Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 11, No. 1, 2023, pp. 42-51
DOI:10.22034/cmde.2022.50833.2109

A robust numerical scheme for singularly perturbed delay parabolic initial-boundary-value problems involving mixed space shifts

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Abstract

This article proposes a parameter uniform numerical method for solving a singularly perturbed delay parabolic initial-boundary-value problem involving mixed space shifts. The model also involves a large delay in time. Taylor's series expansion is applied to approximate the retarded terms in the spatial direction. For the time discretization, the implicit trapezoidal scheme is applied on uniform mesh, and for the spatial discretization, we use a proper combination of the mid-point upwind and the central difference scheme on Shishkin mesh. The proposed scheme provides a second-order convergence rate uniformly with respect to the perturbation parameter. Some comparison results are presented by using the proposed method to support our claim.

Keywords. Time delay parabolic problem, Mixed shifts, Singular perturbation, Boundary layer, Uniform convergence.
2010 Mathematics Subject Classification. 65M06, 65M12, 65M50.

1. Introduction

This article deals with the following singularly perturbed delay parabolic differential-difference equation (SPPDDE) with a large delay in time:

$$\begin{cases} z_s + L_\varepsilon z(x,s) = -f(x,s)z(x,s-\delta) + g(x,s) \text{ for } (x,s) \in \digamma, \\ z\big|_{\Gamma_d} = \Phi_d(x,s),\ z\big|_{\Gamma_l} = \Phi_l(x,s),\ z\big|_{\Gamma_r} = \Phi_r(x,s). \end{cases} \tag{1.1}$$

Here, $L_\varepsilon z = -\varepsilon z_{xx}(x,s) + a(x)z_x(x,s) + b(x)z(x,s) + c(x)z(x-\xi,s) + d(x)z(x+\eta,s)$, $\xi = \eta = o(\varepsilon)$ are the delay and the advance parameters in space, respectively. $\delta > 0$ is the delay parameter which is comapratively large in temporal direction. $\mathfrak{T}$ is assumed to be $\mathfrak{T} = \kappa\delta$ for κ in $\mathbb{N}$. The domains are defined as follows: $\Omega_s = [-\delta, \mathfrak{T}]$, $\Omega_x = [0,1]$, $\overline{\digamma} = \digamma \cup \partial\digamma$, where $\digamma = (0,1) \times (0,\mathfrak{T}]$ and $\partial\digamma = \Gamma_d \cup \Gamma_l \cup \Gamma_r$, with $\Gamma_d = [0,1] \times [-\delta, 0], \Gamma_l = [-\xi, 0] \times (0,\mathfrak{T}]$ and $\Gamma_r = [1, 1+\eta] \times (0,\mathfrak{T}]$. We assume $a(x)$, $b(x)$, $c(x)$, $d(x)$, $f(x,s)$, $g(x,s)$, $\Phi_d(x,s)$, $\Phi_l(x,s)$, $\Phi_r(x,s)$ are sufficient smooth, bounded functions, and independent of ε.

The literature for SPPDDE is quite large. To mention a few: Bansal and Sharma in [1] used a parameter uniform numerical approach based on non-standard finite difference methods for the solution of SPPDDEs. Daba and Dussera [2] described a numerical approximation of a similar kind of model problem using the implicit Euler scheme in time and a cubic B-spline collocation method in space after the application of Taylor's series expansion to the shift terms. Kumar and Kadalbajoo [8] approximated a model problem by using a parameter uniform numerical approach comprised of the standard implicit Euler scheme employing Rothe's method in time and B-spline collocation method in space. [9] described a stable finite difference approximation providing much better approximation than the conventional methods. Lange and Miura [10, 11] carried out a series of investigations on constructing approximate solutions for the existence and uniqueness of singularly perturbed boundary value problems. At the same time, a higher order method was used

Received: 17 March 2022 ; Accepted: 01 June 2022.
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in [12] using Shishkin mesh for solving singularly perturbed problems with space delay. The authors in [13] described a hybrid scheme consisting of the cubic spline in fine mesh, and the mid-point upwind scheme in the coarse mesh region. In [15], the authors used a hybrid numerical approach that combines the implicit Euler in time and the combined finite difference scheme made out of the midpoint upwind and the central difference scheme in space. In [16], a mesh is constructed so that the terms containing shift lie on nodal points after discretization. It further studied the effect of shift on the boundary layer or oscillatory behavior of the solution via a finite difference approach. Almost all the above methods considered space delay and advance terms, however, do not deal with time delay term.

Feedback control systems where time is required to sense the previous information and to act accordingly form time delay models. One can refer to [21], which describes a furnace used to process metal sheets. In this process, the delay occurs due to the finite speed of the controller. In the recent past, several numerical approaches are proposed for SPPs have a time delay. Govindarao and Mohapatra in [6, 7] used hybrid numerical schemes whereas in [5], a fourth-order scheme is proposed to solve the problems arising in population dynamics. In [4], the authors proposed an efficient numerical scheme for SPDDEs containing two small parameters using the implicit Euler scheme for the time and an upwind scheme on the spatial direction. In [14], the NSFD scheme is used. In [18, 19], the authors used the hybrid scheme in the spatial direction on the Shishkin mesh and the implicit Euler scheme on a uniform mesh in the temporal direction. But most of these article dealt with time delay only and no space shift/delay in the model equations.

Up to now, there exists one article [20] in literature, describing the model SPPDDEs having both, a large time lag along with space delay and advance terms which deal with a first-order scheme. In this article, a more accurate and almost second-order scheme is proposed to deal with such specific model (1.1) which is the main contribution of this work.

2. Preliminaries

When the delay and advance terms associated with the spatial variables are of $o(\varepsilon)$, one can assume $\xi = \mu_1\varepsilon$ and $\eta = \mu_2\varepsilon$ where μ_1 and μ_2 are of $o(1)$. Taylor's series expansion is applied to approximate the retarded terms in spatial direction. So, we have

$$\begin{cases} z(x-\xi,s) \approx z(x,s) - \xi z_x(x,s) + \dfrac{\xi^2}{2} z_{xx}(x,s), \\ z(x+\eta,s) \approx z(x,s) + \eta z_x(x,s) + \dfrac{\eta^2}{2} z_{xx}(x,s). \end{cases} \tag{2.1}$$

Now for $(x,s) \in F$, the use of (2.1) converts (1.1) to

$$\begin{cases} Lz(x,t) = z_s + L_{\widetilde{\varepsilon}} z(x,s) = -f(x,s)z(x,s-\delta) + g(x,s), \\ z\big|_{\Gamma_d} = \Phi_d(x,s), \\ z(0,s) = \Phi_l(0,t),\ z(1,s) = \Phi_r(1,s),\ \text{for } s \in \Omega_t. \end{cases} \tag{2.2}$$

Here,

$$L_{\widetilde{\varepsilon}} z(x,s) = -\widetilde{\varepsilon} z_{xx}(x,s) + \big(a(x) - \xi c(x) + \eta d(x)\big) z_x(x,s) + \big(b(x) + c(x) + d(x)\big) z(x,s).$$

Denote $\widetilde{\varepsilon} = \big(\varepsilon - \frac{\xi^2}{2}c(x) - \frac{\eta^2}{2}d(x)\big)$. The choice of smaller ξ and η makes (2.2) a better approximation for (1.1), so we can have $z_s + L_{\widetilde{\varepsilon}} z(x,t) \approx L$ with error of $O(\xi^3, \eta^3)$. Assume that

$$\big(b(x) + c(x) + d(x)\big) > \gamma > 0, \quad c(x) > 2\xi_1 > 0, \quad d(x) > 2\xi_2 > 0 \quad \forall\ x \in \Omega_x.$$

It is evident that, when $\widetilde{\varepsilon} > 0$ and $\big(a(x) - \xi c(x) + \eta d(x)\big) > 2\gamma_1 > 0$, the problem exhibits a layer at right hand side. Similarly, when $\widetilde{\varepsilon} > 0$ and $\big(a(x) - \xi c(x) + \eta d(x)\big) \le -2\gamma_2 < 0$, the layer will be at left hand side of the domain. Here, γ_1, γ_2 and ξ_1, ξ_2 are real constants.

The choice of smaller values to ξ and η shall ensure the layer behaviour of the solution to (2.2), refer [10, 11]. Here, the discussion is made only for the case when the layer appearing at right hand side. A similar numerical approach can be proposed for the case with left hand side layer. As there is a time delay in (2.1), $\Omega_s = [-\delta, k\delta]$. We use $k = 2$

is here. As the solution in $[0,1]\times[0,\delta]$ depends upon the solution in $[0,1]\times[-\delta,0]$, which is known and hence, the analysis is similar to SPP without delay. So, the scheme is proposed in $[0,1]\times[\delta,2\delta]$, where $z(x,s-\delta)$ is the solution in $[0,1]\times[0,\delta]$. For any $k>2$, one can extend this idea. Below, we provide some basic properties which guarantees the existence, uniqueness and stabilty of the solution to the model problem (2.2).

2.1. Properties of the solution.

For the analysis of the proposed scheme, the analytical properties play a vital role. So, we provide here the maximum principle and the stability results for the differential operator which are used later.

Lemma 2.1. *(Maximum principle) For sufficiently smooth function $\chi(x,s)$ satisfying $\chi(x,s)\geq 0,\ \forall(x,s)\in\Gamma=\Gamma_d\cup\Gamma_l\cup\Gamma_r$ and $\chi_s(x,s)+L_{\tilde{\varepsilon}}\chi(x,s)\geq 0\ \forall(x,s)\in F$, then $\chi(x,s)\geq 0,\ \forall(x,s)\in\overline{F}$.*

Lemma 2.2. *Let $\|.\|$ denotes the standard maximum norm. The derivatives of $z(x,s)$ satisfy the following bound:*

$$\left\|\frac{\partial z^k(x,s)}{\partial x^k}\right\|\leq C\Big(1+\tilde{\varepsilon}^{-k}\exp(-\gamma(1-x)/\tilde{\varepsilon})\Big). \tag{2.3}$$

Proof. The proof of these above two lemmas are avilable in [20]. □

3. Finite difference scheme

For the formulation of the numerical scheme, we use uniform mesh in the time domain with step size Λs. The partition of $\overline{F_s}=[-\delta,2\delta=\mathfrak{T}]$ is given as,

$$\begin{aligned} F_s^{\mathcal{P}} &= \{s_n=n\Lambda s, n=0,1\ldots,\mathcal{P}, s_P=0, \Lambda s=\delta/\mathcal{P}\},\\ F_s^{\mathcal{L}} &= \{s_n=n\Lambda s, n=0,1\ldots,\mathcal{L}, s_L=\mathfrak{T}, \Lambda s=\mathfrak{T}/\mathcal{L}\},\end{aligned}$$

where $\mathcal{L}$ and $\mathcal{P}$ are the number of partitions in $[0,\mathfrak{T}]$ and $[-\delta,0]$, respectively. So in total, we have $\mathcal{N}=(\mathcal{L}+\mathcal{P})$ number of partitions in $[-\delta,2\delta]$. To discretize time in (2.2), the implicit Euler method is used, which is given by,

$$L^{\mathcal{N}}z\cong z_s(x,s_{n+0.5})-\tilde{\varepsilon}z_{xx}(x,s_{n+0.5})+\big(a(x)-\xi c(x)+\eta d(x)\big)z_x(x,s_{n+0.5})+\big(b(x)+c(x)+d(x)\big)z(x,s_{n+0.5}) \tag{3.1}$$

$$=-f(x,s_{n+0.5})z(x,s_{n+0.5}-\delta)+g(x,s_{n+0.5}), \tag{3.2}$$

subjected to the conditions:

$$\begin{cases} z\big|_{\Gamma_d}=\Phi_d(x,s_{n+0.5}), z\big|_{\Gamma_l}(x,s_{n+0.5})=\Phi_l(s_{n+0.5}),\\ z\big|_{\Gamma_r}(x,s_{n+0.5})=\Phi_r(s_{n+0.5}).\end{cases} \tag{3.3}$$

Using Taylor's expression about $(x,s_{n+0.5})$ for solutions at s_{n+1} and s_n time levels, we have

$$z(x,s_{n+1})=z(x,s_{n+0.5})+\frac{\Lambda s}{2}z_s(x,s_{n+0.5})+\frac{1}{2!}\left(\frac{\Lambda s}{2}\right)^2 z_{ss}(x,s_{n+0.5})+\frac{1}{3!}\left(\frac{\Lambda s}{2}\right)^3 z_{sss}(x,s_{n+0.5})+\ldots, \tag{3.4}$$

$$z(x,s_n)=z(x,s_{n+0.5})-\frac{\Lambda s}{2}z_s(x,s_{n+0.5})+\frac{1}{2!}\left(\frac{\Lambda s}{2}\right)^2 z_{ss}(x,s_{n+0.5})-\frac{1}{3!}\left(\frac{\Lambda s}{2}\right)^3 z_{sss}(x,s_{n+0.5})+\ldots \tag{3.5}$$

Subtracting (3.5) from (3.4) we have,

$$\begin{aligned} z_s(x,s_{n+0.5}) &= \frac{z(x,s_{n+1})-z(x,s_n)}{\Lambda s}+T_E,\\ T_E &= -\frac{\Lambda s^2}{4!}z_{sss}(x,s_{n+0.5}).\end{aligned} \tag{3.6}$$

The error E of time discretization is bounded in $\overline{\Omega_s}$ and is given by,

$$\|E\|_\infty\leq C\Lambda s^2.$$

A similar argument is given in [19]. Using (3.6) and (3.1) and on further arrangement, we have

$$\begin{aligned} L^N z &= -\big(\varepsilon - \frac{\xi^2}{2}c(x) - \frac{\eta^2}{2}d(x)\big)z_{xx}(x,s_{n+1}) + \big(a(x) - \xi c(x) + \eta d(x)\big)z_x(x,s_{n+1}) \\ &+ \big(b(x) + c(x) + d(x) + \frac{2}{\Lambda s}\big)z(x,s_{n+1}) \\ &= -f(x,s_{n+1})z(x,s_{n+1}-\delta) + g(x,s_{n+1}) + \big(\varepsilon - \frac{\xi^2}{2}c(x) - \frac{\eta^2}{2}d(x)\big)z_{xx}(x,s_n) \\ &- \big(a(x) - \xi c(x) + \eta d(x)\big)z_x(x,s_n) - \big(b(x) + c(x) + d(x) - \frac{2}{\Lambda s}\big)z(x,s_n) - f(x,s_n)z(x,s_n-\delta) + g(x,s_n). \end{aligned} \tag{3.7}$$

Let $\mathcal{M}$ be the number of even partitions in space. The transition parameter ϱ is defined as

$$\varrho = \min\left(\frac{1}{2}, \frac{2}{\xi}(\varepsilon - \xi^2\xi_1 - \eta^2\xi_2)\ln\mathcal{M}\right).$$

The domain $\overline{\Omega}_x$ is divided into two equal sub-domains *i.e.*, $[0, 1-\varrho]$ and $[1-\varrho, 1]$. Now

$$\overline{\Omega}_x = \{x_0 = 0, x_1, x_2, \ldots, x_{\mathcal{M}/2} = 1-\varrho, \ldots x_{\mathcal{M}} = 1\}.$$

Let $x_m = mh_m$ with $h_m = x_m - x_{m-1}$. The standard Shishkin mesh (S-mesh) is denoted by:

$$x_m = \begin{cases} m\dfrac{2(1-\varrho)}{\mathcal{M}}, & \text{if } m = 1, 2, \ldots, \dfrac{\mathcal{M}}{2}, \\ (1-\varrho) + \dfrac{2\varrho}{\mathcal{M}}(m - \mathcal{M}/2), & \text{if } m = \dfrac{\mathcal{M}}{2} + 1, \ldots, \mathcal{M} - 1. \end{cases}$$

One may refer [3] for more details on S-mesh. $\digamma^{M,N}$ is defined as the discretized form of $\digamma$ with $\mathcal{M}$ and $\mathcal{N}$ number of mesh points in spatial and temporal directions, repectively. The operators are defined as

$$\begin{aligned} \delta_x^2 Z_m^n &= \frac{2}{h_m + h_{m+1}}\big(\delta_x^+ Z_m^n - \delta_x^- Z_m^n\big), \\ \delta_x^- Z_m^n &= \frac{Z_m^n - Z_{m-1}^n}{h_m}, \\ \delta_x^0 Z_m^n &= \frac{Z_{m+1}^n - Z_{m-1}^n}{h_m + h_{m+1}}. \end{aligned} \tag{3.8}$$

The fullydiscrete form is a proper combination of the mid-point upwind and the central difference scheme in space, on $\digamma^{M,N}$ at (x_m, s_{n+1}) is given as

$$\begin{cases} L_h^{\mathcal{M},\mathcal{N}} Z_m^{n+1} \cong \tilde{F}, \\ Z_m^{-n} = \Phi_d(x_m, -s_n)\ n = 0, \ldots, P,\ m = 1, \ldots, \mathcal{M}-1, \\ Z_0^{n+1} = \Phi_l(s_{n+1}),\ Z_M^{n+1} = \Phi_r(s_{n+1})\ \forall\ s \in \Omega_t, \end{cases} \tag{3.9}$$

where,

$$L_h^{\mathcal{M},\mathcal{N}} Z_m^{n+1} \cong \begin{cases} L_{mid}^{M,N} Z_m^{n+1} & \text{if } 1 < m \leq \dfrac{\mathcal{M}}{2}, \\ L_{cen}^{M,N} Z_m^{n+1}, & \text{if } \dfrac{\mathcal{M}}{2} < m < \mathcal{M}. \end{cases}$$

Here,

$$\begin{aligned} L_{mid}^{\mathcal{M},\mathcal{N}} Z_m^{n+1} = -\tilde{\varepsilon}_m \xi_x^2 Z_m^{n+1} &+ \big(a(x_{m-1/2}) - \xi c(x_{m-1/2}) + \eta d(x_{m-1/2})\big)\xi_x^- Z_m^{n+1} \\ &+ \big(b(x_{m-1/2}) + c(x_{m-1/2}) + d(x_{m-1/2}) + \frac{2}{\Lambda s}\big) Z_{m-1/2}^{n+1} \end{aligned}$$

and

$$L_{cen}^{M,N} Z_m^{n+1} = -\tilde{\varepsilon}_m \xi_x^2 Z_m^{n+1} + \big(a(x_m) - \xi c(x_m) + \eta d(x_m)\big)\xi_x^0 Z_m^{n+1} + \big(b(x_m) + c(x_m) + d(x_m) + \frac{2}{\Lambda s}\big) Z_m^{n+1}.$$

Also,

$$\tilde{F} = \begin{cases} -f_{m-1/2}^{n+1} Z_m^{n-p+1} + g_{m-1/2}^{n+1} - f_{m-1/2}^{n} Z_m^{n-p} + g_{m-1/2}^{n} - L_{mid}^{\mathcal{M},\mathcal{N}} Z_m^n, & \text{if} \quad 1 < m \le \dfrac{\mathcal{M}}{2}, \\ -f_m^{n+1} Z_m^{n-p+1} + g_m^{n+1} - f_m^n Z_m^{n-p} + g_m^n - L_{cen}^{M,N} Z_m^n, & \text{if} \quad \dfrac{\mathcal{M}}{2} < m < \mathcal{M}, \end{cases}$$

for time levels s_n and s_{n+1}, $\widetilde{\varepsilon}_m = \big(\varepsilon - \frac{\xi^2}{2} c(x_m) - \frac{\eta^2}{2} d(x_m)\big)$, $f_{m-1/2} = \dfrac{f_m + f_{m-1}}{2}$, $Z_{m-1/2} = \dfrac{Z_m + Z_{m-1}}{2}$, with similar definitions for $g_{m-1/2}$, $a_{m-1/2}$, $b_{m-1/2}$, $c_{m-1/2}$ and $d_{m-1/2}$. After simplification, we have the following system of equations

$$\begin{cases} A_m^- Z_{m-1}^{n+1} + A_m^c Z_m^{n+1} + A_m^+ Z_{m+1}^{n+1} = \tilde{F}, \\ Z_m^{-n} = \Phi_d(x_m, -s_n), \ n = 0, \dots, \mathcal{P}, \ m = 1, \dots, \mathcal{M} - 1, \\ Z_0^{n+1} = \Phi_l(s_{n+1}), \ Z_M^{n+1} = \Phi_r(s_{n+1}) \ \forall (x, s) \in F^{M,N}. \end{cases} \tag{3.10}$$

The coefficients for $1 < m \le \dfrac{\mathcal{M}}{2}$ are given by

$$\begin{cases} A_m^+ = \left[\dfrac{-2\widetilde{\varepsilon}_m}{\hbar_m h_{m+1}}\right], \\ A_m^c = \dfrac{1}{\Lambda s} + \left[\dfrac{2\widetilde{\varepsilon}_m}{h_{m+1} h_m}\right] + \left[\dfrac{a_{m-1/2} - \xi c_{m-1/2} + \eta d_{m-1/2}}{h_m}\right] + \left[\dfrac{b_{m-1/2} + c_{m-1/2} + d_{m-1/2}}{2}\right], \\ A_m^- = \dfrac{1}{\Lambda s} - \left[\dfrac{2\widetilde{\varepsilon}_m}{\hbar h_m}\right] - \left[\dfrac{a_{m-1/2} - \xi c_{m-1/2} + \eta d_{m-1/2}}{h_m}\right] + \left[\dfrac{b_{m-1/2} + c_{m-1/2} + d_{m-1/2}}{2}\right], \end{cases}$$

and for $\dfrac{\mathcal{M}}{2} < m \le \mathcal{M}$ are given by

$$\begin{cases} A_m^+ = -\left[\dfrac{2\widetilde{\varepsilon}_m}{\hbar_m h_m}\right] + \left[\dfrac{a_m - \xi c_m + \eta d_m}{\hbar}\right], \\ A_m^c = \dfrac{2}{\Lambda s} + \left[\dfrac{2\widetilde{\varepsilon}_m}{h_{m+1} h_m}\right] + [b_m + c_m + d_m], \\ A_m^- = -\left[\dfrac{2\widetilde{\varepsilon}_m}{\hbar_m h_m}\right] - \left[\dfrac{a_m - \xi c_m + \eta d_m}{\hbar_m}\right]. \end{cases}$$

where $\hbar_m$ is denoted as: $\hbar_m = h_m + h_{m+1}$. To solve the matrix formed on (3.10), we prefer to use the idea of Thomas algorithm [17] which takes less computational time compared to the usual matrix inversion method. In order to attain the stability while using the central difference scheme, the following mild condition is assumed:

$$\frac{\mathcal{M}}{\ln \mathcal{M}} \ge 2 \frac{\max\big(a(x) - \xi c(x) + d(x)\big)}{\min\big(a(x) - \xi c(x) + d(x)\big)}. \tag{3.11}$$

One can find the following error estimate for the proposed numerical scheme on $F^{M,N}$.

Proposition 3.1. *Let z and $\mathbb{Z}$ be the solutions of (2.2) and (3.9), respectively on S-mesh. Then the error bounds at time level s_n is given by*

$$\big|\mathbb{Z}_m^n - z(x_m, s_n)\big| \le C(\mathcal{M}^{-2} \ln^2 \mathcal{M} + (\Lambda s)^2).$$

4. Numerical experiments

The following illustrative example is provided to show the efficacious of our proposed scheme.

Example 4.1. *Consider the time delay SPPDDEs with* $(x,s) \in (0,1) \times (0,2]$ *and*

$$a(x) = 2 - x^2, \qquad b(x) = x - 3, \qquad c(s) = 1,$$
$$d(s) = 2, \qquad f(x,s) = 1, \qquad g(x,s) = 10s^2 exp(-s)x(1-x),$$

subject to $\Phi_d(x,s) = 0,\ \Phi_l(x,s) = 0,\ \Phi_r(x,s) = 0$

The exact solution for the test problem is unknown. The point-wise errors and the rates of convergence are calculated using the double mesh principle. In this process, the problem is solved using the proposed scheme on $F^{2M,2N}$.

The maximum point-wise errors and rates of convergence are computed as,

$$\mathcal{E}_\varepsilon^{\mathcal{M},\Lambda s} = \max_{(x_m,\ s_n)\ \in\ F} \left|\mathbb{Z}^{\mathcal{M},\mathcal{N}}(x_i,\ s_j) - \mathbb{Z}^{\in\mathcal{M},\in\mathcal{N}}(x_i,\ s_j)\right|,$$

$$\mathcal{R}_\varepsilon^{M,\Lambda s} = \log_2\left(\frac{\mathcal{E}_\varepsilon^{\mathcal{M},\Lambda s}}{\mathcal{E}_\varepsilon^{2\mathcal{M},\Lambda s/2}}\right).$$

For computational purposes, $\delta = 1$ is considered. Figure 1 shows the existence of the layer on the right of the domain. It also confirms the sharpness of the layer as ε varies from 1 to 10^{-5}. Figure 2 describes the solution profile at different time zones for given values of ξ and η. Figure 3 depicts the solution profile whereas Figure 4 shows the log log plots for different values of the perturbation parameter. Table 1 consisting of $\mathcal{E}_\varepsilon^{\mathcal{M},\Lambda s}$ and $\mathcal{R}_\varepsilon^{M,\Lambda s}$ which are in agreement of the proposed theoretical bounds. Table 2 shows the maximum point-wise error and the coressponding rate of convergence at fixed time levels with varying values $\mathcal{M}$ and ε. Similarly, Table 3 reflects the maximum point-wise error and the coressponding rate of convergence at different time levels for a fixed values of ε. Hence, from these results, the proposed scheme is shown to be uniformly convergent and robust, providing almost a second order rate of convergence both in space and time variables.

5. Conclusion

An efficient second-order numerical scheme is proposed for solving SPPDDEs having a large time lag in time. The delay and the advance terms in spatial direction are approximated by Taylor's approximation. The temporal direction is dealt with by the implicit trapezoidal scheme. Then, the spatial derivative terms are treated with a hybrid scheme comprising the midpoint upwind and the central difference scheme. Numerical experiments are performed and results are reported for validation of the theoretical error estimates.

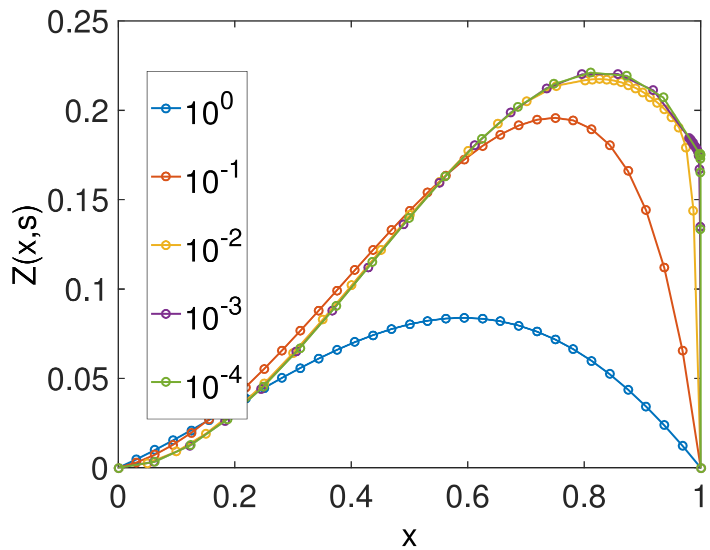

FIGURE 1. Layer behavior with M=32 at s = 1, $\xi = 0.2 * \varepsilon$ and $\eta = 0.4 * \varepsilon$ for Example 4.1

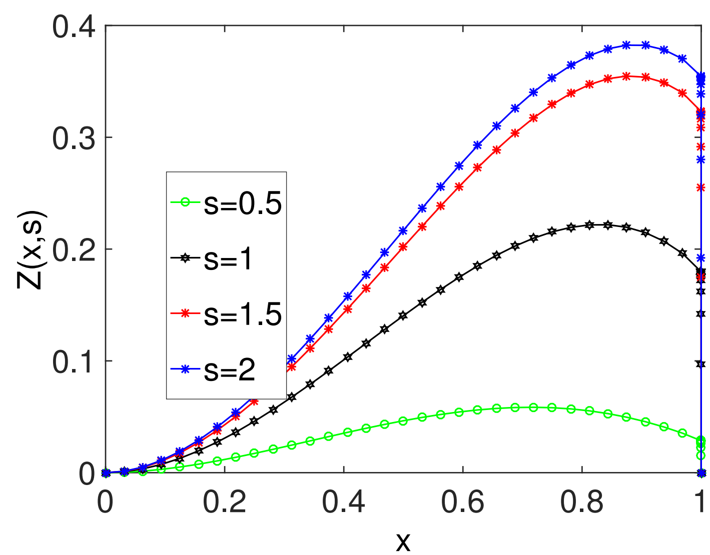

FIGURE 2. Solutions at different time level with $\varepsilon = 10^{-6}$, $\xi = 0.5 * \varepsilon$ and $\eta = 0.8 * \varepsilon$

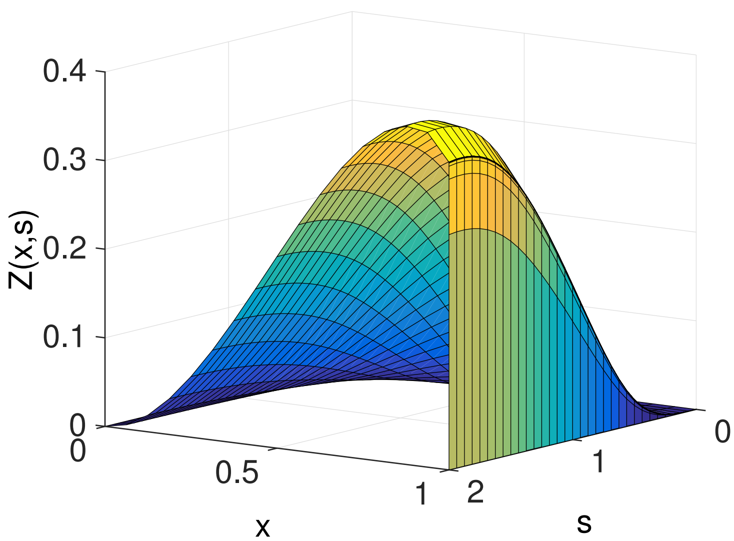

FIGURE 3. Numerical solution profile for Example 4.1 with $\varepsilon = 10^{-4}$, $\xi = 0.5 * \varepsilon$ and $\eta = 0.6 * \varepsilon$

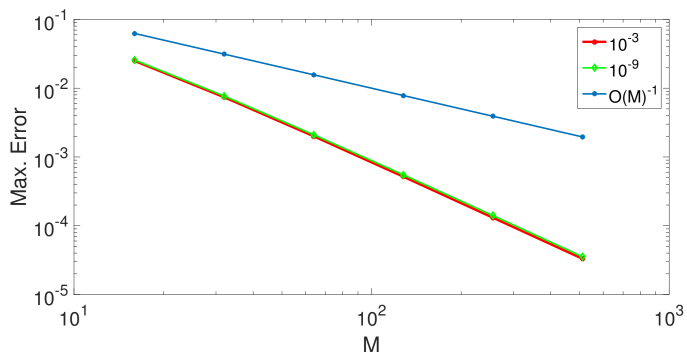

FIGURE 4. Loglog plots with $\xi = 0.4 * \varepsilon$ and $\eta = 0.2 * \varepsilon$ for Example 4.1

TABLE 1. Numerical results with $\xi = \eta = 0.1 * \varepsilon$ for Example 4.1

ε	Method	Number of partitions in space/ temporal mesh size(Λs)					
		$16/\frac{1}{8}$	$32/\frac{1}{16}$	$64/\frac{1}{32}$	$128/\frac{1}{64}$	$256/\frac{1}{128}$	$512/\frac{1}{256}$
10^{-3}	Scheme in [20]	4.1480e-2	2.8933e-2	2.0069e-2	1.3166e-2	8.1760e-3	4.8525e-4
		0.5196	0.5277	0.6081	0.6873	0.7526	
	Proposed scheme	2.9726e-2	1.4681e-2	5.0074e-3	1.6244e-3	5.1460e-4	1.5874e-4
		1.0178	1.5518	1.6242	1.6584	1.6968	
10^{-5}	Scheme in [20]	4.1687e-2	2.9042e-2	2.0150e-2	1.3216e-2	8.2042e-3	4.8694e-3
		0.5214	0.5273	0.6084	0.6878	0.7526	
	Proposed scheme	2.9660e-2	1.4702e-2	5.0373e-3	1.6758e-3	5.4486e-4	1.7267e-4
		1.0125	1.5453	1.5878	1.6209	1.6579	
10^{-7}	Scheme in [20]	4.1689e-2	2.9044e-2	2.0151e-2	1.3216e-2	8.2045e-3	4.8695e-3
		0.5214	0.5273	0.6085	0.6878	0.7526	
	Proposed scheme	2.9660e-2	1.4702e-2	5.0375e-3	1.6762e-3	5.4572e-4	1.7433e-4
		1.0125	1.5452	1.5875	1.6190	1.6463	
10^{-9}	Scheme in [20]	4.1689e-2	2.9044e-2	2.0151e-2	1.3216e-2	8.2045e-3	4.8695e-3
		0.5214	0.5273	0.6085	0.6878	0.7526	
	Proposed scheme	2.9660e-2	1.4702e-2	5.0375e-3	1.6762e-3	5.4573e-4	1.7434e-4
		1.0125	1.5452	1.5875	1.6189	1.6463	

TABLE 2. Numerical results with and $\delta = 0.6 * \varepsilon$, $\eta = 0.3 * \varepsilon$ for Example 4.1

	Number of mesh intervals in space($\mathcal{M}$)					
$\varepsilon \downarrow$	16	32	64	128	256	512
10^{-1}	4.3880e-3	8.6736e-4	4.3958e-4	2.2146e-4	1.1123e-4	5.5744e-5
	2.3389	0.9805	0.9890	0.9934	0.9967	
10^{-3}	2.3980e-2	1.0320e-3	3.8499e-3	1.3270e-3	4.4375e-4	1.4430e-4
	1.2163	1.4226	1.5367	1.5803	1.6207	
10^{-5}	2.6078e-2	9.5397e-3	3.6596e-3	1.2648e-3	4.2444e-4	1.3905e-4
	1.4508	1.3823	1.5327	1.5753	1.6100	
10^{-7}	2.6100e-2	9.5314e-3	3.6575e-3	1.2641e-3	4.2421e-4	1.3898e-4
	1.4533	1.3818	1.5327	1.5753	1.6099	

TABLE 3. Numerical results at different time levels with $\Lambda t = 1/40$, $\varepsilon = 10^{-3}$, $\delta = 0.2 * \varepsilon$, and $\eta = 0.6 * \varepsilon$ for Example 4.1

	Number of mesh intervals in space($\mathcal{M}$)					
$t \downarrow$	16	32	64	128	256	512
0.8	9.2778e-3	2.9525e-3	1.0680e-3	3.6525e-4	1.1569e-4	3.5569e-5
	1.6518	1.4671	1.5479	1.6586	1.7016	
1.2	1.9100e-2	6.3525e-3	2.5071e-3	8.6846e-3	2.8692e-4	8.8675e-5
	1.5882	1.3413	1.5295	1.5978	1.6941	
1.6	2.3911e-2	9.3857e-3	3.5916e-3	1.2452e-3	4.1909e-4	1.3798e-4
	1.3491	1.3858	1.5283	1.5710	1.6028	
2	2.2840e-2	1.0344e-2	3.8529e-3	1.3301e-3	4.4815e-4	1.4914e-4
	1.1428	1.4247	1.5344	1.5695	1.5873	

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