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On the existence of periodic solutions of third order delay differential equations

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Abstract

This work deals with the existence of periodic solutions (EPSs) to a third order nonlinear delay differential equation (DDE) with multiple constant delays. For the considered DDE, a theorem is proved, which includes sufficient criteria related to the EPSs. The technique of the proof depends on Lyapunov-Krasovskiĭ functional (LKF) approach. The obtained result extends and improves some results that can be found in the literature. In a particular case of the considered DDE, an example is provided to show the applicability of the main result of this paper.

Keywords. Existence, Periodic solutions, Differential equation, Third order, Delay, LKF.

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1. Introduction

Over the last five decades, intensive scientific works have been carried out in the field of fundamental properties of solutions of third order ordinary differential equations (ODEs), functional differential equations (FDEs), and so on. For instance, EPSs, the uniqueness of periodic solutions, and some other fundamental properties of solutions of certain third order nonlinear DDEs have been discussed in ([1–6, 8, 9, 14, 16, 17, 19, 20, 23–30, 32–35, 37, 40, 47]). As for fundamental properties of solutions of certain third order nonlinear ODEs, numerous results on ESPs, stability, boundedness, and some other properties of solutions that kind of equations are obtained in ([10–13, 18, 21, 22, 36, 40]). For some other interesting results on the fundamental properties of solutions of various differential equations, in particular, see, also, ([7, 15, 31, 38, 39]). In addition, it is well-known that ODEs of the third order without and with delay can appear during researches in sciences and engineering. So, fundamental behaviors of solutions of ODEs of third order with and without delay attracted the intensive attention of researchers up to now (see, the sources in the references of this paper and that mentioned therein). This means that it deserves to investigate the properties of solutions of ODEs of third order with and without delay.

Next, as we know, it is not always possible to find solutions of ODEs and FDEs explicitly, except numerically. Finding explicit solutions can be more difficult for FDEs rather than ODEs. However, without solving ODEs and FDEs, it can be obtained some qualitative information related to motions of trajectories of solutions of ODEs and FDEs via some well-known methods, called direct Lyapunov method, LKF method, fixed point method, coincidence degree method, variation of parameters method and so forth. This paper does not deal with the details of information related to these methods. However, in this paper, the LKF method will be used as the main technique to prove the main result of this paper. Indeed, the LKF approach is very capable to get qualitative information about behaviors of solutions without prior information of them provided to construct or define proper LKFs for problems under study. In particular cases, as for some interesting and recent real applications, see, the discussions in [42–46], where differential

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equations and some other relations are used as mathematical models. We now outline some reference papers to take into consideration the problem of this paper.

In 1978, Chukwu [9] considered third order DDE with constant delay:

$$x''' + f(x,x',x'')x'' + g(x(t-h),x'(t-h)) + i(x(t-h)) = p(t,x,x',x(t-h),x'(t-h),x''),$$

and the author discussed the EPSs to this DDE by using an LKF.

In 1992, Zhu [47] studied third order DDE with constant delay:

$$x''' + ax'' + \phi(x'(t-r)) + f(x) = p(t),$$

and the author provided a result on the EPSs of this DDE via suitable LKFs.

In 2000, Tejmula ve Tchegnani [27] derived sufficient conditions to the EPSs of third order DDE with constant delay:

$$x''' + f(t,x,x',x'')x'' + g(t,x(t-\tau),x'(t-\tau)) + h(x(t-\tau)) = P_1(t,x,x',x'',x(t-\tau),x'(t-\tau)),$$

by LKF approach.

In 2012, Tunç [29] considered third order DDE with n-multiple constant delays:

$$x''' + \psi(x')x'' + \sum_{i=1}^{n} g_i(x'(t-\tau_i)) + f(x) = p(t,x,x(t-\tau_1),...,x',...,x'(t-\tau_n),x''),$$

and established sufficient conditions for the EPSs of this DDE via the LKF method.

In this paper, motivated by the above results and those in the related literature, we consider third order non-linear DDE with n-multiple delays:

$$x''' + \psi(x')x'' + \sum_{i=1}^{n} g_i(x'(t-\tau_i)) + \sum_{i=1}^{n} f_i(x(t-\tau_i)) = p(t,x,x(t-\tau_1),...,x(t-\tau_n),x'(t-\tau_1),...,x'(t-\tau_n),x''), \tag{1.1}$$

with

$$x(t) = \varphi(t), t \in [-\tau, 0],$$

where $t \in [-\tau,\infty)$,$x \in \mathbb{R}$, $\mathbb{R} = (-\infty,\infty)$, and τ_i are positive constants, i.e., τ_i are fixed constant delays, $\tau = \max\{\tau_1,\tau_2,...,\tau_n\}$ and $\varphi \in C([-\tau,0],\mathbb{R})$ is initial function. It is also assumed that $\psi \in C[\mathbb{R},\mathbb{R}]$, g_i,$f_i \in C^1[\mathbb{R},\mathbb{R}]$ and $p(.) \in C[[-\tau,\infty) \times \mathbb{R}^{2n+2},\mathbb{R}]$, $g_i(0) = f_i(0) = 0$ and $p(.)$ is periodic in t with the period T, $T \geq \tau_i$.

Throughout the paper, it can be used the representations $x = x(t)$, y=$y(t)$, and $z = z(t)$, respectively. From DDE (1.1), it can be written the system:

$$\begin{aligned} x' &= y, y' = z, \\ z' &= -\psi(y)z - \sum_{i=1}^{n} g_i(y) - \sum_{i=1}^{n} f_i(x) + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} {f_i}'(x(s))y(s)ds + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} {g_i}'(y(s))z(s)ds \\ &\quad + p(t,x,x(t-\tau_1),...,y(t-\tau_n),z). \end{aligned} \tag{1.2}$$

In view of the works outlined above and the works in the references of this paper, which are related to the EPSs and some other fundamental properties of solutions of third order ODEs and DDEs, we aim to present some new contributions to the qualitative theory of ODEs and FDEs. Indeed, the DDEs (1.1) extend and improve that of [29, 47] and it is different from those in the references of this paper. Next, providing an example shows the application of the main result of this paper. Finally, the result of this paper is new and original.

2. Existence of periodic solutions

Throughout the paper and above, $p(.)$ represents $p(t, x, x(t-\tau_1), ..., y(t-\tau_n), z)$. The below Theorem 2.1 is main result of this paper.

Theorem 2.1. *Let $a,\ b_i,\ c_i,\ M_0,\ \delta,\ L_i,\ \tau_i \in \mathbb{R},\ (i = 1, 2, ..., n)$, and be positive such that the below conditions (A1) and (A2) are satisfied:*

(A1)

$$\mu b_i - c > 0, \quad ab - c > 0, \quad f_i(0) = 0, \quad f_i(x) sgnx > 0, (x \neq 0),$$

$$\sup\{{f_i}'(x)\} = c_i,\ x \in \mathbb{R}, f_i(x) sgnx \to \infty \ for\ |x| \to \infty,\ (i = 1, 2, ..., n);$$

(A2)

$$g_i(0) = 0, \quad y^{-1} g_i(y) \geq b_i, \quad (y \neq 0),\ y \in \mathbb{R},$$

$$|{g'}_i(y)| \leq L_i, \quad 0 \leq \psi(y) - a \leq \delta,\ y \in \mathbb{R}, \quad |p(.)| \leq M_0.$$

Then, there is a positive constant τ, $\tau = \max\limits_{1 \leq i \leq n} \tau_i$, $\tau \leq \min\left\{\frac{ab-c}{2bN_1}, \frac{ab-c}{4M_1}\right\}$, such that the system (1.2) has at least one T- periodic solution, where

$$b = \sum_{i=1}^{n} b_i, c = \sum_{i=1}^{n} c_i, M_1 = \frac{1}{2} \sum_{i=1}^{n} [(1 + 2\mu)c_i + \mu L_i],$$

$$N_1 = \frac{1}{2} \sum_{i=1}^{n} [(2 + \mu)L_i + c_i] \quad and\ \mu = \frac{ab + c}{2b}, L = \sum_{i=1}^{n} L_i.$$

Proof. We define an LKF $V(.)$ by

$$V(.) = V_1(.) + V_2(.) + 1 + \sum_{i=1}^{n} L_i \int_{-\tau_i}^{0} \int_{t+s}^{t} |z(\theta)|\, d\theta ds + \sum_{i=1}^{n} c_i \int_{-\tau_i}^{0} \int_{t+s}^{t} |y(\theta)|\, d\theta ds, \tag{2.1}$$

where

$$\begin{aligned} V(.) =& V(x_t, y_t, z_t), \\ V_1(.) =& V_1(x, y, z), V_2(.) = V_2(x, y, z), \\ V_1(.) =& \mu \sum_{i=1}^{n} \int_0^x f_i(\xi) d\xi + y \sum_{i=1}^{n} f_i(x) + \sum_{i=1}^{n} \int_0^y g_i(\eta) d\eta + \mu yz + \mu \int_0^y \psi(\eta)\eta d\eta + \frac{1}{2} z^2 \\ &+ \sum_{i=1}^{n} \gamma_i \int_{-\tau_i}^{0} \int_{t+s}^{t} z^2(\theta) d\theta ds + \sum_{i=1}^{n} \beta_i \int_{-\tau_i}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds, \end{aligned}$$

and

$$V_2(.) = \begin{cases} \frac{z}{M} sgnx, & |x| \geq 1, z \leq M, \\ sgnz\ sgnx, & |x| \geq 1, z \geq M, \\ \frac{xz}{M}, & |x| \leq 1, z \leq M, \\ xsgnz, & |x| \leq 1, z \geq M, \end{cases}$$

where $M \in \mathbb{R},\ (M > 1),\ \beta_i,\ \gamma_i \in \mathbb{R}$ such that $\gamma_i > 0, \quad \beta_i > 0$ will be chosen later.

It is obvious that the LKF V is positive definite and two wedge functions can be found that limit this LKF from the bottom and the upper.

Next, we have that $V_1(0,0,0)=0$. Using the conditions $\psi(y)\geq a$, $y^{-1}g_i(y)\geq b_i$, $f_i(0)=0$, $f_i(x)sgn(x)>0$ for all $x\neq 0$, and $\sup\{f_i{}'(x)\}=c_i$, $(i=1,2,...,n)$, we derive that

$$
\begin{aligned}
V_1(.) =&\mu\sum_{i=1}^{n}\int_0^x f_i(\xi)d\xi + y\sum_{i=1}^{n} f_i(x) + \sum_{i=1}^{n}\int_0^y \frac{g_i(\eta)}{\eta}\eta d\eta + \mu yz + \frac{1}{2}\mu a y^2\\
&+\frac{1}{2}z^2 + \sum_{i=1}^{n}\gamma_i\int_{-\tau_i}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds + \sum_{i=1}^{n}\beta_i\int_{-\tau_i}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds\\
\geq&\mu\sum_{i=1}^{n}\int_0^x f_i(\xi)d\xi + y\sum_{i=1}^{n} f_i(x) + \frac{b_1}{2}y^2\\
&+\frac{b_2}{2}y^2+...+\frac{b_n}{2}y^2+\mu yz+\frac{1}{2}\mu a y^2+\frac{1}{2}z^2+\sum_{i=1}^{n}\gamma_i\int_{-\tau_i}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds + \sum_{i=1}^{n}\beta_i\int_{-\tau_i}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds\\
=&\frac{1}{2b_1}[b_1 y + f_1(x)]^2+...+\frac{1}{2b_n}[b_n y+f_n(x)]^2\\
&+\left[4\sum_{i=1}^{n}\frac{1}{2b_i y^2}\int_0^x f_i(\xi)\left\{\int_0^y(\mu b_i - f_i{}'(\xi))\eta d\eta\right\}d\xi\right]\\
&+\frac{1}{2}[\mu y+z]^2+\frac{1}{2}\mu(a-\mu)y^2+\sum_{i=1}^{n}\gamma_i\int_{-\tau_i}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds + \sum_{i=1}^{n}\beta_i\int_{-\tau_i}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds.
\end{aligned}
$$

Using conditions (A1), (A2), the estimates

$$
a-\mu = a-2^{-1}b^{-1}(ab+c)>0, \quad \mu b_i - f_i{}'(\xi)\geq \mu b_i - c_i > 0,
$$

and proceeding some elementary calculations, we can obtain that

$$
V_1(.)\geq K_1x^2+K_2y^2+K_3z^2\geq K_4(x^2+y^2+z^2), \tag{2.2}
$$

where $min\{K_1,K_2,K_3\}=K_4$.

Since V_2 is continuous, then it is clear that

$$
|V_2|\leq 1. \tag{2.3}
$$

Thus, using the inequalities (2.2), (2.3) into (2.1) and the conditions of Theorem 2.1, we can show that the LKF V satisfies the condition (i) of Yoshizawa ([41], Theorem 37.2]).

Differentiating the LKF V_1 with respect to t and using the system (1.2), we derive that

$$
\begin{aligned}
V_1^{'} =&[f_1{}'(x)+f_2{}'(x)+...+f_n{}'(x)]y^2+\mu z^2-\mu y\sum_{i=1}^{n}g_i(y)-\psi(y)z^2\\
&+(\mu y+z)\left[\int_{t-\tau_1}^{t} f_1{}'(x(s))y(s)ds+...+\int_{t-\tau_n}^{t} f_n{}'(x(s))y(s)ds\right]\\
&+(\mu y+z)\left[\int_{t-\tau_1}^{t} g_1{}'(y(s))z(s)ds+...+\int_{t-\tau_n}^{t} g_n{}'(y(s))z(s)ds\right]\\
&+\sum_{i=1}^{n}(\gamma_i\tau_i)z^2-\sum_{i=1}^{n}\gamma_i\int_{t-\tau_i}^{t} z^2(s)ds+\sum_{i=1}^{n}(\beta_i\tau_i)y^2-\sum_{i=1}^{n}\beta_i\int_{t-\tau_i}^{t} y^2(s)ds+(\mu y+z)p(.)
\end{aligned}
$$

$$
\begin{aligned}
=& y^2\sum_{i=1}^{n} {f_i}'(x)+\mu z^2-\mu y\sum_{i=1}^{n} g_i(y)-\psi(y)z^2\\
&+(\mu y+z)\sum_{i=1}^{n}\int_{t-\tau_i}^{t} {f_i}'(x(s))y(s)ds+(\mu y+z)\sum_{i=1}^{n}\int_{t-\tau_i}^{t} {g'}_i(y(s))z(s)ds\\
&+\sum_{i=1}^{n}(\gamma_i\tau_i)z^2-\sum_{i=1}^{n}\gamma_i\int\limits_{t-\tau_i}^{t} z^2(s)ds+\sum_{i=1}^{n}(\beta_i\tau_i)y^2-\sum_{i=1}^{n}\beta_i\int\limits_{t-\tau_i}^{t} y^2(s)ds+(\mu y+z)p(.).
\end{aligned}
$$

Using the condition $-L_i \leq {g'}_i(y) \leq L_i$ and the inequality $2\,|u|\,|v| \leq u^2+v^2$, it follows that

$$
\begin{aligned}
\mu y\sum_{i=1}^{n}\int_{t-\tau_i}^{t} {g_i}'(y(s))z(s)ds \leq& \mu\,|y|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} \left|{g_i}'(y(s))\right|\left|z(s)\right|ds\\
\leq& \mu\,|y|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} L_i\left|z(s)\right|ds\\
\leq& \frac{\mu}{2}\sum_{i=1}^{n}(L_i\tau_i)y^2+\frac{\mu}{2}\sum_{i=1}^{n}L_i\int\limits_{t-\tau_i}^{t} z^2(s)ds,
\end{aligned}
$$

$$
\begin{aligned}
z\sum_{i=1}^{n}\int_{t-\tau_i}^{t} {g_i}'(y(s))z(s)ds \leq& |z|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} \left|{g_i}'(y(s))\right|\left|z(s)\right|ds\\
\leq& |z|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} L_i\left|z(s)\right|ds\\
\leq& \frac{1}{2}\sum_{i=1}^{n}(L_i\tau_i)z^2+\frac{1}{2}\sum_{i=1}^{n}L_i\int\limits_{t-\tau_i}^{t} z^2(s)ds,
\end{aligned}
$$

$$
\begin{aligned}
\mu y\sum_{i=1}^{n}\int_{t-\tau_i}^{t} {f_i}'(x(s))y(s)ds \leq& \mu\,|y|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} \left|{f_i}'(x(s))\right|\left|y(s)\right|ds\\
\leq& \mu\,|y|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} c_i\left|y(s)\right|ds\\
\leq& \frac{\mu}{2}\sum_{i=1}^{n}(c_i\tau_i)y^2+\frac{\mu}{2}\sum_{i=1}^{n}c_i\int\limits_{t-\tau_i}^{t} y^2(s)ds,
\end{aligned}
$$

$$
\begin{aligned}
z\sum_{i=1}^{n}\int_{t-\tau_i}^{t} {f_i}'(x(s))y(s)ds \leq& |z|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} \left|{f_i}'(x(s))\right|\left|y(s)\right|ds\\
\leq& |z|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} c_i\left|y(s)\right|ds\\
\leq& \frac{1}{2}\sum_{i=1}^{n}(c_i\tau_i)z^2+\frac{1}{2}\sum_{i=1}^{n}c_i\int\limits_{t-\tau_i}^{t} y^2(s)ds.
\end{aligned}
$$

On gathering the above inequalities into V_1', we have

$$\begin{aligned}
V_1'(.) \le & y^2 \sum_{i=1}^{n} f_i{}'(x) + \mu z^2 - \mu y \sum_{i=1}^{n} g_i(y) - \psi(y) z^2 + \sum_{i=1}^{n} (\gamma_i \tau_i) z^2 + \frac{\mu}{2} \sum_{i=1}^{n} (L_i \tau_i) y^2 + \frac{1}{2} \sum_{i=1}^{n} (L_i \tau_i) z^2 \\
& + \frac{\mu}{2} \sum_{i=1}^{n} (c_i \tau_i) y^2 + \frac{1}{2} \sum_{i=1}^{n} (c_i \tau_i) z^2 + \sum_{i=1}^{n} (\beta_i \tau_i) y^2 + \sum_{i=1}^{n} \left[\frac{1}{2} L_i (1+\mu) - \gamma_i \right] \int_{t-\tau_i}^{t} z^2(s) ds \\
& + \sum_{i=1}^{n} \left[\frac{1}{2} c_i (1+\mu) - \beta_i \right] \int_{t-\tau_i}^{t} y^2(s) ds + (\mu y + z) p(.).
\end{aligned}$$

Using the conditions $\sup\{f_i{}'(x)\} = c_i$, $(i = 1, 2, ..., n)$, $\psi(y) \ge a$ and $ab - c > 0$, we get

$$\begin{aligned}
V_1'(.) \le & - \left[\mu \sum_{i=1}^{n} \frac{g_i(y)}{y} - \sum_{i=1}^{n} c_i - \frac{\mu}{2} \sum_{i=1}^{n} (L_i + c_i) \tau_i - \sum_{i=1}^{n} (\beta_i \tau_i) \right] y^2 \\
& - \left[a - \mu - \frac{1}{2} \sum_{i=1}^{n} (L_i + c_i) \tau_i - \sum_{i=1}^{n} (\gamma_i \tau_i) \right] z^2 + \sum_{i=1}^{n} \left[\frac{1}{2} c_i (1+\mu) - \beta_i \right] \int_{t-\tau_i}^{t} y^2(s) ds \\
& + \sum_{i=1}^{n} \left[\frac{1}{2} L_i (1+\mu) - \gamma_i \right] \int_{t-\tau_i}^{t} z^2(s) ds + |\mu y + z| \, |p(.)| \\
\le & - \left[\mu \sum_{i=1}^{n} \frac{g_i(y)}{y} - c - \frac{1}{2} \sum_{i=1}^{n} \left[(L_i + c_i) \mu + 2\beta_i \right] \tau_i \right] y^2 - \left[\frac{ab - c}{2b} - \frac{1}{2} \sum_{i=1}^{n} (L_i + c_i + 2\gamma_i) \tau_i \right] z^2 \\
& + \sum_{i=1}^{n} \left[\frac{1}{2} c_i (1+\mu) - \beta_i \right] \int_{t-\tau_i}^{t} y^2(s) ds + \sum_{i=1}^{n} \left[\frac{1}{2} L_i (1+\mu) - \gamma_i \right] \int_{t-\tau_i}^{t} z^2(s) ds + |\mu y + z| \, |p(.)| \, .
\end{aligned}$$

Let $\gamma_i = \frac{1}{2} L_i (1+\mu)$, $\beta_i = \frac{1}{2} c_i (1+\mu)$. Using the condition $|p(.)| \le M_0$, we obtain

$$\begin{aligned}
V_1'(.) \le & - \left[\mu \sum_{i=1}^{n} \frac{g_i(y)}{y} - c - \frac{1}{2} \sum_{i=1}^{n} \left[(1+2\mu) c_i + \mu L_i \right] \tau_i \right] y^2 \\
& - \left[\frac{ab - c}{2b} - \frac{1}{2} \sum_{i=1}^{n} ((2+\mu) L_i + c_i) \tau_i \right] z^2 + \mu M_0 |y| + M_0 |z| \, .
\end{aligned} \tag{2.4}$$

Calculating $V_2' = \frac{dV_2}{dt}$ and using the conditions of Theorem 2.1, we get

$$V_2' = \begin{cases}
\frac{1}{M} \left\{ \begin{array}{c} -\psi(y) z - \sum\limits_{i=1}^{n} g_i(y) - \sum\limits_{i=1}^{n} f_i(x) + \sum\limits_{i=1}^{n} \int\limits_{t-\tau_i}^{t} f'_i(x(s)) y(s) ds \\ + \sum\limits_{i=1}^{n} \int\limits_{t-\tau_i}^{t} g'_i(y(s)) z(s) ds + p(.) \end{array} \right\} sgn x, & |x| \ge 1, z \le M, \\
0, & |x| \ge 1, z \ge M, \\
\frac{yz}{M} + \frac{x}{M} \left\{ \begin{array}{c} -\psi(y) z - \sum\limits_{i=1}^{n} g_i(y) - \sum\limits_{i=1}^{n} f_i(x) + \sum\limits_{i=1}^{n} \int\limits_{t-\tau_i}^{t} f'_i(x(s)) y(s) ds \\ + \sum\limits_{i=1}^{n} \int\limits_{t-\tau_i}^{t} g'_i(y(s)) z(s) ds + p(.) \end{array} \right\}, & |x| \le 1, z \le M, \\
y sgn z, & |x| \le 1, z \ge M,
\end{cases}$$

$$\leq \begin{cases} -\frac{1}{M}\sum\limits_{i=1}^{n} f_i(x)sgnx + \left\{ \begin{array}{c} a+\delta+M_0+\sum\limits_{i=1}^{n}|g_i(y)|+\sum\limits_{i=1}^{n}c_i\int\limits_{t-\tau_i}^{t}|y(s)|\,ds \\ +\sum\limits_{i=1}^{n}L_i\int\limits_{t-\tau_i}^{t}|z(s)|\,ds \end{array}\right\}, & |x|\geq 1, z\leq M,\\ 0, & |x|\geq 1, z\geq M,\\ |y|+a+\delta+m+\sum\limits_{i=1}^{n}|g_i(y)|+\sum\limits_{i=1}^{n}c_i\int\limits_{t-\tau_i}^{t}|y(s)|\,ds+\sum\limits_{i=1}^{n}L_i\int\limits_{t-\tau_i}^{t}|z(s)|\,ds, & |x|\leq 1, z\leq M,\\ |y|, & |x|\leq 1, z\geq M. \end{cases}$$

Let us consider the function V_2 in the region $\max\{|y|-K, |z|-M\}\geq 0$, where K and M are sufficiently large constants, which are determined later. We now consider Case I and Case II to complete the proof of the theorem, respectively.

Case I: We suppose that $|y|\geq K\geq 1$, and x, z are arbitrary. As for the next step, we have

$$V_2^{'}\leq |y|+a+\delta+M_0+\sum_{i=1}^{n}|g_i(y)|+\sum_{i=1}^{n}c_i\int\limits_{t-\tau_i}^{t}|y(s)|\,ds+\sum_{i=1}^{n}L_i\int\limits_{t-\tau_i}^{t}|z(s)|\,ds. \tag{2.5}$$

Using (2.1), (2.4), (2.5), and $\tau=\max\limits_{1\leq i\leq n}\tau_i$, we derive that

$$\begin{aligned} V'(.)\leq &-\left[\mu\sum_{i=1}^{n}\frac{g_i(y)}{y}-c-\frac{1}{2}\sum_{i=1}^{n}\left[(1+2\mu)c_i+\mu L_i\right]\tau_i\right]y^2+\sum_{i=1}^{n}|g_i(y)|\\ &-\left[\frac{ab-c}{2b}-\frac{1}{2}\sum_{i=1}^{n}((2+\mu)L_i+c_i)\tau_i\right]z^2+(\mu M_0+1)|y|+M_0|z|\\ &+(a+\delta+M_0)+\sum_{i=1}^{n}L_i\int\limits_{t-\tau_i}^{t}|z(s)|\,ds+\sum_{i=1}^{n}(L_i\tau_i)|z|\\ &-\sum_{i=1}^{n}L_i\int\limits_{t-\tau_i}^{t}|z(s)|\,ds+\sum_{i=1}^{n}c_i\int\limits_{t-\tau_i}^{t}|y(s)|\,ds\\ &+\sum_{i=1}^{n}(c_i\tau_i)|y|-\sum_{i=1}^{n}c_i\int\limits_{t-\tau_i}^{t}|y(s)|\,ds\\ \leq &-\left[\mu\sum_{i=1}^{n}\frac{g_i(y)}{y}-c-\frac{1}{2}\sum_{i=1}^{n}\left[(1+2\mu)c_i+\mu L_i\right]\tau_i\right]y^2+\sum_{i=1}^{n}|g_i(y)|\\ &-\left[\frac{ab-c}{2b}-\frac{1}{2}\sum_{i=1}^{n}(2+\mu)L_i+c_i)\tau_i\right]z^2+(\mu M_0+1)|y|+M_0|z|\\ &+(a+\delta+M_0)|y|+\sum_{i=1}^{n}(L_i\tau_i)|z|+\sum_{i=1}^{n}(c_i\tau_i)|y|. \end{aligned}$$

Consider the terms

$$\mu\sum_{i=1}^{n}g_i(y)y-\sum_{i=1}^{n}|g_i(y)|-cy^2-\frac{1}{2}\sum_{i=1}^{n}\left[(1+2\mu)c_i+\mu L_i\right]\tau_i y^2,$$

and assume that the constant h satisfies

$$h=\frac{ab+3c}{2(ab+c)}<1.$$

Next, there is a constant $K_1 \in \mathbb{R},\ (K_1 \geq 1)$, with $(1-\mu^{-1}|y|) \geq h$ for $|y| \geq K_1$ such that

$$\begin{aligned}
&\mu\sum_{i=1}^{n} g_i(y)y - \sum_{i=1}^{n}|g_i(y)| - cy^2 - \frac{1}{2}\sum_{i=1}^{n}\left[(1+2\mu)c_i + \mu L_i\right]\tau_i y^2 \\
&\quad=\mu\sum_{i=1}^{n}\frac{g_i(y)}{y}\left(1-\frac{1}{\mu|y|}\right)y^2 - \left(c+\frac{1}{2}\sum_{i=1}^{n}\left[(1+2\mu)c_i+\mu L_i\right]\tau_i\right)y^2 \\
&\quad\geq\frac{(ab+c)b}{2b}\,\frac{ab+3c}{2(ab+c)}y^2 - \left(c+\frac{1}{2}\sum_{i=1}^{n}\left[(1+2\mu)c_i+\mu L_i\right]\tau_i\right)y^2 \\
&\quad=\left(\frac{ab-c}{4}-\frac{1}{2}\sum_{i=1}^{n}\left[(1+2\mu)c_i+\mu L_i\right]\tau_i\right)y^2 \\
&\quad\geq\left(\frac{ab-c}{4}-M_1\tau\right)y^2.
\end{aligned}$$

Hence, we derive that

$$V'(.) \leq -\ \left(\frac{ab-c}{4}-M_1\tau\right)y^2 - \left(\frac{ab-c}{2b}-N_1\tau\right)z^2 + (\mu M_0 + 1 + a + \delta + M_0 + c\tau)\,|y| + (M_0 + L\tau)\,|z|\,,$$

where

$$M_1 = \frac{1}{2}\sum_{i=1}^{n}\left[(1+2\mu)c_i+\mu L_i\right], N_1 = \frac{1}{2}\sum_{i=1}^{n}\left[(2+\mu)L_i + c_i\right], L = \sum_{i=1}^{n}L_i.$$

Since

$$\tau < \min\left\{\frac{ab-c}{2bN_1}, \frac{ab-c}{4M_1}\right\} = \delta_1 > 0,$$

then

$$V'(.) \leq -\delta_1(y^2+z^2) + (\mu M_0 + a + \delta + M_0 + c\tau + 1)\,|y| + (M_0 + L\tau)\,|z|\,.$$

Let

$$\rho_1 = \max\left\{\mu M_0 + a + \delta + M_0 + c\tau + 1,\ M_0 + L\tau\right\}.$$

Next, if $|y| \geq (\sqrt{2}+1)\rho_1{\delta_1}^{-1}$, then

$$\begin{aligned}
V'(.) \leq& -\delta_1(y^2+z^2) + \rho_1(|y|+|z|) \\
=& -\frac{\delta_1}{2}(y^2+z^2) - \frac{\delta_1}{2}\left[(|y|-\frac{\rho_1}{\delta_1})^2 + (|z|-\frac{\rho_1}{\delta_1})^2 - 2\frac{{\rho_1}^2}{{\delta_1}^2}\right] \\
\leq& -\frac{\delta_1}{2}(y^2+z^2).
\end{aligned}$$

Let $K = \max\{(\sqrt{2}+1)\rho_1{\delta_1}^{-1}, K_1\}$. If $|y| \geq K$, then

$$V'(.) \leq -\frac{\delta_1}{2}(y^2+z^2).$$

This is the end of the discussion of Case I.

Case II: Let x, y be arbitrary and $|z| \geq M$. Then, we obtain that

$$V_2^{'}(.) \leq |y|\,.$$

Following a similar way as in Case I, choosing $\gamma_i = \frac{1}{2}L_i(1+\mu)$, $\beta_i = \frac{1}{2}c_i(1+\mu)$ and using the inequality $\tau < \min\left\{\frac{ab-c}{2bN_1}, \frac{ab-c}{4M_1}\right\}$, for some positive constants δ_2, ρ_2 and $|z| \geq M = K$, we derive that

$$\begin{aligned} V'(.) \leq &-\delta_2(y^2+z^2) + (\mu M_0 + c\tau + 1)\,|y| + (M_0 + L\tau)\,|z| \\ \leq &-\delta_2(y^2+z^2) + \rho_2(|y|+|z|) \\ \leq &-\frac{\delta_1}{2}(y^2+z^2). \end{aligned}$$

Consider the function V_2. Let $\max\{|y|-K, |z|-M\} \leq 0$ and $|x| \geq H > 1$. If the derivative of the function V_2 is taken into account under the present conditions, then

$$V_2^{'} \leq -\frac{1}{M}\sum_{i=1}^{n} f_i(x)sgnx + \max_{|y|\leq K}\sum_{i=1}^{n}|g_i(y)| + \sum_{i=1}^{n} c_i \int\limits_{t-\tau_i}^{t} |y(s)|\,ds + \sum_{i=1}^{n} L_i \int\limits_{t-\tau_i}^{t} |z(s)|\,ds + a + \delta + M_0. \tag{2.6}$$

From (2.1), (2.4) and (2.6), we can easily obtain that

$$\begin{aligned} V'(.) \leq &-\left(4^{-1}(ab-c) - M_1\tau\right)y^2 - \left(\frac{ab-c}{2b} - N_1\tau\right)z^2 \\ &+ \mu M_0\,|y| + (M_0+L\tau)\,|z| - \frac{1}{M}\sum_{i=1}^{n} f_i(x)sgnx + \max_{|y|\leq K}\sum_{i=1}^{n}|g_i(y)| \\ &+ \sum_{i=1}^{n} c_i \int\limits_{t-\tau_i}^{t}|y(s)|\,ds + \sum_{i=1}^{n} L_i \int\limits_{t-\tau_i}^{t}|z(s)|\,ds + (a+\delta+m) - \sum_{i=1}^{n} L_i \int\limits_{t-\tau_i}^{t}|z(s)|\,ds - \sum_{i=1}^{n} c_i \int\limits_{t-\tau_i}^{t}|y(s)|\,ds \\ \leq &-\left(\frac{ab-c}{4} - M_1\tau\right)y^2 - \left(\frac{ab-c}{2b} - N_1\tau\right)z^2 - \frac{1}{M}\sum_{i=1}^{n} f_i(x)sgnx + \max_{|y|\leq K}\sum_{i=1}^{n}|g_i(y)| \\ &+ (a+\delta+M_0+LM\tau+\mu M_0 K + M_0 M). \end{aligned} \tag{2.7}$$

Since $\lim\limits_{|x|\to\infty} f_i(x)sgnx = \infty$ and $|x| \geq H > 1$, then

$$\sum_{i=1}^{n} f_i(x)sgnx \geq 2M\left\{\max_{|y|\leq K}\sum_{i=1}^{n}|g_i(y)| + (a+\delta+M_0+LM\tau+\mu M_0K+M_0M)\right\},$$

and accordingly

$$-\frac{\sum\limits_{i=1}^{n} f_i(x)sgnx}{2M} + \left\{\max_{|y|\leq K}\sum_{i=1}^{n}|g_i(y)| + (a+\delta+M_0+LM\tau+\mu M_0K+M_0M)\right\} \leq 0.$$

Substituting the last inequality into (2.7), we derive that

$$V'(.) \leq -\left(\frac{ab-c}{4} - M_1\tau\right)y^2 - \left(\frac{ab-c}{2b} - N_1\tau\right)z^2 - \frac{1}{2M}\sum_{i=1}^{n} f_i(x)sgnx.$$

By the help of the above results, we conclude that there is a sufficiently large positive constant R such that

$$V'(.) \leq -\omega(u),$$

for $u^2 \geq R^2$, where $u = \sqrt{x^2+y^2+z^2}$.

Thus, the proof of Theorem 2.1 was completed since the LKF V satisfies all the conditions of Theorem 37.2 of Yoshizawa [41]. Therefore, DDE (1.1) has at least one T- periodic solution. □

Example 2.2. *Consider the following third order nonlinear DDE with two constant delays, which is included by equation* (1.1)*:*

$$\begin{aligned} &x''' + \left(4 + \frac{1}{1+(x')^2}\right) x'' + 4x'(t-5^{-1}) + \sin\, x'(t-5^{-1}) \\ &\quad + 4x'(t-10^{-1}) + \sin\, x'(t-10^{-1}) + 11x(t-10^{-1}) + 11x(t-10^{-1}) \\ &\quad = \frac{\sin t + \cos t}{2 + x^2 + x^2(t-5^{-1}) + x^2(t-10^{-1}) + x'^2(t-5^{-1}) + x'^2(t-10^{-1}) + x''^2}, \end{aligned} \tag{2.8}$$

where $\tau_1 = 5^{-1}$ and $\tau_2 = 10^{-1}$ are constant delays.

From DDE (2.8)*, we have the system:*

$$\begin{aligned} x' &= y, y' = z, \\ z' &= -\left(4 + \frac{1}{1+y^2}\right) z - (8y + 2\sin y) - 11x(t-5^{-1}) - 11x(t-10^{-1}) \\ &\quad + 4\int_{t-5^{-1}}^{t} z(s)ds + \int_{t-5^{-1}}^{t} \cos y(s)z(s)ds + 4\int_{t-10^{-1}}^{t} z(s)ds + \int_{t-10^{-1}}^{t} \cos y(s)z(s)ds \\ &\quad + \frac{\sin t + \cos t}{2 + x^2 + ... + y^2(t-10^{-1}) + z^2}. \end{aligned} \tag{2.9}$$

Comparing the systems of DDEs (2.9) *and* (1.2) *and applying some elementary calculations, we have the following relations:*

$$\begin{aligned} &\psi(y) = 4 + \frac{1}{1+y^2}, \quad 0 \le \psi(y) - 4 \le 1, a = 4, \delta = 1, \\ &g_1(y) = 4y + \sin y, \quad g_1(0) = 0, \\ &y^{-1}g_1(y) = 4 + y^{-1}\sin y, \quad b_1 = 3, \\ &g_1'(y) = 4 + \cos y, \quad \left|g_1'(y)\right| = |4 + \cos y| \le 5 = L_1, \\ &g_2(y) = 4y + \sin y, \quad g_2(0) = 0, y^{-1}g_2(y) = 4 + y^{-1}\sin y, \quad b_2 = 3, \\ &g_2'(y) = 4 + \cos y, \quad \left|g_2'(y)\right| = |4 + \cos y| \le 5 = L_2, \\ &f_1(x) = 11x, \quad f_1(0) = 0, \quad f_1(x)sgnx = 11xsgnx > 0, \quad (x \ne 0), \\ &f_1(x)sgnx = 11xsgnx \to \infty, \ as \ |x| \to \infty, \quad f_1'(x) = 11, c_1 = 11, \\ &f_2(x) = 11x, \quad f_2(0) = 0, \ f_2(x)sgnx = 11xsgnx > 0, \quad (x \ne 0), \\ &f_2(x)sgnx = 11xsgnx \to \infty \ as \ |x| \to \infty, \quad f'{}_2(x) = 11, \\ &c_2 = 11, \quad c = c_1 + c_2 = 22, \quad b = b_1 + b_2 = 3 + 3 = 6, \\ &ab - c = 2 > 0, \quad h = \frac{ab + 3c}{2(ab+c)} = \frac{90}{92} < 1, \\ &p(t, x, x(t-5^{-1}), ..., y, ..., y(t-10^{-1}), z) = \frac{\sin t + \cos t}{2 + x^2 + ... + y^2(t-10^{-1}) + z^2} \le 1 = M_0, \\ &p(t+2\pi, x, x(t-5^{-1}), ..., y, ..., y(t-10^{-1}), z) = \frac{\sin(t+2\pi) + \cos(t+2\pi)}{2 + ... + z^2} = \frac{\sin t + \cos t}{2 + ... + z^2}, \ T = 2\pi. \end{aligned}$$

By noting the above calculations, we observe that DDE (1.1) *satisfies that all the conditions of Theorem 2.1. Then, DDE* (2.8) *has at least one $T = 2\pi$ -periodic solution.*

3. Conclusion

In this article, a certain nonlinear DDE of third order with multiple constant delays was taken into consideration. A new theorem that has sufficient conditions on the EPSs of the considered DDE was proved. The LKF approach is utilized as the technique in the proof of the theorem. As for the numerical application, in a particular case of DDE (1.1), an example was introduced that satisfies all the conditions of the main result of the article. The new main result of the article has a contribution to the former results those are available in the references of this article and literature and it may be useful to researchers. As for possible future works and the impact of this work in this field of research, fractional models and integro-differential equations versions of DDE (1.1) are suggested to investigate the EPSs and some various qualitative behaviors of their solutions.

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