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Solution of time-fractional equations via Sumudu-Adomian decomposition method

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Abstract

This paper investigates the semi-analytical solutions of linear and non-linear Time Fractional Klein-Gordon equations with appropriate initial conditions to apply the New Sumudu-Adomian Decomposition method (NSADM). This paper shows the semi-analytical as well as a graphical interpretation of the solution by using mathematical software "Mathematica Wolform"and considering Caputo's sense derivatives to semi-analytical results obtained by NSADM.

Keywords. Klein-Gordon equation of fractional order, Adomian Decomposition method, Sumudu Transform, Caputo-derivative.
2010 Mathematics Subject Classification. 26A33, 35R11, 33E12.

1. Introduction

In recent years, Integral transforms and fractional calculus have been a more popular tool in solving the problems of applied science, electromagnetism, mathematical physics and engineering, biology, viscoelasticity and signal processing, mathematical biology, fractional dynamics, and etc [6, 17–19, 32]. Asiru studied the properties of the sumudu transform [4–6, 29] which is used to solve integral equations of convolution type.

There are plenty of analytical techniques for evaluating the differential, partial differential, fractional partial differential equations, and systems of functional equations such as The variational iteration method (VIM), homotopy perturbation transform method (HPTM) [1, 3, 20, 24], homotopy perturbation method(HAM) [5, 14, 15]. Iterative Laplace transforms, Rahmat darzi et al applied sumudu transform for solving a fractional diffusion-wave equation and fractional differential equations. kilbas and shrivasta [10–12, 19, 22, 26] derived the formulae for sumudu transform of R-L, Caputo, and miller-ross sequential fractional derivatives by using Laplace-sumudu duality. Modified variational iterative method (MVIP)[8, 25] for solving klein-Gordon equations.

In 2001, a novel approach is introduced by khuri, evaluate the problems of solution of nonlinear differential equations by Laplace Adomian Decomposition Method(LADM) [21, 23, 31], and LADM process has been used to find Volterra differntial equations[16, 27], the newton-homotopy method for solving nonlinear equations [3]. Many problems in fractional derivatives [25], hydrodynamics [2], chemical diffusion [34], option pricing [13], computational fluid dynamics [28], and control theory [33] can be modeled using partial differential equations (PDEs). Now a day, a lot of attention has been devoted to the study of nonlinear PDEs and methods for numerical solutions of nonlinear problems. Our goal in this paper is to approximate the solution of linear and nonlinear time fractional Klein-Gordon equations with appropriate initial conditions to apply the New Sumudu-Adomian Decomposition method (NSADM).

2. Preliminaries:

In this section, some fractional calculus definitions, notation, and basic results are needed for the rest of the work will be introduced.

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Definition 2.1. Consider the set[30]

$$Z = \left\{ K(\wp)/\exists M, Q_1, Q_2 > 0, \text{ such that } |K(\wp)| \leq Me^{\frac{|q|}{Q_j}}, if\ \wp \in (-1)^j \times [0, \infty) \right\},$$

for all real $\wp \geq 0$, the Sumudu transform of $K(\wp) \in A$ denoted by $S[K(\wp)] = K(u)$ is defined as

$$S[K(\wp)] = K(u) = \int_0^\infty e^{\wp} k(u - \wp) d\wp, u \in (-Q_1, Q_2). \tag{2.1}$$

The function $K(\wp)$ in equation (2.1) is called the inverse Sumudu transform of $K(u)$ and is denoted by $K(\wp) = S^{-1}[K(u)]$.

Theorem 2.2. *[19] Let $F(u)$ is the Sumudu Transform of the function $f(\wp)$ then the Sumudu transform of Riemann Liouville fractional integral of $f(\wp)$ of order λ, $\left(I_{a+}^{\lambda} f\right)(\wp)$ is given by*

$$S\left[\left({}_aD_\Omega^{-\lambda} f\right)(\Omega)\right](u) = S\left[\left(I_{a+}^{\lambda} f\right)(\Omega)\right](u) = u^\lambda F(u) \quad, Re(\lambda) > 0. \tag{2.2}$$

Definition 2.3. [25] The Caputo Fractional derivative of function $f(\wp)$ of order λ denoted by ${}_a^cD_\wp^\lambda f(\wp)$ is defined as

$${}_a^cD_\wp^\lambda f(\wp) = \frac{1}{\Gamma(r-\lambda)} \int_a^t \frac{f^n(\wp)}{(\wp - T)^{\lambda+1-r}} dT, \tag{2.3}$$

where $f \in \mathbb{C}_{-1}^r, r-1 < \lambda \leq r, r \in \mathbb{N}$.

Theorem 2.4. *[4] Let $A(\wp)$ and $B(\wp)$ be the Sumudu transform of $a(\wp)$and $b(\wp)$ respectively. If*

$$h(\wp) = (a * b)(\wp) = \int_0^t a(\wp) b(\wp - T) dT, \tag{2.4}$$

where $*$ denotes the convolution of a and b then the $S[h(\wp)] = \wp A(\wp)B(\wp)$ is Sumudu transform of $h(\wp)$.

Theorem 2.5. *[19] Let $r \geq 1$ and $F(u)$ be the Sumudu Transform of the function $f(\wp)$ and the Sumudu transform of r^{th} derivative of $f(\wp)$ defined by*

$$S[f^r(\wp)](u) = F_r(u) = \frac{F(u)}{u^r} - \sum_{q=0}^{r-1} \frac{f^q(0)}{u^{r-q}}. \tag{2.5}$$

Lemma 2.6. *[5] Let $\lambda > 0, \beta > 0, \lambda \in \mathbb{R}$ and $u^{-\lambda} > |\lambda|$ then*

$$S[\wp^{\beta-1} E_{\lambda,\beta}(\lambda \wp^\lambda)](u) = \left[\frac{u^{\beta-1}}{1 - \lambda u^\lambda}\right], \tag{2.6}$$

where $E_{\lambda,\beta}$ is Mittag-Leffler function in two parameters.

3. Process of The New Sumudu-Adomian Decomposition Method (NSADM)

We studied the linear time-fractional Klein-Gordon equation in this article of the following type

$$D_\wp^\lambda \Lambda(\Omega, \wp) - \Lambda_{\Omega\Omega}(\Omega, \wp) + p\Lambda(\Omega, \wp) = f(\Omega, \wp), \quad 1 < \lambda \leq 2, \tag{3.1}$$

with initial conditions

$$\Lambda(\Omega, 0) = j_1(\Omega) \text{ and } \Lambda_\wp(\Omega, 0) = j_2(\Omega). \tag{3.2}$$

Time-fractional klien-Gordon equation of the non-linear form,

$$D_\wp^\lambda \Lambda(\Omega, \wp) - \Lambda_{\Omega\Omega}(\Omega, \wp)) + p\Lambda(\Omega, \wp)) + qg(\Lambda(\Omega, \wp)) = f(\Omega, \wp), \quad 1 < \lambda \leq 2, \tag{3.3}$$

with initial conditions

$$\Lambda(\Omega, 0) = j_1(\Omega) \text{ and } \Lambda_\wp(\Omega, 0) = j_2(\Omega), \tag{3.4}$$

f is a known analytic function and $j(u)$ is a non-linear function,where p and q are real. In the Caputo sense, fractional derivatives are examined.

We assume and write a general fractional partial differentail equation in an operator form as below for illustrating the process of SADM [18]

$$D_{\wp}^{\lambda}\Lambda(\Omega,\wp)+R(\Lambda(\Omega,\wp))+N(\Lambda(\Omega,\wp))=g(\Omega,\wp), \quad p-1<\lambda\leq p, p\in\mathbb{N}, \tag{3.5}$$

$$\Lambda^{(k)}(\Omega,0)=h_k(\Omega), \quad k=0,1,2,\dots,p-1, \tag{3.6}$$

where $D_{\wp}^{\lambda}\Lambda(\Omega,\wp)$ is the caputo fractional derivative of the order λ, defined by the equation (3.1), where $p-1<\lambda\leq p$, R and N are a linear and non-linear operators respectively which might include the other fractional derivatives of order less than λ, and $g(\Omega,\wp)$ is known as an analytic function.
Applying the Sumudu transform to the equation (3.5), we have

$$S\left[D_{\wp}^{\lambda}\Lambda(\Omega,\wp)\right]+S\left[R(\Lambda(\Omega,\wp))+N(\Lambda(\Omega,\wp))\right]=S\left[g(\Omega,\wp)\right], \tag{3.7}$$

using the equation (2.6), we get

$$S\left[\Lambda(\Omega,\wp)\right]=\frac{1}{u^{-\lambda}}\sum_{k=0}^{r-1}u^{k-\lambda}\left[\Lambda^{k}(\Omega,0)\right]-\frac{1}{u^{-\lambda}}S\left[R(\Lambda(\Omega,\wp))+N(\Lambda(\Omega,\wp))\right]+\frac{1}{u^{-\lambda}}S\left[g(\Omega,\wp)\right]. \tag{3.8}$$

Apply inverse Sumudu transform to the equation (3.8), we get

$$\Lambda(\Omega,\wp)=\begin{bmatrix} S^{-1}\left[\frac{1}{u^{-\lambda}}\sum_{k=0}^{r-1}u^{k-\lambda}\left[\Lambda^{k}(\Omega,0)\right]+\frac{1}{u^{-\lambda}}S\left[g(\Omega,\wp)\right]\right] \\ -S^{-1}\left[\frac{1}{u^{-\lambda}}S\left[R(\Lambda(\Omega,\wp))+N(\Lambda(\Omega,\wp))\right]\right]\end{bmatrix}. \tag{3.9}$$

The ADM solution $\Lambda(\Omega,\wp)$ is represented by the following infinite series

$$\Lambda(\Omega,\wp)=\sum_{r=0}^{\infty}\Lambda_r(\Omega,\wp), \tag{3.10}$$

and the non linear term deccompoed as follows

$$N(\Lambda(\Omega,\wp))=\sum_{r=0}^{\infty}A_r, \tag{3.11}$$

where A_r are the Adomian polynomials given by

$$A_r=\frac{1}{r!}\left[\frac{d^r}{d\lambda^r}N\left(\sum_{i=0}^{r}\lambda^i\Lambda_i\right)\right]_{\lambda=0}, r=1,2,3,\dots, \tag{3.12}$$

substituting equation (3.10) and equation (3.11) into equation (3.9) we get,

$$\sum_{r=0}^{\infty}\Lambda_r(\Omega,\wp)=S^{-1}\left[\frac{1}{u^{-\lambda}}\sum_{k=0}^{r-1}u^{k-\lambda}\left[\Lambda^{k}(\Omega,0)\right]+\frac{1}{u^{-\lambda}}S\left[g(\Omega,\wp)\right]\right]-S^{-1}\left[\frac{1}{u^{-\lambda}}S\left[R(\sum_{r=0}^{\infty}\Lambda_r(\Omega,\wp))+\sum_{n=0}^{\infty}A_r\right]\right]. \tag{3.13}$$

Apply the technique of the Adomian we find the recursive equation in the following form,

$$\begin{aligned}\Lambda_0(\Omega,\wp)&=S^{-1}\left[\frac{1}{u^{-\lambda}}\sum_{k=0}^{r-1}u^{k-\lambda}\left[\Lambda^{k}(\Omega,0)\right]+\frac{1}{u^{-\lambda}}S\left[g(\Omega,\wp)\right]\right],\\ \Lambda_r(\Omega,\wp)&=-S^{-1}\left[\frac{1}{u^{-\lambda}}S\left[R(\sum_{r=0}^{\infty}\Lambda_r(\Omega,\wp))+\sum_{r=0}^{\infty}A_r\right]\right].\end{aligned} \tag{3.14}$$

The solution of the given series is convergent rapidly and The classical approach to converge to this series has been represented by Cherruault and Adomian [9]

4. Implication of Method

Example 4.1. Suppose the linear time fractional Klien-Gordon equation [7]

$$D_{\wp}^{\lambda}\Lambda(\Omega,\wp)-\Lambda_{\Omega\Omega}+\Lambda=0,\qquad 1<\lambda\leq 2, \tag{4.1}$$

with initial conditions

$$\Lambda(\Omega,0)=0 \text{ and } \Lambda_{\wp}(\Omega,0)=\Omega^2. \tag{4.2}$$

Taking the Sumudu the transform equation (4.1), we have

$$S\left[D_{\wp}^{\lambda}\Lambda(\Omega,\wp)\right]-S\left[\Lambda_{\Omega\Omega}(\Omega,\wp)+\Lambda(\Omega,\wp)\right]=0,\qquad 1<\lambda\leq 2, \tag{4.3}$$

using the fractional derivatives properties of the Sumudu transform, we get

$$S\left[\Lambda(\Omega,\wp)\right]=\frac{1}{u^{-\lambda}}\sum_{k=0}^{2-1}u^{k-\lambda}\left[\Lambda^{k}(\Omega,0)\right]-\frac{1}{u^{-\lambda}}S\left[\frac{\partial^2\Lambda(\Omega,\wp)}{\partial\Omega^2}+\Lambda(\Omega,\wp)\right]. \tag{4.4}$$

Applying insverse Sumudu transform to the equation (4.4), we get

$$\Lambda(\Omega,\wp)=\sum_{k=0}^{2-1}\left(\frac{\partial^k\Lambda(\Omega,\wp)}{\partial\wp^k}\right)u^k-S^{-1}\left[\frac{1}{u^{-\lambda}}S\left[\frac{\partial^2\Lambda(\Omega,\wp)}{\partial\Omega^2}+\Lambda(\Omega,\wp)\right]\right], \tag{4.5}$$

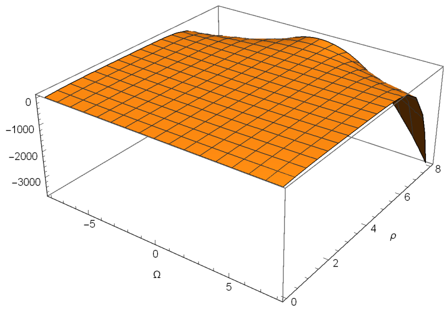
putting the results from equation (3.10) and equation (3.11) in equation (4.5) and applying equation (3.14) we determined the components of the SADM solution as follows

$$\begin{aligned}
\Lambda_0(\Omega,\wp)&=\sum_{k=0}^{2-1}\left(\frac{\partial^k\Lambda(\Omega,\wp)}{\partial\wp^k}\right)u^k\\
&=\Omega^2 t, && (4.6)\\
\Lambda_1(\Omega,\wp)&=-S^{-1}\left[\frac{1}{u^{-\lambda}}S\left[\frac{\partial^2\Lambda_0(\Omega,\wp)}{\partial\Omega^2}+\Lambda_0(\Omega,\wp)\right]\right]\\
&=-\frac{\Omega^2\wp^{\lambda+1}}{\Gamma(\lambda+2)}, && (4.7)\\
\Lambda_2(\Omega,\wp)&=-S^{-1}\left[\frac{1}{u^{-\lambda}}S\left[\frac{\partial^2\Lambda_1(\Omega,\wp)}{\partial\Omega^2}+\Lambda_1(\Omega,\wp)\right]\right]\\
&=(-1)^2\frac{\Omega^2\wp^{2\lambda+1}}{\Gamma(2\lambda+2)}, && (4.8)\\
\Lambda_3(\Omega,\wp)&=-S^{-1}\left[\frac{1}{u^{-\lambda}}S\left[\frac{\partial^2\Lambda_2(\Omega,\wp)}{\partial\Omega^2}+\Lambda_2(\Omega,\wp)\right]\right]\\
&=(-1)^3\frac{\Omega^2\wp^{3\lambda+1}}{\Gamma(3\lambda+2)}, && (4.9)\\
&\vdots
\end{aligned}$$

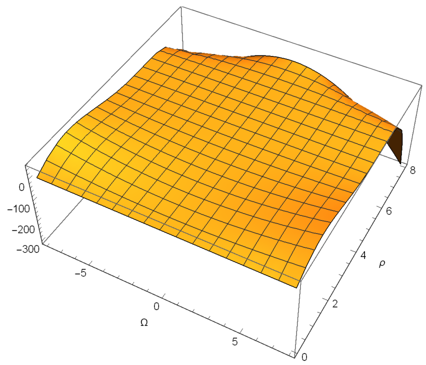
and so on. Thus, the series solution form is given by

$$\begin{aligned}
\Lambda(\Omega,\wp)&=\Lambda_0(\Omega,\wp)+\Lambda_1(\Omega,\wp)+\Lambda_2(\Omega,\wp)+\dots\\
&=\Omega^2\left[\wp-\frac{\wp^{\lambda+1}}{\Gamma(\lambda+2)}+\frac{\wp^{2\lambda+1}}{\Gamma(2\lambda+2)}-\frac{\wp^{3\lambda+1}}{\Gamma(3\lambda+2)}+\dots\right], && (4.10)
\end{aligned}$$

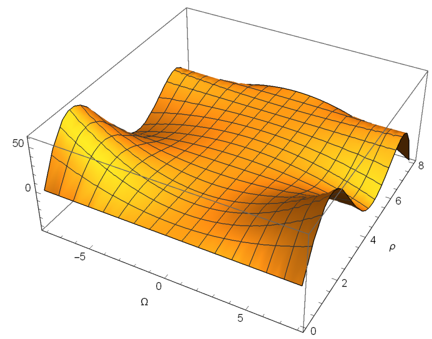
if we put $\lambda=2$, in equation (4.10) we have, $\Lambda(\Omega,\wp)=\Omega^2\sin\wp$ which is solution.



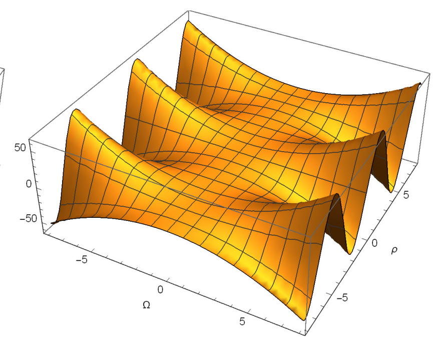
(a) $\lambda = 1.2$



(b) $\lambda = 1.6$



(c) $\lambda = 1.8$



(d) $\lambda = 2$

FIGURE 1. Estimated solution of equation (4.1) at 10^{th} order for different values of λ.

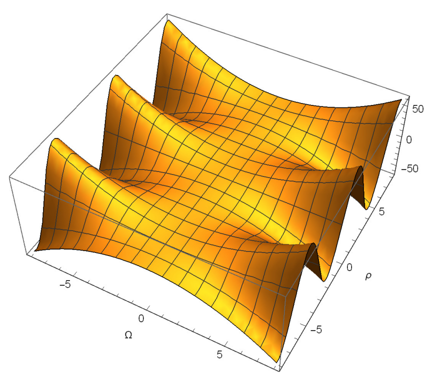


FIGURE 2. Exact solution of equation (4.1) at 10^{th} order for $\lambda = 2$.

Remark 4.2. The linear homogeneous time fractional Klien-Gordon equation shown above. The estimated solutions of the linear homogeneous time fractional Klien-Gordon equation at different values for λ=1.2, 1.6, 1.8, 2, and the accurate solution for λ=2 are shown in Figures 1 and 2 respectively. The solution is so simple to discover that it is constantly dependent on the values of time-fractional derivatives.

Example 4.3. Suppose the linear nonhomogeneous time fractional Klien-Gordon equation [7]

$$D_{\wp}^{\lambda}\Lambda(\Omega,\wp) - \Lambda_{\Omega\Omega} + \Lambda = 2\sin\Omega, \qquad 1 < \lambda \le 2, \tag{4.11}$$

with initial conditions

$$\Lambda(\Omega,0) = \sin\Omega \text{ and } \Lambda_{\wp}(\Omega,0) = 1. \tag{4.12}$$

Apply the Sumudu transform equation (4.11), we have

$$S\left[D_{\wp}^{\lambda}\Lambda(\Omega,\wp)\right] - S\left[\Lambda_{\Omega\Omega}(\Omega,\wp) - \Lambda(\Omega,\wp) + 2\sin\Omega\right] = 0, \qquad 1 < \lambda \le 2, \tag{4.13}$$

using the fractional derivatives properties of the Sumudu transform, we get

$$S\left[\Lambda(\Omega,\wp)\right] = \frac{1}{u^{-\lambda}}\sum_{k=0}^{2-1} u^{k-\lambda}\left[\Lambda^{k}(\Omega,0)\right] + \frac{1}{u^{-\lambda}}S\left[\frac{\partial^2\Lambda(\Omega,\wp)}{\partial\Omega^2} - \Lambda(\Omega,\wp) + 2\sin\Omega\right]. \tag{4.14}$$

Applying insverse Sumudu transform to the equation (4.14) we get,

$$\Lambda(\Omega,\wp) = \sum_{k=0}^{2-1}\left(\frac{\partial^k\Lambda(\Omega,\wp)}{\partial\wp^k}\right)u^k + S^{-1}\left[\frac{1}{u^{-\lambda}}S\left[\frac{\partial^2\Lambda(\Omega,\wp)}{\partial\Omega^2} - \Lambda(\Omega,\wp) + 2\sin\Omega\right]\right], \tag{4.15}$$

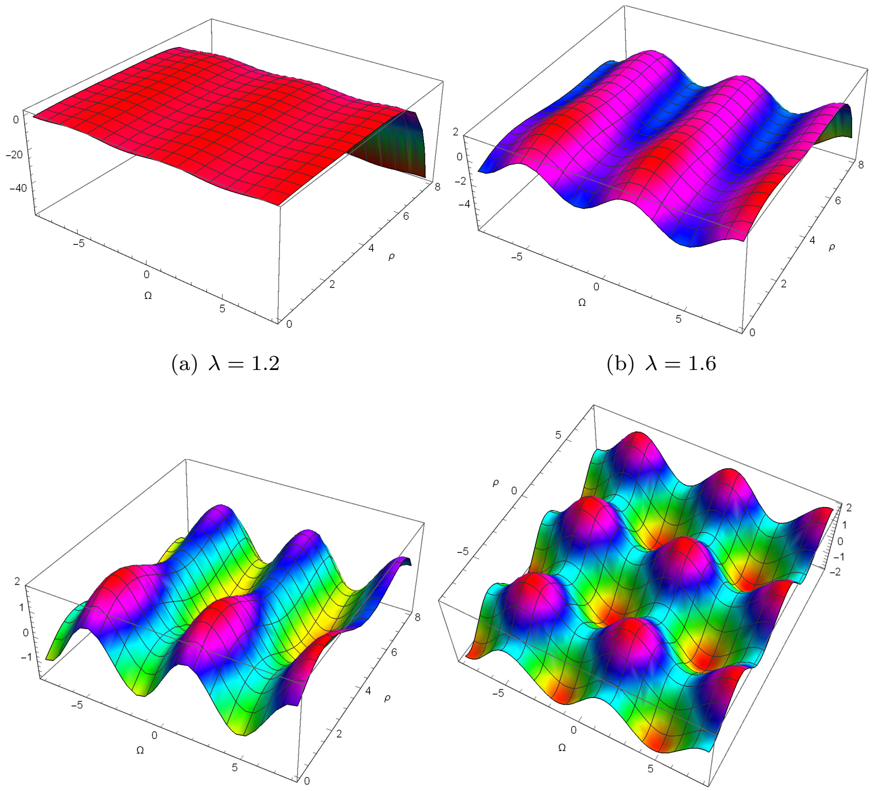
putting the results from equation (3.10) and equation (3.11) in equation (4.15) and applying equation (3.14) we determined the components of the SADM solution as follows

$$\begin{aligned}
\Lambda_0(\Omega,\wp) &= \sum_{k=0}^{2-1}\left(\frac{\partial^k\Lambda(\Omega,\wp)}{\partial\wp^k}\right)u^k + S^{-1}\left[u^{\lambda}S[2\sin\Omega]\right] \\
&= \sin\Omega + \wp + 2\sin\Omega\frac{\wp^{\lambda}}{\Gamma(\lambda+1)}, && (4.16)\\
\Lambda_1(\Omega,\wp) &= S^{-1}\left[\frac{1}{u^{-\lambda}}S\left[\frac{\partial^2\Lambda_0(\Omega,\wp)}{\partial\Omega^2} - \Lambda_0(\Omega,\wp)\right]\right] \\
&= -2\sin\Omega\frac{\wp^{\lambda}}{\Gamma(\lambda+1)} - \frac{\wp^{\lambda+1}}{\Gamma(\lambda+2)} - 4\sin\Omega\frac{\wp^{2\lambda}}{\Gamma(2\lambda+1)}, && (4.17)\\
\Lambda_2(\Omega,\wp) &= S^{-1}\left[\frac{1}{u^{-\lambda}}S\left[\frac{\partial^2\Lambda_1(\Omega,\wp)}{\partial\Omega^2} - \Lambda_1(\Omega,\wp)\right]\right] \\
&= 4\sin\Omega\frac{\wp^{2\lambda}}{\Gamma(2\lambda+1)} + \frac{\wp^{2\lambda+1}}{\Gamma(2\lambda+2)} + 8\sin\Omega\frac{\wp^{3\lambda}}{\Gamma(3\lambda+2)}, && (4.18)\\
\Lambda_3(\Omega,\wp) &= S^{-1}\left[\frac{1}{u^{-\lambda}}S\left[\frac{\partial^2\Lambda_2(\Omega,\wp)}{\partial\Omega^2} - \Lambda_2(\Omega,\wp)\right]\right] \\
&= -8\sin\Omega\frac{\wp^{3\lambda}}{\Gamma(3\lambda+1)} - \frac{\wp^{3\lambda+1}}{\Gamma(3\lambda+2)} - 16\sin\Omega\frac{\wp^{3\lambda}}{\Gamma(3\lambda+1)}, && (4.19)\\
&\ \vdots
\end{aligned}$$

and so on. Thus,the series solution form is given by

$$\begin{aligned}
\Lambda(\Omega,\wp) &= \Lambda_0(\Omega,\wp) + \Lambda_1(\Omega,\wp) + \Lambda_2(\Omega,\wp) + \dots \\
&= \sin\Omega + \left[\wp - \frac{\wp^{\lambda+1}}{\Gamma(\lambda+2)} + \frac{\wp^{2\lambda+1}}{\Gamma(2\lambda+2)} - \frac{\wp^{3\lambda+1}}{\Gamma(3\lambda+2)} + \dots\right]. && (4.20)
\end{aligned}$$

if we put $\lambda = 2$, in equation (4.20) we have, $\Lambda(\Omega,\wp) = \sin\Omega + \sin\wp$ which is solution.



(a) $\lambda = 1.2$ (b) $\lambda = 1.6$

(c) $\lambda = 1.8$ (d) $\lambda = 2$

FIGURE 3. Estimated solution of equation (4.11)at 10^{th} order for different values of λ.

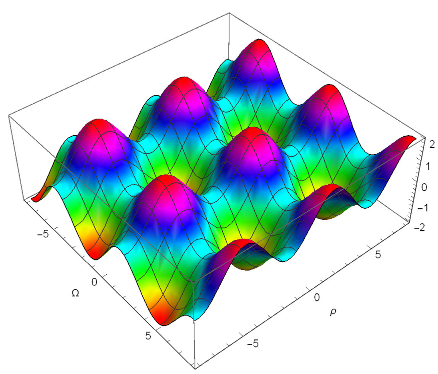


FIGURE 4. Exact solution of equation (4.11) at 10^{th} order for $\lambda = 2$.

Remark 4.4. The linear nonhomogeneous time fractional Klien-Gordon equation shown above. The Estimated solutions of the linear nonhomogeneous time fractional Klien-Gordon equation at different values for λ=1.2, 1.6, 1.8, 2 and the accurate solution for λ=2 are shown in Figures 3 and 4 respectively. The solution is so simple to discover that it is constantly dependent on the values of time-fractional derivatives.

Example 4.5. Suppose the Non linear time Fractional Klien-Gordon equation [7]

$$D_{\wp}^{\lambda}\Lambda(\Omega,\wp)-\Lambda_{\Omega\Omega}-(\Lambda_{\Omega})^2-\Lambda^2=0, 1<\lambda\leq 2, \tag{4.21}$$

with initial conditions

$$\Lambda(\Omega,0)=0 \text{ and } \Lambda_{\wp}(\Omega,0)=e^{\Omega}. \tag{4.22}$$

Taking the Sumudu the transform equation (4.21) , we have

$$S\left[D_{\wp}^{\lambda}\Lambda(\Omega,\wp)\right]-S\left[\Lambda_{\Omega\Omega}(\Omega,\wp)+(\Lambda_{\Omega})^2+\Lambda^2\right]=0, \tag{4.23}$$

using the fractional derivatives properties of the Sumudu transform, we get

$$S\left[\Lambda(\Omega,\wp)\right]=\frac{1}{u^{-\lambda}}\sum_{k=0}^{2-1}u^{k-\lambda}\left[\Lambda^{k}(\Omega,0)\right]+\frac{1}{u^{-\lambda}}S\left[\frac{\partial^2\Lambda(\Omega,\wp)}{\partial\Omega^2}+\left(\frac{\partial\Lambda(\Omega,\wp)}{\partial x}\right)^2+(\Lambda(\Omega,\wp))^2\right]. \tag{4.24}$$

Applying Inverse Sumudu Transform to the equation (4.24), we get

$$\Lambda(\Omega,\wp)=\sum_{k=0}^{2-1}\left(\frac{\partial^k\Lambda(\Omega,\wp)}{\partial\wp^k}\right)u^k+S^{-1}\left[\frac{1}{u^{-\lambda}}S\left[\frac{\partial^2\Lambda(\Omega,\wp)}{\partial\Omega^2}+\left(\frac{\partial\Lambda(\Omega,\wp)}{\partial x}\right)^2+(\Lambda(\Omega,\wp))^2\right]\right], \tag{4.25}$$

putting the results from equation (3.10) and equation (3.11) in equation (4.25) and applying equation (3.14) we determined the components of the SADM solution as follows

$$\begin{aligned}\Lambda_0(\Omega,\wp)&=\sum_{k=0}^{2-1}\left(\frac{\partial^k\Lambda(\Omega,\wp)}{\partial\wp^k}\right)u^k\\&=e^{2\Omega}t,\end{aligned} \tag{4.26}$$

$$\Lambda_{n+1}(\Omega,\wp)=S^{-1}\left[\frac{1}{u^{-\lambda}}\left(\frac{\partial^2\Lambda_n(\Omega,\wp)}{\partial\Omega^2}\right)\right]+S^{-1}\left[\frac{1}{u^{-\lambda}}\left(A_n\right)\right],\ n=0,1,2,3,\ldots \tag{4.27}$$

where, A_n are the Non linear terms of Adomian Polynomials
$N(\Lambda(\Omega,\wp))=\left(\frac{\partial\Lambda(\Omega,\wp)}{\partial x}\right)^2+(\Lambda(\Omega,\wp))^2$.
now for $n=0,1,2,3,\ldots$ and using equation (3.12) and equation (4.27) we have,

$$A_0=0, \tag{4.28}$$

$$\Lambda_1(\Omega,\wp)=\frac{e^{\Omega}\wp^{\lambda+1}}{\Gamma(\lambda+2)}, \tag{4.29}$$

$$A_1=0, \tag{4.30}$$

$$\Lambda_2(\Omega,\wp)=\frac{e^{\Omega}\wp^{2\lambda+1}}{\Gamma(2\lambda+2)}, \tag{4.31}$$

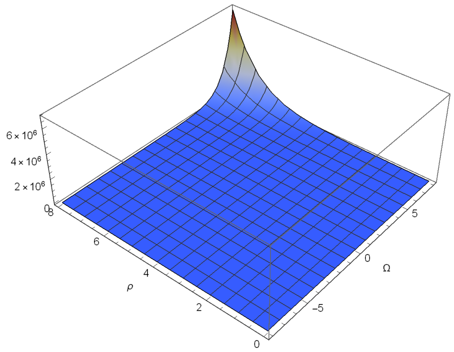
$$A_2=0, \tag{4.32}$$

$$\Lambda_3(\Omega,\wp)=\frac{e^{\Omega}\wp^{3\lambda+1}}{\Gamma(3\lambda+2)}, \tag{4.33}$$

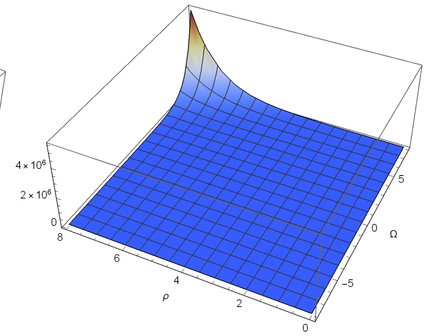
and so on. Thus,the series solution form is given by

$$\begin{aligned}\Lambda(\Omega,\wp)&=\Lambda_0(\Omega,\wp)+\Lambda_1(\Omega,\wp)+\Lambda_2(\Omega,\wp)+\ldots\\&=e^{\Omega}\left[\wp+\frac{\wp^{\lambda+1}}{\Gamma(\lambda+2)}+\frac{\wp^{2\lambda+1}}{\Gamma(2\lambda+2)}+\frac{\wp^{3\lambda+1}}{\Gamma(3\lambda+2)}+\ldots\right],\end{aligned} \tag{4.34}$$

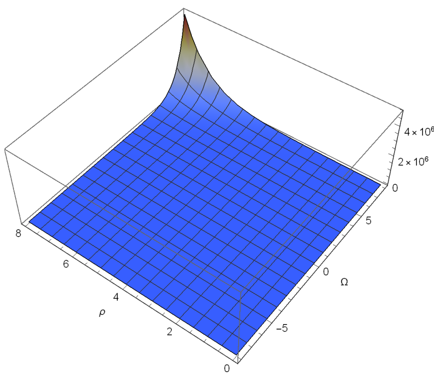
if we put $\lambda=2$, in equation (4.34) we have, $\Lambda(\Omega,\wp)=e^{\Omega}\sinh t$ which is solution.



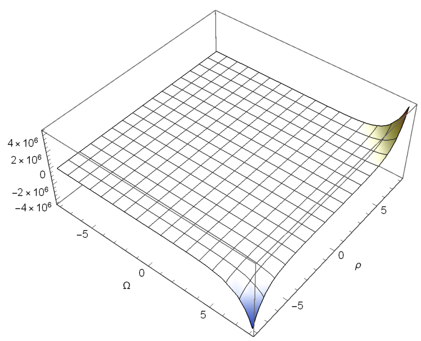
(a) $\lambda = 1.2$



(b) $\lambda = 1.6$



(c) $\lambda = 1.8$



(d) $\lambda = 2$

FIGURE 5. Estimated solution of equation (4.21) at 10^{th} order for different values of λ.

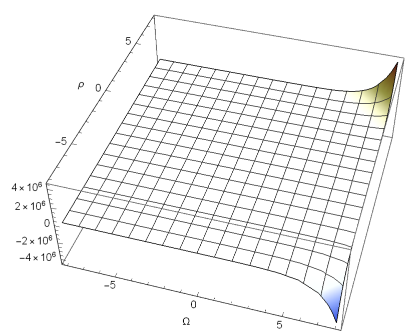


FIGURE 6. Exact solution of equation(4.21) at 10^{th} order for $\lambda = 2$.

Remark 4.6. The Non linear time fractional Klien-Gordon equation shown above. The Estimated solutions of the Non linear time fractional Klien-Gordon equation at different values for λ=1.2,1.6,1.8,2 and the accurate solution for λ=2 are shown in Figures 5 and 6 respectively. The solution is so simple to discover that it is constantly dependent on the values of time-fractional derivatives.

TABLE 1. Compare the solution with the 10th order approximation solution of equation (4.1) and the accurate solution for $\lambda = 2$.

Ω	$\wp$	Λ_{NSADM}	Λ_{Exact}	$\lvert\Lambda_{NSADM} - \Lambda_{Exact}\rvert$
0.2	0.3	0.0118208	0.0118208	0
0.4	0.5	0.0472832	0.0767081	0.0294249
0.6	0.7	0.106387	0.231918	0.125531
0.8	0.9	0.189133	0.501329	0.312196

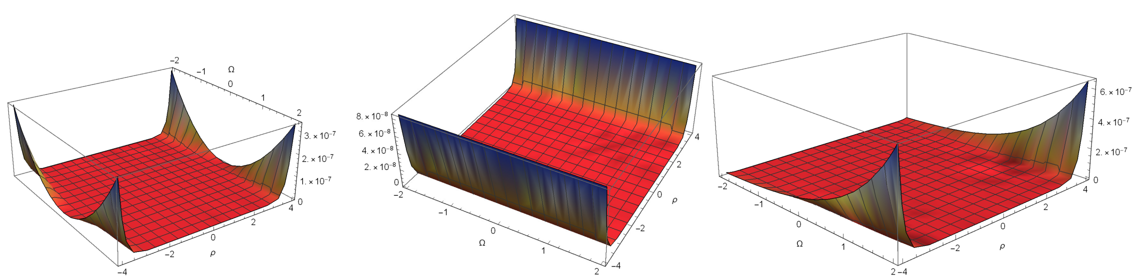
TABLE 2. Compare the solution with the 10th order approximation solution of(4.11) and the accurate solution for $\lambda = 2$.

Ω	$\wp$	Λ_{NSADM}	Λ_{Exact}	$\lvert\Lambda_{NSADM} - \Lambda_{Exact}\rvert$
0.2	0.3	0.49419	0.49419	0
0.4	0.5	0.684939	0.868844	0.183905
0.6	0.7	0.860163	1.20886	0.348697
0.8	0.9	1.01288	1.50068	0.4878

TABLE 3. Compare the solution with the 10th order approximation solution of equation(4.21) and the accurate solution for $\lambda = 2$.

Ω	$\wp$	Λ_{NSADM}	Λ_{Exact}	$\lvert\Lambda_{NSADM} - \Lambda_{Exact}\rvert$
0.1	0.2	0.122344	0.122344	0
0.2	0.3	0.135211	0.271775	0.136564
0.3	0.4	0.149431	0.454291	0.30486
0.4	0.5	0.165147	0.677216	0.512069

Remark 4.7. By comparison, it is clear that by computing additional terms, the efficiency and accuracy of this method (NSADM)can be greatly improved. We just use a few terms in this post. The precision of the estimated solution will be substantially enhanced if we employ additional terms. As a result, the recommended method for solving the linear differential equation is precise and efficient.



(a) Eq. (4.1) (b) Eq. (4.11) (c) Eq. (4.21)

FIGURE 7. The absolute error$\lvert\Lambda_{NSADM} - \Lambda_{Exact}\rvert$ of the equations at 10^{th} order for $\alpha = 2$.

Remark 4.8. Figures 7, 8, and 9 depict the absolute error between approximate and accurate solutions for α=2. By comparison, it is clear that by computing additional terms, the efficiency and accuracy of this method (NSADM)can be greatly improved. We just use a few terms in this post. The precision of the estimated solution will be substantially enhanced if we employ additional terms. As a result, the recommended method for solving the linear differential equation is precise and efficient.

Conclusion

The New Sumudu-Adomian Decomposition method approach was successfully employed in this research to get an Estimated solution for the time-fractional equations. The new Sumudu-Adomian Decomposition method (NSADM) to achieve accurate and Estimated semi-analytical solutions for time-fractional equations. It is also shown graphically. It is revealed that the NSADM is more efficient and accurate and requires less calculation.

Conflict of interests

The authors declare that there is no conflict of interests.

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