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Linear B-spline finite element Method for solving delay reaction diffusion equation

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Abstract

This paper is concerned with the numerical treatment of delay reaction-diffusion with the Dirichlet boundary condition. The finite element method with linear B-spline basis functions is utilized to discretize the space variable. The Crank-Nicolson method is used for the processes of time discretization. Sufficient and necessary conditions for the numerical method to be asymptotically stable are investigated. The convergence of the numerical method is studied. Some numerical experiments are performed to verify the applicability of the numerical method.

Keywords. Delay reaction diffusion equation, Crank Nicolson, Linear B-spline, Finite element method, Asymtotic stability, Convergence.
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1. Introduction

In this paper, we consider a class of the delay reaction-diffusion equation of the form [20]

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = a_1 \frac{\partial^2 u(x,t)}{\partial x^2} + a_2 u\left(x, t-\tau\right), t > 0, x \in \Omega := [0, \pi], \\ u(x,t) = \varphi\left(x,t\right), -\tau \leq t \leq 0, x \in \Omega, \\ u(0,t) = u(\pi,t) = 0, t > 0, \end{cases} \tag{1.1}$$

with $a_1, a_2 \in \mathbb{R}$ with $a_1 > 0$ and $\tau > 0$ is a delay constant.
Reaction-diffusion equations with delay are widely applied to model natural phenomena in many areas of sciences [15, 19, 30, 33, 35, 36, 39]. Numerous types of numerical methods are available in the literature for solving delay reaction-diffusion equations (for details one may refer to [1, 14, 20, 25, 34, 40]). As far as we know, some of the numerical methods available to approximate the diffusion term are based on the classical numerical methods, such as finite difference method [32], finite element method [21], spectral method [7], Haar Wavelet [5], variational method [9], and so on. Several types of partial differential equations are solved using finite element method [12, 13, 24]. The use of various degrees of B-spline functions to obtain the numerical solutions of some partial differential equations has been shown to provide easy and simple algorithms, for instance, B-spline finite elements have been widely applied to solve elliptic equations [17, 28], Korteweg De Vries equation [2, 3, 37, 38], Burgers equation [4, 16, 23, 29], Regularized Long Wave equation [10, 27], Fokker-Planck equation [18], advection-diffusion equation [11], and generalized equal width wave equation [6], etc., successfully. On the other hand, the application of B-spline finite element method for solving reaction-diffusion with delay receives less attention. In this paper, we have applied a linear B-spline finite element method to find numerical solutions to the problem under consideration.

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Notations: Let $H^r = H^r(\Omega) = \omega_2^r(\Omega)$ denote the Sobolev spaces of order r with respective to norm $\|.\|_r$ defined as

$$\|\nu\| = \|\nu\|_{L_2} := \left(\int_\Omega \nu(x)^2 dx\right)^{\frac{1}{2}}, \tag{1.2}$$

and

$$\|\nu\|_r = \quad \|\nu\|_{H^r} := \left(\sum_{i\leq r} \quad \left\|\frac{\partial^i \nu(x)}{\partial x^i}\right\|^2\right)^{\frac{1}{2}}. \tag{1.3}$$

Let $\nu(x), w(x) (x \in \Omega)$ be real valued functions.

$$(\nu(x), w(x)) := \int_\Omega \nu(x)w(x)dx,\ (\nabla\nu(x), \nabla w(x)) := \int_\Omega \frac{\partial \nu(x)}{\partial x}\frac{\partial w(x)}{\partial x}dx. \tag{1.4}$$

Assumptions: Assume $u(t) := u(.,t), u_t(t) := u_t(.,t), u_{tt}(t) := u_{tt}(.,t), u_{ttt}(t) := u_{ttt}(.,t), \varphi(t) := \varphi\ (.,t)$, and $\varphi_t(t) := \varphi_t(.,t)$.

2. Stability Property of the Continuous Problem

In this section, based on [26], we give a sufficient condition for the trivial solution of the problem to be asymptotically stable.

Definition 2.1. The solution of Eq. (1.1) is called asymptotically stable if the solution $u(x,t)$, of Eq. (1.1), corresponding to a sufficiently smooth function $\varphi(x,t)$ satisfies

$$\lim_{t\to\infty} u(x,t) = 0, \quad x \in [0,\pi]. \tag{2.1}$$

Theorem 2.2. *[26] Given the solution of the form $u(x,t) = e^{st}e^{ikx}$,where $s \in \mathbb{C}$ and $k \in \mathbb{R}$ for $x \in [0,\pi]$ and $t \geq 0$. The sufficient condition for the zero solution of Eq. (1.1) to be asymptotically stable is that*

$$|a_2| < a_1k^2, a_1 > 0, \quad \textit{and} \quad k = n, \quad n = 1, 2, \ldots. \tag{2.2}$$

Proof. The zero solution of Eq. (1.1) is asymptotically stable only if all roots of the characteristic equation

$$s - a_2e^{-s\tau} = -a_1k^2 \tag{2.3}$$

have negative real parts. Substituting $s = \beta + \gamma i, \beta, \gamma \in \mathbb{R}$ in Eq. (2.3) gives

$$(\beta + \gamma i) - a_2e^{-(\beta+\gamma i)\tau} + a_1k^2 = 0. \tag{2.4}$$

Separating real and imaginary parts yields

$$\begin{cases} \beta = a_2e^{-\beta\tau}cos\gamma\tau - a_1k^2 \\ \gamma = a_2e^{-\beta\tau}sin\gamma\tau. \end{cases} \tag{2.5}$$

Assume that $a_1, a_2 \in \mathbb{R}$ and $a_1 > 0$, then β is always negative when

$$a_2cos\gamma\tau < a_1k^2e^{\beta\tau} \Rightarrow |a_2| < a_1k^2. \tag{2.6}$$

That means that when $|a_2| < a_1k^2$, then all zeros of the characteristic Eq. (2.3) have negative real part and hence the trivial solution of Eq. (1.1) is delay -independently asymptotically stable. On the other hand, if $|a_2| > a_1k^2$, there exists a root (2.3) with positive real part for some $\tau > 0$, which implies unstable trivial solution. □

3. Description of the Method

For the spatial discretization of system (1.1), we divide the interval $\Omega = [0, \pi]$ with a mesh: $0 = x_0 < x_1 < \cdots < x_N = \pi$ with the space step size $h = \pi/N$.
The linear B-spline basis functions is chosen as follows:

$$Q_j(x) = \begin{cases} \frac{x-x_{j-1}}{h}, x \in [x_{j-1}, x_j] \\ \frac{x_{j+1}-x}{h}, x \in [x_j, x_{j+1}] \\ 0, \quad x \notin [x_{j-1}, x_{j+1}] \end{cases} \quad j = 1, 2, \ldots, N-1. \tag{3.1}$$

Introduce an arbitrary test function ν. Multiply Eq. (1.1) by this test function and integrate over the domain to obtain the weak formulation

$$\int_\Omega u_t(x,t)\nu dx + a_1 \int_\Omega \frac{\partial u\,(x,t)}{\partial x}\frac{\partial \nu}{\partial x} dx - a_2 \int_\Omega u_(x, t-\tau)\nu dx = 0, \forall \nu \in H_0^1(\Omega), t > 0. \tag{3.2}$$

Equivalently, applying Green's formula to the second term of equation Eq. (1.1) we can also write as

$$(u_t(x,t), \nu) + a_1(\nabla u(x,t), \nabla \nu) - a_2(u(x, t-\tau), \nu) = 0, \forall \nu \in H_0^1(\Omega), t > 0. \tag{3.3}$$

Define the space

$$S_h = \{\zeta : \zeta \in C^2([0,\pi]), \zeta|_{[x_{n-1},x_n]} \in P, 1 \le n \le N, \zeta(0) = \zeta(\pi) = 0\}, \tag{3.4}$$

where P is the space of all polynomials of degree less or equal to 1. We can find the approximate solution $u_h(t) := u_h(.,t)$ belonging to S_h for each t, so that

$$\begin{cases} (u_{h,t}(t,\zeta)) + a_1(\nabla u_h(t), \nabla\zeta) - a_2(u_h(t-\tau), \zeta) = 0, \forall \zeta \in S_h, t > 0, \\ u_h(x,t) = \varphi_h(x,t) = 0, t \ge -\tau, \end{cases} \tag{3.5}$$

where $\varphi_h(.,t)$ is an approximation of $\varphi(.,t)$ in S_h.
Let $\Delta t = \tau/m$ be a given step size with $m \ge 1$, the grid points $t_n = n\Delta t (n = 0, 1, \ldots)$ and U^n be the approximation in S_h of $u(t)$ at $t = t_n = n\Delta t$.
Application of Galerkin Crank-Nicloson method to (1.1) gives a numerical scheme of the following type

$$\left(\frac{U^n - U^{n-1}}{\Delta t}, \zeta\right) + a_1\left(\frac{\nabla U^{n-1} + \nabla U^n}{2}, \nabla\zeta\right) - a_2\left(\frac{U^{n-m-1} + U^{n-m}}{2}, \zeta\right) = 0, \quad \forall \zeta \in S_h, \tag{3.6}$$

where $U^n(.) = \varphi(., t_n)$ for $-m \le n \le 0$.
Let

$$U^n(x) := \sum_{j=1}^{N-1} Q_j(x)\alpha_j^n. \tag{3.7}$$

Substituting Eq. (3.7) into Eq. (3.6) and choosing $\zeta = Q_i, i = 1, \ldots, N-1$, we get

$$\frac{1}{\Delta t}\sum_{j=1}^{N-1}(\alpha_j^n - \alpha_j^{n-1})(Q_i(x), Q_j(x)) = -\frac{a_1}{2}\sum_{j=1}^{N-1}(\alpha_j^n + \alpha_j^{n-1})(\nabla Q_i(x), \nabla Q_j(x)) \tag{3.8}$$

$$+ \frac{a_2}{2}\sum_{j=1}^{N-1}(\alpha_j^{n-m-1} + \alpha_j^{n-m})(Q_i(x), Q_j(x)), \tag{3.9}$$

which can be rewritten as

$$\frac{1}{\Delta t}\sum_{j=1}^{N+1}(\alpha_j^n-\alpha_j^{n-1})\int_0^\pi Q_i(x)Q_j(x)dx = -\frac{a_1}{2}\sum_{j=1}^{N-1}(\alpha_j^n+\alpha_j^{n-1})\int_0^\pi Q_i'(x)Q_j'(x)dx \tag{3.10}$$

$$+\frac{a_2}{2}\sum_{j=1}^{N-1}(\alpha_j^{n-m-1}+\alpha_j^{n-m})\int_0^\pi Q_i(x)Q_j(x)dx. \tag{3.11}$$

Define the following matrices:

$$A=(a_{i,j})_{i,j=1}^{N-1}=\int_0^\pi Q_i'(x)Q_j'(x)dx, \tag{3.12}$$

$$B=(b_{i,j})_{i,j=1}^{N-1}=\int_0^\pi Q_i(x)Q_j(x)dx. \tag{3.13}$$

We can explicitly write the entries of the matrices A and B in Eq. (3.12) and Eq. (3.13) as $A=\frac{1}{h}(2I-S)$ and $B=\frac{h}{6}(4I+S)$,

where I is an identity matrix and

$$S=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0\\ 1 & 0 & 1 & \dots & 0 & 0\\ 0 & 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 0 & 1\\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}_{(N-1)\times(N-1)} \tag{3.14}$$

with its eigenvalues $\lambda_j^S=2cos(j\pi/N), j=1,2,\dots,N-1$.

$$\begin{cases} (B+\frac{1}{2}a_1\Delta tA)\alpha^n=(B-\frac{1}{2}a_1\Delta tA)\alpha^{n-1}+\frac{1}{2}a_2\Delta tB(\alpha^{n-m-1}+\alpha^{n-m}),\\ \alpha^n=\gamma^n,\ for\ -m\le n\le 0, \end{cases} \tag{3.15}$$

with $\gamma^n=\varphi(t_n)$ an initial approximation and $\alpha^n:=[\alpha_1^n,\alpha_2^n,\dots,\alpha_{N-1}^n]^T$, and $B+\frac{1}{2}a_1\Delta tA$ is positive definite and hence, in particular, invertible. Therefore, α^n can be obtained recursively by using the matrix inversion method.

4. Stability Analysis

In this section, the asymptotic stability analysis of the numerical scheme is investigated.

Definition 4.1. [22] If the solution U^n of Eq. (3.6) corresponding to any sufficiently differentiable function $\varphi_h(x,t)$ with $\varphi_h(0,t)=\varphi_h(\pi,t)=0$ satisfies

$$\lim_{n\to\infty}U^n=0, x\in[0,\pi], \tag{4.1}$$

then the zero solution of Eq. (3.6) is called asymptotically stable.

The fully discrete numerical scheme can be written in the matrix form

$$\psi_0(S)U^{n+1}=\psi_1(S)U^n-\psi_m(S)U^{n+1-m}-\psi_{m+1}(S)U^{n-m}, \tag{4.2}$$

where

$$a=\frac{a_1\Delta t}{\Delta x^2}, b=a_2\Delta t, \tag{4.3}$$

$$\psi_0(\eta)=\frac{2}{3}+a+(\frac{1}{6}-\frac{1}{2}a)\eta, \tag{4.4}$$

$$\psi_1(\eta) = \frac{2}{3} - a + (\frac{1}{6} + \frac{1}{2}a)\eta, \tag{4.5}$$

$$\psi_m(\eta) = \frac{1}{3}b + \frac{1}{12}b\eta, \tag{4.6}$$

$$\psi_{m+1}(\eta) = \frac{1}{3}b + \frac{1}{12}b\eta. \tag{4.7}$$

It is well known that the trivial solution of Eq. (4.2) is asymptotically stable if and only if the characteristic polynomial

$$P_m(z) \equiv \det[\psi(S)z^{m+1} - \psi_1(S)z^m + \psi_m(S)z + \psi_{m+1}(S)], \tag{4.8}$$

is a Schur polynomial (that is, the modulus of all zeros of the characteristic polynomial is less than 1). Simple calculation yields

$$P_m(z) = \prod_{\lambda_j \in \varrho[\lambda_j]} [\psi(\lambda_j)z^{m+1} - \psi_1(\lambda_n)z^m + \psi_m(\lambda_j)z + \psi_{m+1}(\lambda_j)]. \tag{4.9}$$

Therefore, the numerical scheme (3.6) is asymptotically stable with respect to the trivial solution if and only if

$$P_{m,j}(z) = \psi_0(\lambda_j)z^{m+1} - \psi_1(\lambda_j)z^m + \psi_m(\lambda_j)z + \psi_{m+1}(\lambda_j), \tag{4.10}$$

is a Schur polynomial for all $m \geq 1$, $\lambda_j \in \varrho[S], j = 1, 2, \ldots, N-1$.
Substituting $\psi_0(\lambda_j), \psi_1(\lambda_j), \psi_m(\lambda_j), \psi_{m+1}(\lambda_j)$ into $P_{m,j}$, we have

$$P_{m,j} = \left[\frac{2}{3} + a + (\frac{1}{6} - \frac{1}{2}a)\lambda_j\right]z^{m+1} - \left[\frac{2}{3} - a + (\frac{1}{6} + \frac{1}{2}a)\lambda_j\right]z^m + \left[\frac{1}{3}b + \frac{1}{12}b\lambda_j\right]z + \left[\frac{1}{3}b + \frac{1}{12}b\lambda_j\right] \tag{4.11}$$

$$= z^m\left(\left[\frac{2}{3} + a + (\frac{1}{6} - \frac{1}{2}a)\lambda_j\right]z - \left[\frac{2}{3} - a + (\frac{1}{6} + \frac{1}{2}a)\lambda_j\right]\right) - \left[-\frac{1}{3}b - \frac{1}{12}b\lambda_j\right](z+1), \tag{4.12}$$

Denote

$$\alpha_j(z) = \left[\frac{2}{3} + a + (\frac{1}{6} - \frac{1}{2}a)\lambda_j\right]z - \left[\frac{2}{3} - a + (\frac{1}{6} + \frac{1}{2}a)\lambda_j\right], \tag{4.13}$$

$$\beta_j(z) = \left[-\frac{1}{3}b - \frac{1}{12}b\lambda_j\right](z+1), \tag{4.14}$$

then Eq. (4.12) can be written as

$$P_{m,j} = \alpha_j(z)z^m - \beta_j(z). \tag{4.15}$$

In order to prove that the characteristic polynomial is a Schur polynomial, we need the following lemma.

Lemma 4.2. *[32] Let $\kappa_m(z) = \alpha(z)z^m - \beta(z)$ be a polynomial, with $\alpha(z)$ and $\beta(z)$ are polynomials of zero degree. Then $\kappa_m(z)$ is a Schur polynomial for $m \geq 1$ if and only if the following conditions are satisfied*

(i) $\alpha(z) = 0 \Rightarrow |z| < 1$,
(ii) $|\beta(z)| \leq |\alpha(z)|, \forall z \in \mathbb{C}, |z| = 1,$ *and*
(iii) $\kappa_m(z) \neq 0, \forall z \in \mathbb{C}, |z| = 1$.

With the help of [26], we obtain the following theorem that leads to the sufficient and necessary conditions for the numerical scheme (3.6) to be asymptotically stable.

Theorem 4.3. *Suppose that $a_1 > 0$ and $|a_2| < -a_1\lambda^*$ (where $\lambda^* = -k^2(\approx \lambda_j)$). Then the zero solution of the Linear B-spline finite element method is the delay independently asymptotically stable.*

Proof. Denote

$$\alpha_j(z) = [\frac{2}{3} + a + (\frac{1}{6} - \frac{1}{2}a)\lambda_j]z - [\frac{2}{3} - a + (\frac{1}{6} + \frac{1}{2}a)\lambda_j], \tag{4.16}$$

$$\beta_j(z) = (-\frac{1}{3}b - \frac{1}{12}b\lambda_j)(z+1). \tag{4.17}$$

Then

$$P_{m,j}(z) = \alpha_j(z)z^m - \beta_j(z). \tag{4.18}$$

We prove the theorem with Lemma 4.2. First, it follows from $\alpha_j(z) = 0$ that

$$|z| = \left| \frac{\frac{2}{3} + a + (\frac{1}{6} - \frac{1}{2}a)\lambda_j}{\frac{2}{3} - a + (\frac{1}{6} + \frac{1}{2}a)\lambda_j} \right|, \tag{4.19}$$

which implies $\alpha_j(z) = 0 \Rightarrow |z| < 1$. Thus, condition (i) of lemma 4.2 holds for $j = 1, 2, \ldots, N-1$. In order to show that (ii) and (iii), we define the following complex variable function

$$\omega = \frac{\alpha_j(z)}{\beta_j(z)} \tag{4.20}$$

$$= \frac{[\frac{2}{3} + a + (\frac{1}{6} - \frac{1}{2}a)\lambda_j]z - [\frac{2}{3} - a + (\frac{1}{6} + \frac{1}{2}a)\lambda_j]}{(-\frac{1}{3}b - \frac{1}{12}b\lambda_j)(z+1)}. \tag{4.21}$$

Set $\omega = x + yi$ and $|z| = 1$, after some manipulations, we find

$$\min_{|z|=1, z\in\mathbb{C}} \left| \frac{\alpha_j(z)}{\beta_j(z)} \right| = \left| \frac{6a(2-\lambda_j)}{b(4+\lambda_j)} \right|. \tag{4.22}$$

It follows from the assumptions $a_1 > 0$ and $|a_2| < -a_1\lambda^*$, that $6a(2-\lambda_j) > b(4+\lambda_j)$. Then, for all $z \in \mathbb{C}, |z| = 1$, we find that

$$\left| \frac{\alpha_j(z)}{\beta_j(z)} \right| \geq \min_{|z|=1, z\in\mathbb{C}} \left| \frac{\alpha_j(z)}{\beta_j(z)} \right| = \frac{6a(2-\lambda_j)}{b(4+\lambda_j)} > 1, \tag{4.23}$$

which indicates that (ii) and (iii) of lemma 4.2 hold. □

5. Convergence Analysis

In this section, we present the convergence analysis for the proposed method.

The Ritz projection $R_h : H_0^1(\Omega) \to S_h$ is a mapping for any $\nu \in H_0^1(\Omega)$ such that

$$(\nabla R_h \nu - \nu, \nabla w) = 0, \forall w \in S_h. \tag{5.1}$$

Lemma 5.1. *[31] Assume that for any $v \in H^s(\Omega) \cap H_0^1(\Omega)$,*

$$\inf_{\zeta \in S_h} \{\|\nu - \zeta\| + h\|\nabla(\nu - \zeta)\|\} \leq Ch^s \|\nu\|_s, \mathit{for}\ 1 \leq s \leq r. \tag{5.2}$$

holds. Then, with R_h defined by Eq. (5.1), we have

$$\|R_h\nu - \nu\| + h\|\nabla(R_h\nu - \nu)\| \leq Ch^s \|\nu\|_s, \tag{5.3}$$

for any $\nu \in H^s(\Omega) \cap H_0^1(\Omega), 1 \leq s \leq r$.

The number r is referred to as the order of accuracy of the family $\{S_h\}$. For the case of piecewise linear B-spline function, $r = 2$.
Define $u(t) := u(., t)$ and $u : [0, +\infty) \to H_0^1(\Omega)$. Let $D_h : H_0^1(\Omega) \to S_h$ by

$$a_1(\nabla D_h u(t) - \nabla u(t), \nabla\zeta) - a_2(u(t-\tau), \zeta) = 0, \forall\zeta \in S_h, \tag{5.4}$$

$$D_h u(t) = R_h u(t) = R_h \varphi(t), for\ \ -\tau \le t \le 0. \tag{5.5}$$

Based on [22], we have the following convergence theorem.

Theorem 5.2. *Let u and U^n be the solution of Eq. (3.3) and Eq. (3.6), respectively. Assume that $\|u(t) - R_h u(t)\| \le Ch^2 \|u(t)\|_2$, $\|u_t(t) - R_h u_t(t)\| \le Ch^2 \|u_t(t)\|_2$, $-\tau \le t \le 0$ and $\|\varphi_h(t) - \varphi(t)\| \le Ch^2$, then*

$$\|U^n - u(t_n)\| \le C(h^2 + (\Delta t)^2), for\ n = 1, 2, ... \tag{5.6}$$

where C is a positive constant independent of h and Δt.

Proof. Define

$$e^n = U^n - u(t_n) = (U^n - D_h u(t_n)) + (D_h u(t_n) - u(t_n)) = \mu^n + \sigma^n, \tag{5.7}$$

where
$\mu^n = U^n - D_h u(t_n)$, $\sigma^n = D_h u(t_n) - u(t_n)$, so that

$$\|U^n - u(t_n)\| \le \|\mu^n\| + \|\sigma^n\|. \tag{5.8}$$

The term $\sigma^n(t) = \sigma(t_n)$ is easily bounded by lemma 5.1

$$\left(\frac{\mu^n - \mu^{n-1}}{\Delta t}, \zeta\right) + a_1\left(\frac{\nabla\mu^n + \nabla\mu^{n-1}}{2}, \nabla\zeta\right) - a_2\left(\frac{\mu^{n-m} + \mu^{n-m-1}}{2}, \zeta\right) = -(\omega^n, \zeta), \quad \forall\zeta \in S_h, \tag{5.9}$$

with

$$\omega^n = \frac{D_h u(t_n) - D_h u(t_{n-1})}{\Delta t} - \frac{u_t(t_n) + u_t(t_{n-1})}{2} \tag{5.10}$$

$$= (D_h - I)\bar{\partial} u(t_n) + \left(\bar{\partial} u(t_n) - \frac{u_t(t_n) + u_t(t_{n-1})}{2}\right) =: \omega_1^n + \omega_2^n. \tag{5.11}$$

Setting $\zeta = \frac{\mu^n + \mu^{n-1}}{2}$, gives

$$\left(\frac{\mu^n - \mu^{n-1}}{\Delta t}, \frac{\mu^n + \mu^{n-1}}{2}\right) + a_1\left\|\frac{\mu^n + \mu^{n-1}}{2}\right\|_1^2 - a_2\left(\frac{\mu^{n-m} + \mu^{n-m-1}}{2}, \frac{\mu^n + \mu^{n-1}}{2}\right) = -\left(\omega^n, \frac{\mu^n + \mu^{n-1}}{2}\right). \tag{5.12}$$

By applying Schwartz inequality,

$$\left(\frac{\mu^n - \mu^{n-1}}{\Delta t}, \frac{\mu^n + \mu^{n-1}}{2}\right) + \left\|\frac{\mu^n + \mu^{n-1}}{2}\right\|_1^2 \le C\left(\left\|\frac{\mu^{n-m} + \mu^{n-m-1}}{2}\right\|_1^2 + \|\omega^n\| \left\|\frac{\mu^n + \mu^{n-1}}{2}\right\|\right). \tag{5.13}$$

Hence

$$\|\mu^n\|^2 + \Delta t\left\|\frac{\mu^n + \mu^{n-1}}{2}\right\|_1^2 \le C\left(\|\mu^{n-1}\|^2 + \Delta t\left\|\frac{\mu^{n-m} + \mu^{n-m-1}}{2}\right\|_1^2 + (\Delta t)^2 \|\omega^n\|^2\right). \tag{5.14}$$

We can assume that $n \in ((k-1)m, km], k \in N$. Then

$$\Delta t\left\|\frac{\mu^n + \mu^{n-1}}{2}\right\|_1^2 \le C\left(\|\mu^{n-1}\|^2 + \Delta t\left\|\frac{\mu^{n-m} + \mu^{n-m-1}}{2}\right\|_1^2 + (\Delta t)^2 \|\omega^n\|^2\right) \tag{5.15}$$

$$\leq C\left(\left\|\mu^{n-1}\right\|^2 + \left\|\mu^{n-m-1}\right\|^2 + \Delta t \left\| \frac{\mu^{n-2m} + \mu^{n-2m-1}}{2} \right\|_1^2 + (\Delta t)^2 (\|\omega^n\|^2 + \|\omega^{n-m}\|^2) \right) \tag{5.16}$$

$$\leq ... \leq C\left(\sum_{i=0}^{k-1} \left\|\mu^{n-im-1}\right\|^2 + \Delta t \left\| \frac{\mu^{n-km} + \mu^{n-km-1}}{2} \right\|_1^2 + (\Delta t)^2 \sum_{i=0}^{k-1} \left\|\omega^{n-im}\right\|^2 \right). \tag{5.17}$$

So

$$\|\mu^n\|^2 \leq C\left(\sum_{i=0}^{k-1} \left\|\mu^{n-im-1}\right\|^2 + \Delta t \left\| \frac{\mu^{n-km} + \mu^{n-km-1}}{2} \right\|_1^2 + (\Delta t)^2 \sum_{i=0}^{k-1} \left\|\omega^{n-im}\right\|^2 \right). \tag{5.18}$$

By applying Gronwall inequality (see [8, 17]),

$$\|\mu^n\|^2 \leq C\left(\left\|\mu^0\right\|^2 + \Delta t \left\| \frac{\mu^{n-km} + \mu^{n-km-1}}{2} \right\|_1^2 + (\Delta t)^2 \sum_{i=0}^{k-1} \left\|\omega^{n-im}\right\|^2 \right). \tag{5.19}$$

We write

$$\omega_1^n = (D_h - I)\tilde{\partial} u(t_n) = \Delta t^{-1} \int_{t_{n-1}}^{t_n} (D_h - I) u_t(t) dt, \tag{5.20}$$

hence

$$(\Delta t)^2 \sum_{i=1}^{k-1} \left\|\omega_1^{n-im}\right\|^2 \leq \sum_{i=1}^{k-1} \left(\int_{t_{n-im-1}}^{t_{n-im}} Ch^2 \left\|u_t(t)\right\|_2 dt \right)^2 \leq C(h^2)^2. \tag{5.21}$$

Further

$$\left\|\Delta t \omega_2^i\right\| = \left\| u(t_i) - u(t_{i-1}) - \Delta t \frac{u_t(t_i) + u_t(t_{i-1})}{2} \right\| \leq C(\Delta t)^2 \int_{t_{i-1}}^{t_i} \left\|u_{ttt}(t)\right\| dt, \tag{5.22}$$

such that

$$(\Delta t)^2 \sum_{i=1}^{k-1} \left\|\omega_2^{n-im}\right\|^2 \leq C(\Delta t)^4 \sum_{i=1}^{k-1} \left(\int_{t_{n-im-1}}^{t_{n-im}} \left\|u_{ttt}(S)\right\| dt \right)^2 \leq C(\Delta t)^4. \tag{5.23}$$

From Eqs. (5.21) and (5.23), we have

$$\|U^n - u(t_n)\| \leq C(h^2 + (\Delta t)^2), for \quad n = 1, 2, \dots. \tag{5.24}$$

□

6. Numerical Experiments

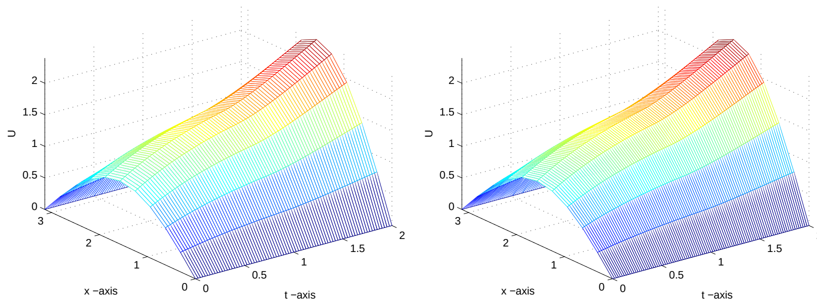
The performance of the proposed numerical method is tested by using numerical experiments. To evaluate errors, L_∞ and L_2 error norms are applied as follows:

$$L_\infty = \max_{1 \leq n \leq N} |u(t_n) - (U^n)|, L_2 = \sqrt{h \sum_{i=1}^{N} |u(t_n) - (U^n)|^2}. \tag{6.1}$$

Order of convergence is obtained by

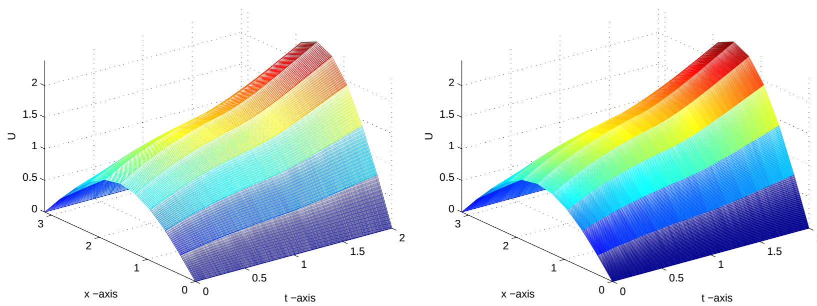
$$Order = \frac{\log(E^{h_1}/E^{h_2})}{\log(h_1/h_2)}, \tag{6.2}$$

where E^{h_1} and E^{h_2} represent the errors at step sizes h_1 and h_2, respectively.



(A) (B)

FIGURE 1. The numerical solution of Example 6.1 with a) $N = 10$, $m = 40$, and $\tau = 1$ b) $N = 10$, $m = 50$, and $\tau = 1$.



(A) (B)

FIGURE 2. The numerical solution of Example 6.1 with a) $N = 10$, $m = 100$, and $\tau = 1$ b) $N = 10$, $m = 200$, and $\tau = 1$.

Example 6.1. Consider Eq. (1.1) with parameter values $a_1 = 1, a_2 = 2, \tau = 1$, initial function

$$u(x,t) = \varphi(x,t) = sin(x), \tag{6.3}$$

and boundary conditions of the Dirichlet type

$$u(0,t) = u(\pi,t) = 0, t > 0. \tag{6.4}$$

The numerical results are obtained and plotted at time $T = 2$ using different step sizes ($\Delta t = \tau/m, h = \pi/N$) in Figures 1 and 2. These figures show that the numerical solution is asymptotically stable. And these confirm the theoretical analysis in Theorem 4.3.

Example 6.2. We consider the reaction-diffusion equation with the delay

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + u(x,t) - e^{-\tau} u(x,t-\tau), x \in [0,\pi], t > 0, \tag{6.5}$$

subject to the boundary conditions

$$u(0,t) = u(\pi,t) = 0, t > 0, \tag{6.6}$$

initial condition

$$u(x,t) = \varphi(x,t) = sin(x), \tag{6.7}$$

and with the exact solution

$$u(x,t) = \exp(-t)sin(x). \tag{6.8}$$

Here, we solve the problem on $[0,\pi] \times [0,2]$ with different temporal and spatial step sizes ($\Delta t = \tau/m$ $h = \pi/N$).

TABLE 1. The error norms and convergence order for different m and N at fixed $\tau = 0.5$ and $T = 1$ for example 6.2.

N	m	L_2	Order	L_∞	Order
5	50	1.2560e-02	-	9.5312e-03	-
10	100	3.1306e-03	2.0043	2.4979e-03	1.9319
20	200	7.8196e-04	2.0013	6.2391e-04	2.0013
40	400	1.9545e-04	2.0003	1.5594e-04	2.0003
80	800	4.8858e-05	2.0001	3.8983e-05	2.0001

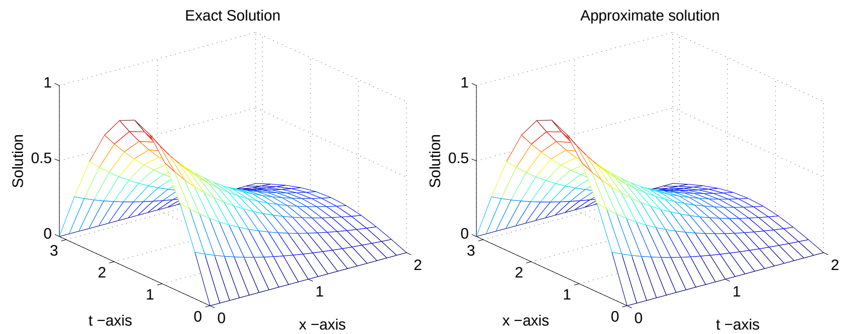
TABLE 2. The error norms and convergence order for different m and N at fixed $\tau = 0.5$ and $T = 1$ for example 6.2.

N	m	L_2	Order	L_∞	Order
4	40	1.9659e-02	-	1.5685e-02	-
8	80	4.8947e-03	2.0059	3.9054e-03	2.0058
16	160	1.2220e-03	2.0020	9.7503e-04	2.0020
32	320	3.0540e-04	2.0005	2.4367e-04	2.0005
64	640	7.6342e-05	2.0002	6.0912e-05	2.0001

TABLE 3. Comparison of the numerical solutions obtained with various values of m for $N = 10, T = 1$, and $\tau = 0.5$ with the exact solution for Example 6.2.

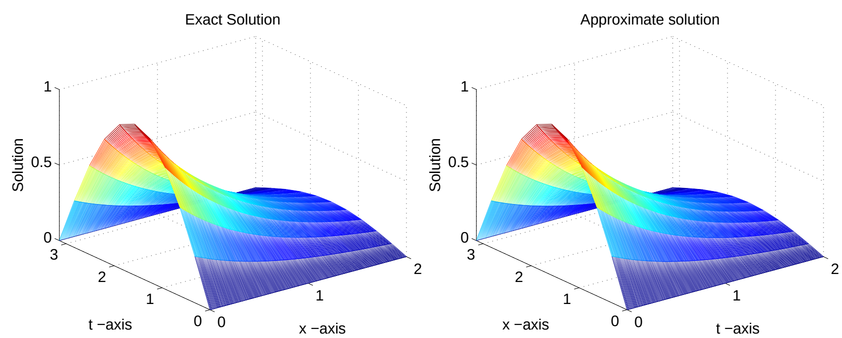
x	Numerical solutions					Exact solution
	m =10	m =20	m =40	m =60	m =80	
0.1π	0.18741	0.18742	0.18743	0.18743	0.18743	0.18743
0.2π	0.35647	0.35650	0.35651	0.35651	0.35651	0.35651
0.3π	0.49064	0.49068	0.49069	0.49069	0.49069	0.49069
0.4π	0.57678	0.57683	0.57684	0.57684	0.57684	0.57684
0.5π	0.60647	0.60651	060652	0.60652	0.60653	0.60653
0.6π	0.57678	0.57683	0.57674	0.57684	0.57684	0.57684
0.7π	0.49064	0.49068	0.49069	0.49069	0.49069	0.49069
0.8π	0.35647	0.35650	0.35651	0.35651	0.35651	0.35651
0.9π	0.18741	0.18742	0.18742	0.18742	0.18743	0.18743

Numerical errors and the corresponding orders are listed in Tables 1 and 2. As it can be seen from these tables, there is a noticeable decrease in both error norms when mesh sizes decrease. These results confirm the convergence



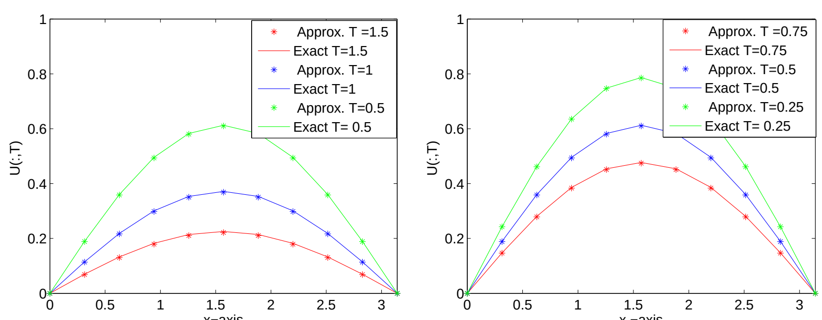
(A) (B)

FIGURE 3. Comparison between approximate and exact solutions of Example 6.2 ($N = 10, m = 10$, $\tau = 1$, and $T = 2$).



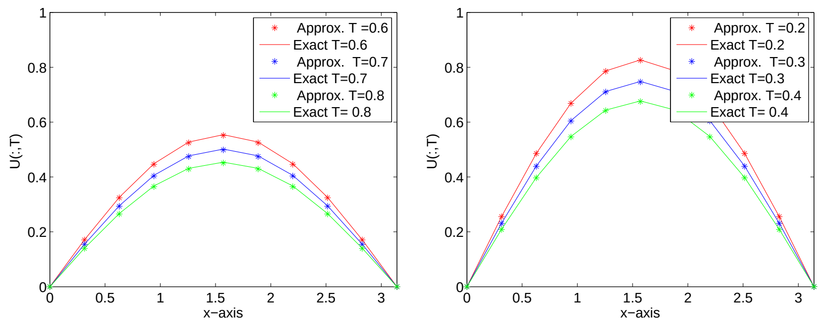
(A) (B)

FIGURE 4. Comparison between approximate and exact solutions of Example 6.2 ($N = 10, m = 100$, $\tau = 1$, and $T = 2$).



(A) (B)

FIGURE 5. Approximate and exact solutions for different time levels for Example 6.2 ($N = 10, m = 200$, $\tau = 1$, and $T = 2$).



(A) (B)

FIGURE 6. Approximate and exact solutions for different time levels for Example 6.2 ($N = 10, m = 100$, $\tau = 1$, and $T = 2$).

of the numerical scheme. In Figures 3 and 4, the exact and numerical solutions are depicted. By comparing the two solutions, we observe that the solution obtained by the presented method is comparable with the one obtained by analytical method. Figures 5 and 6 correspond to the exact and approximate solutions at different time levels.

7. CONCLUSION

In this paper, a finite element method is constructed based on linear B-spline basis functions for solving reaction-diffusion equations with delay. The detailed description of results through tables and graphs proves that the proposed numerical method is working efficiently. For all the test cases, simulations at a different set of data points are carried out to check the applicability of the numerical scheme. Based on these observations, our expectation that the given method is well suited to reaction diffusion with the delay is confirmed.

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