & \mathrm{M} & \ldots & \mathrm{S} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathrm{M} \end{bmatrix}_{\theta\times\theta},$$

where $\theta = PQ$, $\mathbf{0}$ is the $Q \times Q$ zero matrix, and S and M have the following forms:

$$\mathrm{S} = \frac{1}{P}\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{Q\times Q},$$

$$\mathrm{M} = \frac{1}{2P}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & \frac{1}{5} & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & 0 & \frac{1}{7} & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{9} & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2Q-9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{-1}{2Q-7} & 0 & \frac{1}{2Q-7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{-1}{2Q-5} & 0 & \frac{1}{2Q-5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{-1}{2Q-3} & 0 & \frac{1}{2Q-3} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \frac{-1}{2Q-1} & 0 \end{bmatrix}_{Q\times Q}.$$

The integration of the cross product of two hybrid function vectors $\mathbf{\Phi}(t)$ can be obtained as [22]

$$\mathbf{C} = \int_0^1 \mathbf{\Phi}(\varsigma)\mathbf{\Phi}^T(\varsigma)\,\mathrm{d}\varsigma = \begin{bmatrix} \mathrm{D} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathrm{D} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{D} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathrm{D} \end{bmatrix}_{\theta\times\theta}, \tag{2.4}$$

where D is a diagonal matrix as

$$\mathrm{D} = \frac{1}{P}\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{3} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{5} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2Q-1} \end{bmatrix}_{Q\times Q}.$$

The matrix $\mathbf{C}$ is used to convert the integral terms of the boundary conditions to the algebraic equations. It should be noted that since the matrix $\mathbf{C}$ is a diagonal matrix, in addition to saving memory storage, it also increases the calculating speed. Also, we need to evaluate $\mathbf{\Phi}(t)\mathbf{\Phi}^T(t)$. To do this, let

$$\mathbf{\Phi}(t)\mathbf{\Phi}^T(t)\mathbf{G} = \tilde{\mathbf{G}}\mathbf{\Phi}(t), \tag{2.5}$$

where matrix $\tilde{\mathbf{G}}$ is defined in [17].

3. Method implementation

In this section, we construct a numerical method to approximate the solutions of the Duffing equations (1.1)-(1.4) using the considered hybrid functions. For this purpose, we put

$$v''(t) \simeq \mathbf{V}_z^T\mathbf{\Phi}(t), \tag{3.1}$$

where $\mathbf{V}_z = [v_1, v_2, \dots, v_\theta]^T$ is the unknown vector.
By applying integral operator to both sides of (3.1) on $(0, t)$ and using Eq. (2.3), we will have

$$v'(t) \simeq \mathbf{V}_z^T\Upsilon\mathbf{\Phi}(t) + v'(0) = (\mathbf{V}_z^T\Upsilon + \mathbf{V}_{\alpha(0)}^T)\mathbf{\Phi}(t), \tag{3.2}$$

where

$$\mathbf{V}_{\alpha(0)} = \Big[\underbrace{v'(0)\ \overbrace{0\ \ 0\ \ \ldots\ \ 0}^{Q-1}\ \ v'(0)\ \ \overbrace{0\ \ 0\ \ \ldots\ \ 0}^{Q-1}\ \ \ldots\ \ v'(0)\ \ \overbrace{0\ \ 0\ \ \ldots\ \ 0}^{Q-1}}_{\theta=PQ}\Big]^T.$$

In the same way, by using equations (3.2) and (2.3), we get

$$v(t) \simeq \mathbf{V}_z^T \Upsilon^2 \mathbf{\Phi}(t) + v'(0)t + v(0) = (\mathbf{V}_z^T \Upsilon^2 + v'(0)\mathbf{E}^T + \mathbf{V}_{(0)}^T)\mathbf{\Phi}(t), \tag{3.3}$$

where $\mathbf{E} = [e_{pq}]_{\theta\times 1}$ and e_{pq}s are the hybrid coefficients for function $g(t) = t$, which can be calculated by Eq. (2.2). Also, the vector $\mathbf{V}_{(0)}$ is considered as

$$\mathbf{V}_{(0)} = \Big[\underbrace{v(0)\ \overbrace{0\ \ 0\ \ \ldots\ \ 0}^{Q-1}\ \ v(0)\ \ \overbrace{0\ \ 0\ \ \ldots\ \ 0}^{Q-1}\ \ \ldots\ \ v(0)\ \ \overbrace{0\ \ 0\ \ \ldots\ \ 0}^{Q-1}}_{\theta=PQ}\Big]^T.$$

Next, it is necessary to expand the non-linear terms in terms of the hybrid functions. For this purpose, we consider Eqs. (2.1) and (2.5) and obtain

$$u(t)v(t) \simeq (\mathbf{U}^T\mathbf{\Phi}(t))(\mathbf{V}^T\mathbf{\Phi}(t)) = \mathbf{U}^T\mathbf{\Phi}(t)\mathbf{\Phi}^T(t)\mathbf{V} = \mathbf{U}^T\tilde{\mathbf{V}}\mathbf{\Phi}(t),$$

and

$$v^3(t) = v^2(t)v(t) \simeq (\mathbf{V}^T\tilde{\mathbf{V}}\mathbf{\Phi}(t))(\mathbf{V}^T\mathbf{\Phi}(t)) = \mathbf{V}^T\tilde{\mathbf{V}}\mathbf{\Phi}(t)\mathbf{\Phi}^T(t)\mathbf{V} = \mathbf{V}^T\tilde{\mathbf{V}}^2\mathbf{\Phi}(t),$$

and in the same way, $v^a(t) \simeq \mathbf{V}^T\tilde{\mathbf{V}}^{a-1}\mathbf{\Phi}(t)$, where a is a positive integer number.

3.1. **Implementation for Version 1.** In this version of the Duffing equation, we consider the integral term as

$$\int_0^t h(t,x,v(x))\,\mathrm{d}x = h_1(t)\int_0^t h_2(x)h_3(v(x))\,\mathrm{d}x.$$

The functions h_2 and h_3 can be approximated in terms of the vector $\mathbf{\Phi}(t)$ as

$$h_2(x) \simeq \mathbf{H}_2^T\mathbf{\Phi}(t),\ \ h_3(v(x)) \simeq \mathbf{H}_3^T\mathbf{\Phi}(t). \tag{3.4}$$

By using Eqs. (2.3), (2.5) and (3.4), the last integral can be written as

$$\int_0^t h_2(x)h_3(v(x))\,\mathrm{d}x \simeq \mathbf{H}_2^T\tilde{\mathbf{H}}_3\Upsilon\mathbf{\Phi}(t), \tag{3.5}$$

where $\tilde{\mathbf{H}}_3$ can be calculated from Eq. (2.5). By substituting Eqs. (3.1), (3.2), (3.3) and (3.5) in Eq. (1.1), we get

$$\left(\mathbf{V}_z^T + \mu(\mathbf{V}_z^T\Upsilon + \mathbf{V}_{\alpha(0)}^T) + \mathbf{F}^T + h_1(t)\mathbf{H}_2^T\tilde{\mathbf{H}}_3\Upsilon\right)\mathbf{\Phi}(t) = 0. \tag{3.6}$$

Also, by substituting Eqs. (3.2) and (3.3) in Eq. (1.2), we have

$$r_0(\mathbf{V}_z^T\Upsilon^2 + v'(0)\mathbf{E}^T + \mathbf{V}_{(0)}^T)\mathbf{\Phi}(0) - s_0(\mathbf{V}_z^T\Upsilon + \mathbf{V}_{\alpha(0)}^T)\mathbf{\Phi}(0) - a = 0, \tag{3.7}$$

$$r_1(\mathbf{V}_z^T\Upsilon^2 + v'(0)\mathbf{E}^T + \mathbf{V}_{(0)}^T)\mathbf{\Phi}(1) - s_1(\mathbf{V}_z^T\Upsilon + \mathbf{V}_{\alpha(0)}^T)\mathbf{\Phi}(1) - b = 0, \tag{3.8}$$

Now, by collocating Eq. (3.6) at the points $t = t_j = \frac{2j-1}{2Q}$; $j = 1, 2, \ldots, \theta$, we obtain the following algebraic equations:

$$\left(\mathbf{V}_z^T + \mu(\mathbf{V}_z^T\Upsilon + \mathbf{V}_{\alpha(0)}^T) + \mathbf{F}^T + h_1(t_j)\mathbf{H}_2^T\tilde{\mathbf{H}}_3\Upsilon\right)\mathbf{\Phi}(t_j) = 0,\ \ j = 1, 2, \ldots, \theta. \tag{3.9}$$

Eqs. (3.7), (3.8) and (3.9) form a system of algebraic equations with unknowns $v_1, v_2, \ldots, v_\theta$, $v(0)$ and $v'(0)$, which can be solved by a proper numerical method. After obtaining the vector $\mathbf{V}_z = [v_1, v_2, \ldots, v_\theta]^T$, the approximate solution of $v(t)$ can be calculated by Eq. (3.3).

3.2. **Implementation for Version 2.** For this version, we substitute Eqs. (3.1), (3.2), (3.3) in Eq. (1.3) and obtain

$$\left(\mathbf{V}_z^T+\mu(\mathbf{V}_z^T\Upsilon+\mathbf{V}_{\alpha(0)}^T)+\mathbf{F}^T\right)\mathbf{\Phi}(t)=0. \tag{3.10}$$

If we collocate Eq. (3.10) at the points $t=t_j=\frac{2j-1}{2Q}$; $j=1,2,\dots,\theta$, the algebraic equations are obtained as

$$\left(\mathbf{V}_z^T+\mu(\mathbf{V}_z^T\Upsilon+\mathbf{V}_{\alpha(0)}^T)+\mathbf{F}^T\right)\mathbf{\Phi}(t_j)=0,\ j=1,2,\dots,\theta. \tag{3.11}$$

Also, by using Eqs. (2.1) and (2.4), the integral terms in Eq. (1.4) can be written as

$$\int_0^1 k_i(x)v(x)\,\mathrm{d}x\simeq\int_0^1\mathbf{K}_i^T\mathbf{\Phi}(x)\mathbf{\Phi}^T(x)\mathbf{V}\,\mathrm{d}x=\mathbf{K}_i^T\mathbf{C}\mathbf{V},\ i=1,2. \tag{3.12}$$

where $\mathbf{V}^T=\mathbf{V}_z^T\Upsilon^2+v'(0)\mathbf{E}^T+\mathbf{V}_{(0)}^T$.
By substituting Eqs. (3.2), (3.3) and (3.12) in Eq. (1.4), we get

$$(\mathbf{V}_z^T\Upsilon^2+v'(0)\mathbf{E}^T+\mathbf{V}_{(0)}^T)\mathbf{\Phi}(0)-\lambda_0(\mathbf{V}_z^T\Upsilon+\mathbf{V}_{\alpha(0)}^T)\mathbf{\Phi}(0)-\mathbf{K}_1^T\mathbf{C}\mathbf{V}=0, \tag{3.13}$$

$$(\mathbf{V}_z^T\Upsilon^2+v'(0)\mathbf{E}^T+\mathbf{V}_{(0)}^T)\mathbf{\Phi}(1)+\lambda_1(\mathbf{V}_z^T\Upsilon+\mathbf{V}_{\alpha(0)}^T)\mathbf{\Phi}(1)-\mathbf{K}_2^T\mathbf{C}\mathbf{V}=0. \tag{3.14}$$

Eqs. (3.11), (3.13) and (3.14) form a system of algebraic equations with unknowns $v_1,v_2,\dots,v_\theta$, $v(0)$ and $v'(0)$, which can be solved by a proper numerical method. After obtaining the vector $\mathbf{V}_z=[v_1,v_2,\dots,v_\theta]^T$, the approximate solution of $v(t)$ can be calculated by Eq. (3.3).

4. Error analysis

In current section, we obtain an upper bound of the error for the present method using a Hilbert space called the Sobolev space as well as the relevant norm.

Definition 4.1. [13] Let β be a non-negative integer and $(a,b)\subset\mathbb{R}^2$ be a bounded interval. The vector space of the functions $\omega\in L^2(a,b)$ with the following definition is called the Sobolev space:

$$H^\beta(a,b)=\left\{\omega|\,\omega^{(m)}\in L^2(a,b),\text{ for }0\le m\le\beta\right\}.$$

For the Sobolev space, the relevant norm is defined as

$$\|\omega\|_{H^\beta(a,b)}=\left(\sum_{m=0}^{\beta}\|\omega^{(i)}\|^2_{L^2(a,b)}\right)^{\frac{1}{2}}.$$

Some properties of the Sobolev spaces are considered as follows:

- $H^0(a,b)\equiv L^2(a,b)$,
- $\dots H^{\beta+1}(a,b)\subset H^\beta(a,b)\subset\dots\subset H^1(a,b)\subset L^2(a,b)$.

Lemma 4.2. [13] *Let $\{L_q\}_{q=0}$ be the sequence of Legendre polynomials and $\Lambda_d(t)=\sum_{q=0}^d l_qL_q(t)$ the best polynomial approximation of degree d for $\omega\in L^2(-1,1)$. Then, for $\gamma\ge1$, there exists a constant $\delta>0$ such that*

$$\|\omega-\Lambda_d\|_{L^\infty(-1,1)}\le\delta d^{\frac{3}{4}-\gamma}\|\omega\|_{H^\gamma(-1,1)},$$

for all functions ω in $H^\gamma(-1,1)$.

The following lemma is obviously obtained using Lemma 4.2.

Lemma 4.3. *Let $\omega\in H^\gamma[0,1)$ and let ω_θ be the polynomial approximation of ω defined in Eq. (2.1). Then,*

$$\|\omega-\omega_\theta\|_{L^\infty[0,1)}\le\delta(\theta)^{\frac{3}{4}-\gamma}\max_{1\le k\le P}\|\omega\|_{H^\gamma(I_k)},$$

where $I_k=[\frac{k-1}{P},\frac{k}{P})$ and δ is a positive constant.

Before explaining the following theorem, we convert equation (1.1) into a system of ordinary differential equations of order one. Hence, we define functions $w_1(t)$ and $w_2(t)$ as

$$w_1(t) = v(t),\ \ w_2(t) = v'(t),$$

and obtain the following system of ordinary differential equations:

$$\begin{cases} w_1'(t) = w_2(t) = f_1(t, w_1(t), w_2(t)), \\ w_2'(t) = -\mu w_2(t) - f(t, w_1(t), w_2(t)) - \displaystyle\int_0^t h(t, x, w_1(x))\,\mathrm{d}x = f_2(t, w_1(t), w_2(t)). \end{cases} \tag{4.1}$$

The same can be done for equation (1.3). We note that since the functions f and h are continuous, the functions f_1 and f_2 are continuous, too. Now, we consider system 4.1 in the matrix form as

$$W(t) = W(0) + \int_0^t F(x, W(x))\,\mathrm{d}x, \tag{4.2}$$

where

$$W(t) = [w_1(t), w_2(t)]^T,\ \ F(x, W(x)) = \begin{pmatrix} f_1(x, W(x)) \\ f_2(x, W(x)) \end{pmatrix}.$$

Theorem 4.4. *Let $W \in H^{\gamma}[0,1)$ and $W_{app}(t) = W_{\theta}(t)$ be the exact solution and the approximate solution obtained by the hybrid Legendre Block-pulse method for equation (4.2), respectively. Furthermore, assume that $f_i(x, W(x))$, $i = 1, 2$ be a continuous function for $0 \le x \le t < 1$ and that satisfies the Lipschitz condition*

$$|f_i(x, W_1(x)) - f_i(x, W_2(x))| \le L_i \left\| W_1(x) - W_2(x) \right\|_{\infty}, \tag{4.3}$$

where $L_i > 0$, $i = 1, 2$, is a Lipschitz constant. There exists a positive constant ϵ such that

$$\|W - W_{app}\|_{\infty} \le \epsilon \max_{1 \le k \le P} \|w_s\|_{H^{\gamma}(I_k)}.$$

Proof. Let $e_i(t) = w_i(t) - w_{i(app)}(t)$ be the error term, where $w_{i(app)}(t)$, $i = 1, 2$, is the approximate solution of the system (4.2). We can write

$$e_i(t) = w_i(t) - w_{i(app)}(t) = \int_0^t \Big(f_i(x, W(x)) - f_i(x, W_{app}(x)) \Big)\,\mathrm{d}x,\ \ i = 1, 2.$$

By using Eq. (4.3) and that $0 < t < 1$, we get

$$|e_i(t)| \le L_i \left\| W - W_{app} \right\|_{\infty} = L_i \max_{1 \le i \le 2} \|w_i - w_{i(app)}\|_{\infty}.$$

If $L = \max\{L_1, L_2\}$; then

$$\|e_i\|_{\infty} \le L \left\| w_s - w_{s(app)} \right\|_{\infty}, \qquad s \in \{1, 2\}. \tag{4.4}$$

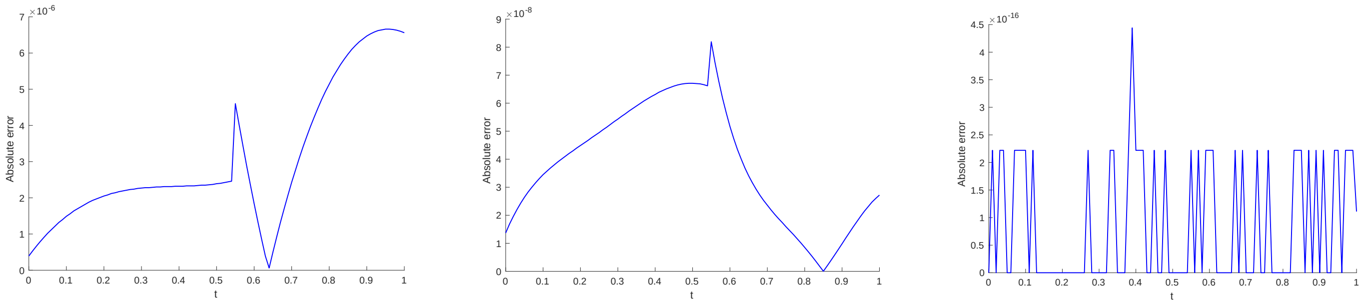
Now, by considering Eq. (4.4) and using Lemma 4.3, we have

$$\|e_i\|_{\infty} \le L \left\| w_s - w_{s(app)} \right\|_{\infty} = L \left\| w_s - w_{s(app)} \right\|_{L^{\infty}[0,1)} \le \epsilon \max_{1 \le k \le P} \|w_s\|_{H^{\gamma}(I_k)},$$

where $\epsilon = L\delta(\theta)^{\frac{3}{4} - \gamma}$. □

5. Numerical results

In this section, in order to evaluate the effectiveness of the present method, some examples of two versions of the Duffing equation are presented. The first two examples are problems of Version 1 and the other two examples are problems of Version 2. The implementation of the present method for each of these two versions are described in Section 3. We point out that in each of the references mentioned in the following examples, the proposed methods were used only to solve one of these versions while the present method is able to solve both. For all computations, a Matlab 2017a software package is used.



(A) $P = 2, Q = 4$ (B) $P = 2, Q = 6$ (C) $P = 2, Q = 8$

FIGURE 1. The absolute errors for some different values of P and Q in Example 1.

Example 1. The first example is a problem of Version 1 as [19, 35, 45]

$$\begin{cases} v''(t) + v'(t) + v(t)v'(t) + \displaystyle\int_0^t txv^2(x)\,\mathrm{d}x = g(t), \ 0 < t < 1, \\ v(0) - v'(0) = 0, \ v(1) - v'(1) = 0, \end{cases} \tag{5.1}$$

where

$$g(t) = \frac{1}{6}t^7 - \frac{2}{5}t^6 - \frac{1}{4}t^5 + \frac{2}{3}t^4 + \frac{5}{2}t^3 - 3t^2 - 3t.$$

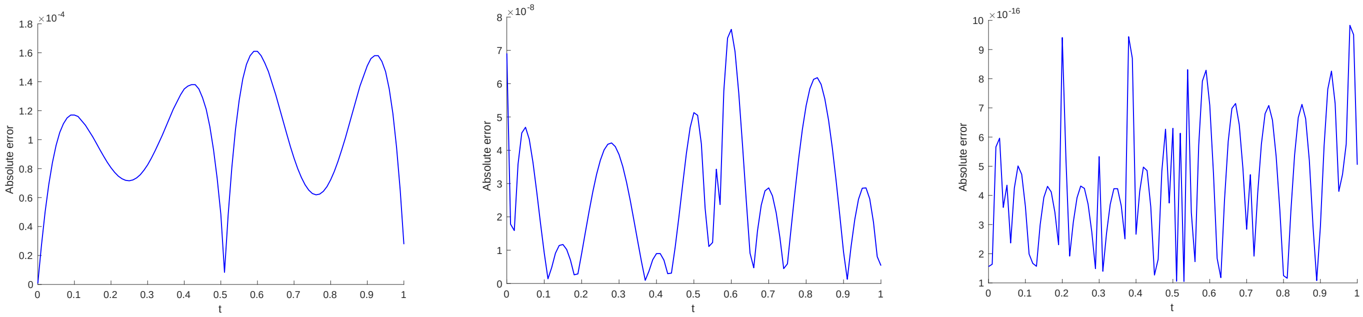
The exact solution of problem (5.1) is given by $v(t) = -t^2 + t + 1$. In [45], an iteration method in the reproducing kernel space proposed to solve this version of the Duffing equation. Also, Geng [19] using an improved variational iteration method and Saadatmandi and Yeganeh [35] using the sinc-collocation method solved this type of problems. Here, we applied the present method for solving equation (5.1) with $P = 2$ and $Q = 8$. In Table 1, the obtained results are reported and compared with the numerical results in [19, 35, 45]. Also, the absolute errors for some different values of P and Q are compared in Figure 1.

TABLE 1. The numerical results for Example 1.

t	Exact $v(t)$	Absolute error of $v(t)$			
		Method of [45]	Method of [19]	Method of [35]	Present method
0.1	1.09	9.47E-6	6.66E-16	8.97E-09	2.22E-16
0.2	1.16	9.78E-6	4.44E-16	2.03E-08	0
0.3	1.21	9.82E-6	4.44E-16	8.03E-08	0
0.4	1.24	9.67E-6	2.22E-16	1.17E-07	2.22E-16
0.5	1.25	9.37E-6	4.44E-16	1.77E-07	0
0.6	1.24	8.98E-6	4.44E-16	4.53E-08	0
0.7	1.21	8.51E-6	4.44E-16	2.01E-07	0
0.8	1.16	7.98E-6	4.44E-16	1.78E-08	0
0.9	1.09	7.40E-6	6.66E-16	3.05E-08	0
1.0	1.00	6.75E-6	4.44E-16	2.00E-08	1.11E-16

Example 2. Another problem of Version 1 is considered as [19, 45]

$$\begin{cases} v''(t) + v'(t) + v(t)(1 + v'(t)) + \displaystyle\int_0^t x^2 v(x)\,\mathrm{d}x = g(t), \ 0 < t < 1, \\ 2v(0) - v'(0) = 0, \ 3v(1) + v'(1) = 0, \end{cases} \tag{5.2}$$



(A) $P = 2, Q = 4$ (B) $P = 2, Q = 6$ (C) $P = 2, Q = 8$

FIGURE 2. The absolute errors for some different values of P and Q in Example 2.

where

$$g(t) = \frac{1}{24(1+e)^2}\Big(8e^2(3t^4 + 2t^3 + 60t + 42) + 726e^t - 132e^{t+1}(11 + t^2 + 2t) + e(15t^4 + 32t^3 - 228t + 540) - 66e^t(11 + 2t^2 - 7t) - 9t^4 + 16t^3 + 18t + 204\Big).$$

For this problem, the exact solution is given by $v(t) = 2 - \frac{t(3-8e)+11e^t}{2(1+e)}$. Similar to the previous example, we take $P = 2$, $Q = 8$ and solve problem (5.2) by the present method. The comparisons between the absolute errors obtained by the present method and the methods in [45] and [19] are shown in Table 2. In Figure 2, the absolute errors for some different values of P and Q are compared.

TABLE 2. The numerical results for Example 2.

t	Exact $v(t)$	Absolute error of $v(t)$		
		Method of [45]	Method of [19]	Present method
0.1	0.6173379116	3.37E-05	1.11E-16	3.62E-16
0.2	0.6974925703	3.35E-05	2.22E-16	9.41E-16
0.3	0.7595654498	3.06E-05	2.22E-16	5.32E-16
0.4	0.8016548726	2.59E-05	2.22E-16	2.67E-16
0.5	0.8216591603	2.05E-05	0	6.30E-16
0.6	0.8172555990	1.52E-05	4.44E-16	7.09E-16
0.7	0.7858771928	1.05E-05	2.22E-16	2.84E-16
0.8	0.7246869725	6.78E-06	0	1.25E-16
0.9	0.6305496023	4.17E-06	6.66E-16	2.93E-16
1.0	0.5	2.72E-06	7.77E-16	5.05E-16

Example 3. The following example is considered as a problem of Version 2 [20, 31]:

$$\begin{cases} v''(t) + v'(t) + t(1-t)v^3(t) = g(t), \ 0 < t < 1, \\ v(0) - \dfrac{2}{\pi^2}v'(0) = -\displaystyle\int_0^1 v(x)\,\mathrm{d}x, \ v(1) + \dfrac{1}{\pi^2}v'(1) = -\displaystyle\int_0^1 xv(x)\,\mathrm{d}x, \end{cases} \tag{5.3}$$

where

$$g(t) = -\sin(\pi t)(\pi^2 + t(t-1)\sin^2(\pi t)) + \pi\cos(\pi t).$$

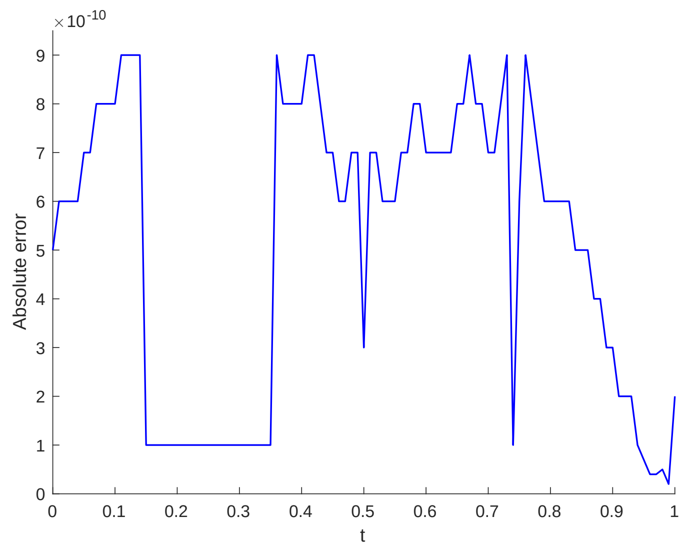


FIGURE 3. The absolute errors of $v(t)$ for $P = 4, Q = 8$ in Example 3.

The exact solution of the problem (5.3) is $v(t) = \sin(\pi t)$. Geng and Cui [20] have solved this version of the Duffing equation by combining the homotopy perturbation method and the reproducing kernel Hilbert space method. Here, we solved it using the proposed method, once for $P = 2, Q = 8$ and once again for $P = 4, Q = 8$. In Table 3, the obtained relative errors are compared. Also, in [31] by using Legendre multiwavelets method for solving this problem, the maximum relative error is obtained 6.64E-07. Figure 3 shows the absolute errors by the present method with $P = 4$ and $Q = 8$.

TABLE 3. The numerical results for Example 3.

t	Exact $v(t)$	Relative error of $v(t)$		
		Method of [20]	Present Method	
		n=100, m=5	$P = 2, Q = 8$	$P = 4, Q = 8$
0.01	0.0314107591	7.49E-05	1.91E-05	2.02E-08
0.08	0.2486898872	8.17E-05	3.15E-06	3.29E-09
0.16	0.4817536741	8.11E-05	2.24E-06	2.12E-10
0.32	0.8443279255	8.09E-05	1.80E-06	1.29E-10
0.48	0.9980267284	8.08E-05	1.90E-06	7.04E-10
0.64	0.9048270525	8.03E-05	1.50E-06	8.12E-10
0.80	0.5877852523	7.97E-05	1.33E-06	9.83E-10
0.96	0.1253332336	7.85E-05	2.45E-06	3.58E-10

Example 4. As the last example, we consider the following problem of Version 2 [20]:

$$\begin{cases} v''(t) - v'(t) - 2v(t) + \sin(v(t)) = g(t), \ 0 < t < 1, \\ v(0) - \dfrac{4}{3\pi^2}v'(0) = -\displaystyle\int_0^1 \cos(\frac{\pi x}{2})v(x)\,\mathrm{d}x, \ v(1) + \dfrac{6}{\pi^2}v'(1) = -\displaystyle\int_0^1 (2x+2)v(x)\,\mathrm{d}x, \end{cases} \tag{5.4}$$

where

$$g(t) = \sin(\sin(\pi t)) - (2 + \pi^2)\sin(\pi t) - \pi\cos(\pi t).$$

The exact solution of the problem (5.4) is $v(t) = \sin(\pi t)$. The problem is solved using the proposed method, once for $P = 3, Q = 6$ and once again for $P = 6, Q = 6$ and the obtained absolute errors are reported in Table 4. We point out that in [20], the maximum absolute error is 2.0E-4.

Table 4. The numerical results for Example 4.

t	Exact $v(t)$	Absolute error of $v(t)$	
		$P = 3, Q = 6$	$P = 6, Q = 6$
0.1	0.3090169944	2.69E-04	3.04E-05
0.2	0.5877852523	7.67E-04	1.12E-05
0.3	0.8090169944	1.18E-03	2.94E-05
0.4	0.9510565163	1.29E-03	3.82E-05
0.5	1	1.42E-03	4.83E-05
0.6	0.9510565163	1.42E-03	4.31E-05
0.7	0.8090169944	1.42E-03	3.75E-05
0.8	0.5877852523	1.09E-03	2.03E-05
0.9	0.3090169944	5.81E-04	1.05E-05
1.0	0	8.27E-05	1.20E-05

6. Conclusions

In the present paper, a hybrid functions method based on the combination of Legendre polynomials and Block-pulse functions were successfully implemented for solving two versions of the Duffing equation: one involving both integral and non-integral forcing terms with separated boundary conditions and the other involving non-integral forcing terms with integral boundary conditions. It is noteworthy that in each of the references mentioned in the examples, the proposed methods have been used only to solve one of the above versions while the present method is able to solve both. By using this method, these two second-order boundary value problems were reduced to the system of algebraic equations, which we solved them by the Newton's method. Also, an upper bound of the error for the proposed method was obtained. Furthermore, some numerical examples were simulated to show the applicability and effectiveness of the method. The obtained results of the proposed method were compared with the exact solutions and with the reported results in the literature. The reported results and comparisons confirm that the obtained numerical solutions by the present method are in desired agreement with the exact solutions and also, the proposed method gives better approximations than the previous methods.

References

[1] B. Ahmad and B. S. Alghamdi, *Approximation of solutions of the non-linear Duffing equation involving both integral and non-integral forcing terms with separated boundary conditions*, Comput. Phys. Comm., *179* (2008), 409–416.

[2] B. Ahmad and A. Alsaedi, *Existence of approximate solutions of the forced Duffing equation with discontinuous type integral boundary conditions*, Anal. Real World Appl., *10* (2009), 358–367.

[3] B. Ahmad, A. Alsaedi, and B. Alghamdi, *Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions*, Anal. Real World Appl., *9* (2008), 1727–1740.

[4] B. Ahmad, S. K. Ntouyas, and A. Alsaedi, *On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions*, Chaos Solitons and Fractals, *83* (2016), 234–241.

[5] A. M. Alenezi, *Hybrid Orthonormal Bernstein and Block-Pulse Method for the Solution of New System of Volterra Integro-differential Equations*, Journal of Mathematics Research, *12*(5) (2020), 1–20.

[6] M. Alshammari, M. Al-Smadi, O. A. Arqub, I. Hashim, and M. A. Alias, *Residual Series representation algorithm for solving fuzzy duffing oscillator equations*, Symmetry, *12*(4) (2020), 572.

[7] R. A. Attia, D. Lu, and M. MA Khater, *Chaos and relativistic energy-momentum of the nonlinear time fractional Duffing equation*, Mathematical and Computational Applications, *24*(1) (2019), 10.

[8] Z. Azimzadeh, A. R. Vahidi, and E. Babolian, *Exact solutions for non-linear Duffing's equations by He's homotopy perturbation method*, Indian J. Phys., *86* (2012), 721–726.

[9] S. Balaji, *A new approach for solving Duffing equations involving both integral and non-integral forcing terms*, Ain Shams Engineering Journal, *5*(3) (2014), 985–990.

[10] S. L. Bichi and S. U. Faruk, *Numerical Solution of System of Linear Volterra Integral Equations via Hybrid of Taylor and Block-pulse Functions*, Bayero Journal of Pure and Applied Sciences, *12*(1) (2019), 462–469.

[11] B. Bilgehan and A. Özyapıcı, *Direct solution of nonlinear differential equations derived from real circuit applications*, Analog Integrated Circuits and Signal Processing, *101*(3) (2019), 441–448.

[12] A. Boucherif, *Second order boundary value problems with integral boundary condition*, Nonlinear Anal. Theor., *70* (2009), 364–371.

[13] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral methods in fluid dynamics*, Springer Verlag, Berlin Heidelberg, 1988.

[14] M. Chai and L. Ba, *Application of EEG Signal Recognition Method Based on Duffing Equation in Psychological Stress Analysis*, Advances in Mathematical Physics, *2021* (2021).

[15] Z. Cheng and Q. Yuan, *Damped superlinear Duffing equation with strong singularity of repulsive type*, Journal of Fixed Point Theory and Applications, *22*(2) (2020), 1–18.

[16] A. A. Cherevko, E. E. Bord, A. K. Khe, V. A. Panarin, and K. J. Orlov, *The analysis of solutions behaviour of Van der Pol Duffing equation describing local brain hemodynamics*, Journal of Physics: Conference Series, *894* (1) (2017), 012012.

[17] M. R. Doostdar, A. R. Vahidi, T. Damercheli, and E. Babolian, *A hybrid functions method for solving linear and non-linear systems of ordinary differential equations*, Math. Commun., *26* (2021), 197–213.

[18] J. Du and M. Cui, *Solving the forced Duffing equation with integral boundary conditions in the reproducing kernel space*, Int. J. Comput. Math., *87* (2010), 2088–2100.

[19] F. Geng, *Numerical solutions of Duffing equations involving both integral and non-integral forcing terms*, Comput. Math. Appl., *61* (2011), 1935–1938.

[20] F. Geng and M. Cui, *New method based on the HPM and RKHSM for solving forced Duffing equations with integral boundary conditions*, J. Comput. Appl. Math., *233* (2009), 165–172.

[21] J. H. He, *A short review on analytical methods for a fully fourth-order nonlinear integral boundary value problem with fractal derivatives*, International Journal of Numerical Methods for Heat and Fluid Flow, *30* (2020), 4933–4943.

[22] E. Hesameddini and M. Riahi, *Hybrid Legendre Block-Pulse functions method for solving partial differential equations with non-local integral boundary conditions*, Journal of Information and Optimization Sciences, *40*(7) (2019), 1–13.

[23] R. Jafari, R. Ezzati, and K. Maleknejad, *A new operational matrix of derivative for hybrid third kind Chebyshev polynomials and Block-pulse functions and its applications in solving second-order differential equations*, Tbilisi Mathematical Journal, *14* (2021), 163–179.

[24] T. Jankowski, *Positive solutions for second order impulsive differential equations involving Stieltjes integral conditions*, Nonlinear Anal., *74*(11) (2011), 3775–3785 .

[25] D. Jiang, J. J. Nieto, and W. Zuo, *On monotone method for first order and second periodic boundary value problems and periodic solutions of functional differential equation*, J. Math. Anal. Appl., *289* (2004), 691–699.

[26] E . Kreyszig, *Introductory functional analysis with applications*, Wiley, New York, 1989.

[27] C. W. Lim and B. S. Wu, *A new analytical approach to the Duffing-harmonic oscillator*, Phys. Lett. A, *311*(4–5) (2003), 365–373.

[28] L. Liu, B. Liu, and Y. Wu, *Nontrivial solutions of m-point boundary value problems for singular second-order differential equations with a sign-changing nonlinear term*, J. Comput. Appl. Math., *224*(1) (2009), 373–382 .

[29] R. Ma and H. Wang, *Positive solutions of nonlinear three-point boundary-value problems*, Electron. J. Differ. Equ., *279*(1) (2003), 216–227.

[30] N. Mollahasani, M. M. Moghadam, and G. Chuev, *Hybrid Functions of Lagrange Polynomials and Block-Pulse Functions for Solving Integro-partial Differential Equations*, Iran. J. Sci. Technol. Trans. Sci., (2018), 1–9.

[31] R. Najafi and B. Nemati Saray, *Numerical solution of the forced Duffing equations using Legendre multiwavelets*, Comput. Methods Differ. Equ., *5* (2017), 43–55.

[32] J. J. Nieto and R. Rodriguez-Lopez, *Monotone method for first-order functional differential equations*, Comput. Appl. Math., *52* (2006), 471–480.

[33] R. B. Potts, *Best difference equation approximation to Duffing's equation*, J. Aust. Math. Soc. Ser. B, *23*(4) (1981/82), 349–356.

[34] A. Saadatmandi and F. Mashhadi-Fini, *A pseudospectral method for nonlinear Duffing equation involving both integral and non-integral forcing terms*, Mathematical Methods in the Applied Sciences, *38*(7) (2015), 1265–1272.

[35] A. Saadatmandi and S. Yeganeh, *New approach for the duffing equation involving both integral and non-integral forcing terms*, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, *79* (2017), 43–52.

[36] A. H. Salas, S. A. El-Tantawy, and N. H. Aljahdaly, *An exact solution to the quadratic damping strong nonlinearity Duffing oscillator*, Mathematical Problems in Engineering, (2021), Article ID 88755892021.

[37] L. J. Sheu, H. K. Chen, J. H. Chen, and L. M. Tam, *Chaotic Dynamics of the Fractionally Damped Duffing Equation*, Chaos Solitons and Fractals, *32* (2007), 1459–1468.

[38] J. J. Stokes, *Nonlinear Vibrations*, Intersciences, New York, 1950.

[39] M. Sun and Y. Xing, *Existence results for a kind of fourth-order impulsive integral boundary value problems*, Boundary Value Problems, *2016*(1) (2016), 1–15.

[40] C. Tunç and O. Tunç, *New qualitative criteria for solutions of Volterra integro-differential equations*, Arab Journal of Basic and Applied Sciences, *25*(3) (2018), 158–165.

[41] C. Tunç and O. Tunç, *On the stability, integrability and boundedness analyses of systems of integro-differential equations with time-delay retardation*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, *115*(3) (2021), 1–17.

[42] C. Tunç and O. Tunç, *New results on the qualitative analysis of integro-differential equations with constant time-delay*, J. Nonlinear Convex Anal., *23*(3) (2022), 435–448.

[43] A. R. Vahidi, E. Babolian, and Z. Azimzadeh, *An improvement to the homotopy perturbation method for solving nonlinear Duffing's equations*, Bull. Malaysian Math. Sci. Soc., *41* (2017), 1105–1117.

[44] A. S. Vatsala and J. Yang, *Monotone iterative technique for semi linear elliptic systems*, Boundary Value Problems, *2* (2005), 93–106.

[45] H. M. Yao, *Solution of the Duffing equation involving both integral and non-integral forcing terms*, Comput. Phys. Commun., *180* (2009), 1481–1488.