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Numerical solution of space-time fractional PDEs with variable coefficients using shifted Jacobi collocation method

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Abstract

The paper reports a spectral method for generating an approximate solution for the space-time fractional PDEs with variable coefficients based on the spectral shifted Jacobi collocation method in conjunction with the shifted Jacobi operational matrix of fractional derivatives. The spectral collocation method investigates both temporal and spatial discretizations. By applying the shifted Jacobi collocation method, the problem reduces to a system of algebraic equations, which greatly simplifies the problem. Numerical results are given to establish the validity and accuracy of the presented procedure for space-time fractional PDE.

Keywords. Jacobi polynomials, Operational matrices, Space-time PDEs, Collocation method.
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1. Introduction

Fractional calculus has become the focus of many researchers due to the accuracy of fractional differential and integral equations in modeling different natural phenomena. Oldham and Spanier [32], Miller and Ross [31] and Podlubny [33] obtained the history and a general treatment of this subject. Fractional order differential equations (FDEs) have been used in engineering, physics, chemistry, and other branches of science such as surface hydrology [3, 16], finance [12], epidemiology [11], biology [25], etc. The main advantage of fractional order models over classical integer ones is their non-local property.

In recent years, many researchers have paid attention to obtain the solution of fractional order differential equations. Since, the kernel of the differential equations is fractional, it is very difficult to find analytical solutions. Then, the analytical methods are not adequate to the majority of FDEs. Accordingly, in the last decade, several numerical methods have been proposed to solve fractional PDEs such as Fourier transforms [13], Laplace transforms [34], the finite difference [38], the finite element [43], Adomian decomposition method [23], homotopy perturbation method [1], He's variation iteration method [15], spectral method [5, 7, 8] and so on.

The fractional PDEs have been studied by many authors. Li and Xu [27] solved the time-fractional diffusion equation with the Jacobi PS method. Wang et al. [37] applied the wavelet method for solving fractional partial differential equations numerically. Chen et al. [14] presented a wavelet method for a class of fractional convection-diffusion equation with variable coefficients. Hanert [21] presented a pseudospectral method to discretize the space-time fractional diffusion equation. Rehman and Khan [35] obtained numerical solutions to initial and boundary value problems for linear fractional partial differential equations. Kumar and Piret [26] obtained numerical solutions of space-time fractional PDEs, based on the radial basis functions (RBF) and pseudospectral (PS) methods. Saadatmandi and Dehghan in [36] presented an efficient numerical technique to solve fractional differential equations. Bayrak and Demir [2] used the residual power series method for solving space-time fractional PDEs.

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Recently, numerical schemes based on operational matrices such as spectral methods, have attracted many researchers. The mentioned techniques provide accurate numerical solutions to both linear and nonlinear ordinary as well as partial differential equations of classical and fractional order. Spectral methods are one of the efficient schemes to solve fractional partial differential equations (FPDEs). Wherein, the approximate solution is written as a finite sum of basis functions, which may be orthogonal polynomials. So that, the approximate solution is generated by solving the existing algebraic system [5, 7–10, 39–42].

In this paper, we present the shifted Jacobi collocation method. We have used shifted Jacobi polynomials as a basis function for the construction of operational matrices. In our suggested method, first the unknown function and their derivatives are approximated by taking the shifted Jacobi orthogonal polynomials. Then, by using these approximations in the following space-time fractional PDE with variable coefficients, we obtain a system of equations. Finally, by collocating this system, we get an approximate solution for the problem. The proposed scheme is applicable to all types of boundary conditions. Examples with existing procedure are proposed to illustrate the applicability and accuracy of method. In the present paper, we propound the following space-time fractional PDE with variable coefficients:

$$a_1(\iota,\xi)\frac{\partial^{\alpha}\nu(\iota,\xi)}{\partial\xi^{\alpha}} = a_2(\iota,\xi)\frac{\partial^{\beta_1}\nu(\iota,\xi)}{\partial\iota^{\beta_1}} + a_3(\iota,\xi)\frac{\partial^{\beta_2}\nu(\iota,\xi)}{\partial\iota^{\beta 2}} + a_4(\iota,\xi)\nu(\iota,\xi) + a_5(\iota,\xi), \tag{1.1}$$

$$(\iota,\xi)\in[0,L]\times[0,T], \qquad \alpha,\beta_2\in(0,1], \qquad \beta_1\in(1,2],$$

with initial and boundary conditions

$$\begin{aligned} &\nu(\iota,0)=\nu_0(\iota),\\ &\nu(0,\xi)=\nu_0(\xi),\\ &\nu(L,\xi)=\nu_L(\xi). \end{aligned} \tag{1.2}$$

Where, the fractional derivatives are described in the Caputo sense. The important goal of this paper, is to use the shifted Jacobi polynomials and the operational matrix of the fractional derivative together with collocation method to solve Eqs. (1.1)-(1.2) to get the approximate solution.

The reminder of this paper is organized as follows: In section 2, we introduce some necessary definitions and give some relevant properties of Jacobi polynomials. In section 3, our method is used to solve the space-time fractional PDE with variable coefficients. In section 4, numerical results are provided to show the accuracy of the presented scheme. Finally, conclusions are given in section 5.

2. Basic Definitions and Notation

To begin with, we describe some necessary definitions and mathematical preliminaries of the fractional derivative theory.

Definition 2.1. The Caputo fractional derivative of order α for a two variables function $\nu(\iota,\xi)$ with respect to variable ι, is defined as [31, 33]

$$\frac{\partial^{\alpha}\nu(\iota,\xi)}{\partial\iota^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^{\iota}\frac{\partial^m\nu(\tau,\xi)}{\partial\tau^m(\iota-\tau)^{\alpha-m+1}}d\tau & if\ m-1<\alpha<m,\\ \dfrac{\partial^m\nu(\iota,\xi)}{\partial\iota^m} & if\ \alpha=m\in N, \end{cases} \tag{2.1}$$

where, the Caputo fractional derivative of $\iota^j, j\geq 0$ is determined by [31, 33]

$${}_0^C D_\iota^{\alpha}\iota^j = \begin{cases} 0 & for\ j\in N_0\ and\ j<\lceil\alpha\rceil,\\ \dfrac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}\iota^{j-\alpha} & for\ j\in N_0\ and\ j\geq\lceil\alpha\rceil. \end{cases} \tag{2.2}$$

Some few properties of the shifted Jacobi polynomials are introduced in this part. The Jacobi polynomials are defined by $P_j^{(\theta,\eta)}(z)$ and $\theta > -1, \eta > -1$, over $I = [-1, 1]$. The following recurrence relation of the Jacobi polynomials is defined as [6, 8]

$$\begin{aligned} P_i^{(\theta,\eta)}(z) &= \frac{(\theta+\eta+2i-1)\left\{\theta^2-\eta^2+z(\theta+\eta+2i)(\theta+\eta+2i-2)\right\}}{2i(\theta+\eta+i)(\theta+\eta+2i-2)}P_{i-1}^{(\theta,\eta)}(z) \\ &-\frac{(\theta+i-1)(\eta+i-1)(\theta+\eta+2i)}{i(\theta+\eta+i)(\theta+\eta+2i-2)}P_{i-2}^{(\theta,\eta)}(z), \\ &i = 2, 3, ... \end{aligned} \tag{2.3}$$

where

$$P_0^{(\theta,\eta)}(z) = 1, \qquad and \qquad P_1^{(\theta,\eta)}(z) = \frac{\theta+\eta+2}{2}z + \frac{\theta-\eta}{2}.$$

The orthogonality condition of shifted Jacobi polynomials is

$$\int_{-1}^{1} P_j^{(\theta,\eta)}(z)P_k^{(\theta,\eta)}(z)w^{(\theta,\eta)}(z)dz = \delta_{jk}\gamma_k^{(\theta,\eta)}, \tag{2.4}$$

where δ_{jk} is the Kronecker function. In addition, the weight function is

$$w^{(\theta,\eta)}(z) = (1-z)^{\theta}(1+z)^{\eta},$$

also,

$$\gamma_k^{(\theta,\eta)} = \frac{2^{\theta+\eta+1}\Gamma(k+\theta+1)\Gamma(k+\eta+1)}{(2k+\theta+\eta+1)k!\Gamma(k+\theta+\eta+1)}.$$

In order to construct the shifted Jacobi polynomials in $\iota \in [0, L]$, we applied the change of variable $z = \frac{2\iota}{L} - 1$. So that, the shifted Jacobi polynomials $P_i^{(\theta,\eta)}\left(\frac{2\iota}{L} - 1\right)$ be denoted by $P_{L,i}^{(\theta,\eta)}(\iota)$. The analytical form of the shifted Jacobi polynomials $P_{L,i}^{(\theta,\eta)}(\iota)$ can be determined as follows [6, 8]

$$P_{L,i}^{(\theta,\eta)}(\iota) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+\eta+1)\Gamma(i+k+\theta+\eta+1)}{\Gamma(k+\eta+1)\Gamma(i+\theta+\eta+1)(i-k)!k!L^k}\iota^k. \tag{2.5}$$

Also, the following orthogonality condition is useful

$$\int_0^L P_{L,j}^{(\theta,\eta)}(\iota)P_{L,k}^{(\theta,\eta)}(\iota)w_L^{(\theta,\eta)}(\iota)d\iota = h_{L,k}^{(\theta,\eta)}\delta_{jk}, \tag{2.6}$$

where, $w_L^{(\theta,\eta)}(\iota) = (L-\iota)^{\theta}\iota^{\eta}$ and

$$h_{L,k}^{(\theta,\eta)} = \frac{L^{\theta+\eta+1}\Gamma(k+\theta+1)\Gamma(k+\eta+1)}{(2k+\theta+\eta+1)\Gamma(k+1)\Gamma(k+\theta+\eta+1)}. \tag{2.7}$$

Shifted Jacobi polynomials have some properties as :

- $P_n^{(\theta,\eta)}(0) = (-1)^n\dfrac{\Gamma(n+\eta+1)}{\Gamma(\eta+1)n!}$,
- $P_n^{(\theta,\eta)}(L) = \dfrac{\Gamma(n+\theta+1)}{\Gamma(\theta+1)n!}$,
- $\dfrac{d^i}{d\iota^i}P_n^{(\theta,\eta)}(\iota) = \dfrac{\Gamma(n+\theta+\eta+i+1)}{\Gamma(n+\theta+\eta+1)}P_{n-i}^{(\theta+1,\eta+1)}(\iota)$.

A function $\nu(\iota) \in L^2_{w_L^{(\theta,\eta)}(x)}(0, L)$, can be demonstrated in terms of $P_{L,j}^{(\theta,\eta)}(\iota)$ with respect to the weight function $w_L^{(\theta,\eta)}(\iota)$ as

$$\nu(\iota) = \sum_{j=0}^{\infty} c_j P_{L,j}^{(\theta,\eta)}(\iota), \tag{2.8}$$

where the coefficients $c_j, j = 0, 1, 2, \dots$ are given by

$$c_j = \frac{1}{h_{L,j}^{(\theta,\eta)}} \int_0^L P_{L,j}^{(\theta,\eta)}(\iota) w_L^{(\theta,\eta)}(\iota)\nu(\iota)d\iota, \quad j = 0, 1, 2, \dots. \tag{2.9}$$

Since, we approximate $\nu(\iota)$ by the first $(M+1)-$terms, then we have

$$\nu(\iota) \approx \nu_M(\iota) \equiv \sum_{j=0}^{M} c_j P_{L,j}^{(\theta,\eta)}(\iota) = C^T \Phi_{L,M}(\iota), \tag{2.10}$$

where

$$C = [c_0, c_1, ..., c_M]^T,$$

$$\Phi_{L,M}(\iota) = \left[P_{L,0}^{(\theta,\eta)}(\iota), P_{L,1}^{(\theta,\eta)}(\iota), ..., P_{L,M}^{(\theta,\eta)}(\iota)\right]^T.$$

Equivalently, a function $\nu(\iota, \xi)$ with two independent variables over the interval $D = [0, L] \times [0, T]$ can be expanded as

$$\nu(\iota, \xi) \approx \nu_{N,M}(\iota, \xi) = \sum_{i=0}^{N} \sum_{j=0}^{M} k_{ij} P_{T,i}^{(\theta,\eta)}(\xi) P_{L,j}^{(\theta,\eta)}(\iota) = \Phi_{T,N}^T(\xi) K \Phi_{L,M}(\iota), \tag{2.11}$$

with

$$K = \begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0N} \\ k_{10} & k_{11} & \cdots & k_{1N} \\ \vdots & \vdots & & \vdots \\ k_{M0} & k_{M1} & \cdots & k_{MN} \end{bmatrix}, \tag{2.12}$$

and

$$k_{ij} = \frac{1}{h_{L,j}^{(\theta,\eta)} h_{T,i}^{(\theta,\eta)}} \int_0^T \int_0^L \nu(\iota, \xi) P_{T,i}^{(\theta,\eta)}(\xi) P_{L,j}^{(\theta,\eta)}(\iota) w_T^{(\theta,\eta)}(\xi) w_L^{(\theta,\eta)}(\iota) d\iota d\xi. \tag{2.13}$$

Lemma 2.2. *If the first order derivative of $\Phi_{L,M}(\iota)$ is expressed as*

$$\frac{d\Phi_{L,M}(\iota)}{d\iota} = D_+^{(1)} \Phi_{L,M}(\iota), \tag{2.14}$$

then $D_+^{(1)}$ is the $(M+1) \times (M+1)$ operational matrix of derivative as

$$D_+^{(1)} = (\varrho_{ij}) = \begin{cases} \mu(i,j) & j < i \quad i, j = 1, 2, ..., M+1 \\ 0 & \qquad o.w \end{cases},$$

and

$$\begin{aligned} \mu(i,j) = & \frac{L^{\theta+\eta}(i+\theta+\eta+1)(i+\theta+\eta+2)_j(i+\theta+2)_{i-j-1}\Gamma(i+\theta+\eta+1)}{(i-j-1)!\Gamma(2j+\theta+\eta+1)} \\ & \times {}_3F_2 \left(\begin{array}{c} j-i+1,\ i+j+\theta+\eta+2,\ j+\theta+1 \\ \\ j+\theta+2,\ 2j+\theta+\eta+2 \end{array} ; \iota \right), \end{aligned} \tag{2.15}$$

where ${}_pF_q(z)$ is the generalized hypergeometric function defined as

$$
{}_pF_q(a_1,...,a_p;b_1,...,b_q;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_p)_n}\frac{z^n}{n!},
$$

in which

$$
\begin{aligned}
&(a)_0 = 1,\\
&(a)_n = a(a+1)(a+2)...(a+n-1), \qquad n \geq 1,
\end{aligned}
$$

is the Pochhammer symbol [20, 30].

Proof. See [17] p. 41 and [29], pp. 103-104. □

By using the relation (2.14), we have

$$
\frac{d^n \Phi_{L,M}(\iota)}{d\iota^n} = \left(D^{(1)}\right)^n \Phi_{L,M}(\iota) = D^{(n)}\Phi_{L,M}(\iota), \quad n = 1,2,.... \tag{2.16}
$$

Theorem 2.3. *Let the shifted Jacobi vector is $\Phi_{L,M}(\iota)$ and $\alpha \in R^+$. Thereupon, the Caputo fractional derivative of the mentioned vector can be defined as*

$$
D^{\alpha}\Phi_{L,M}(\iota) \approx D^{(\alpha)}\Phi_{L,M}(\iota), \tag{2.17}
$$

such that, $D^{(\alpha)}$ denotes the Jacobi operational matrix of the Caputo fractional derivative and is obtained from

$$
D^{(\alpha)} = \begin{bmatrix}
0 & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots\\
0 & 0 & & 0\\
\Lambda_\alpha(\lceil\alpha\rceil,0) & \Lambda_\alpha(\lceil\alpha\rceil,1) & \dots & \Lambda_\alpha(\lceil\alpha\rceil,M)\\
\vdots & \vdots & & \vdots\\
\Lambda_\alpha(i,0) & \Lambda_\alpha(i,1) & \cdots & \Lambda_\alpha(i,M)\\
\vdots & \vdots & & \vdots\\
\Lambda_\alpha(M,0) & \Lambda_\alpha(M,1) & \cdots & \Lambda_\alpha(M,M)
\end{bmatrix}, \tag{2.18}
$$

where

$$
\Lambda_\alpha(i,j) = \sum_{k=\lceil\alpha\rceil}^{i} \delta_{ijk}, \quad i = \lceil\alpha\rceil,\dots,M, \quad j = 0,1,2,\dots,M,
$$

and δ_{ijk} is specified as

$$
\begin{aligned}
\delta_{ijk} = \;& \frac{(-1)^{i-k}L^{\theta+\eta-\alpha+1}\Gamma(j+\eta+1)\Gamma(i+\eta+1)\Gamma(i+k+\theta+\eta+1)}{h_j\Gamma(j+\theta+\eta+1)\Gamma(k+\eta+1)\Gamma(i+\theta+\eta+1)\Gamma(k-\alpha+1)(i-k)!}\\
&\times \sum_{l=0}^{j} \frac{(-1)^{j-l}\Gamma(k+l+\theta+\eta+1)\Gamma(\theta+1)\Gamma(l+k+\eta-\alpha+1)}{\Gamma(l+\eta+1)\Gamma(l+k+\theta+\eta-\alpha+2)(j-l)!l!}.
\end{aligned} \tag{2.19}
$$

Consider that the first $\lceil\alpha\rceil$ rows in $D^{(\alpha)}$ are all zeros.

Proof. For the evidence see [19]. □

3. Jacobi Spectral Collocation Method

Since, the Jacobi spectral collocation method approximates the initial boundary problems in physical space and it is a global scheme, it is very easy to implement and adapt it to different problems, including variable coefficients and nonlinear problems [4, 18]. In this section, Jacobi collocation method is applied for solving space-time fractional PDEs with variable coefficients.

To solve problems (1.1)-(1.2), we approximate $\nu(\iota,\xi)$ by the shifted Jacobi polynomials as

$$\nu(\iota,\xi) \approx \nu_{N,M}(\iota,\xi) = \Phi^T_{T,N}(\xi) A \Phi_{L,M}(\iota), \tag{3.1}$$

note that A is a $(N+1)\times(M+1)$ unknown matrix. Utilizing Eqs. (2.17) and (3.1), yields

$$\begin{aligned} \frac{\partial^{\alpha}\nu(\iota,\xi)}{\partial\xi^{\alpha}} &= \Phi^T_{T,N}(\xi) {D^{(\alpha)}_{\xi}}^T A \Phi_{L,M}(\iota),\\ \frac{\partial^{\beta_1}\nu(\iota,\xi)}{\partial\iota^{\beta_1}} &= \Phi^T_{T,N}(\xi) A D^{(\beta_1)}_{\iota} \Phi_{L,M}(\iota),\\ \frac{\partial^{\beta_2}\nu(\iota,\xi)}{\partial\iota^{\beta_2}} &= \Phi^T_{T,N}(\xi) A D^{(\beta_2)}_{\iota} \Phi_{L,M}(\iota), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\nu(\iota,0) = \Phi^T_{T,N}(0) A \Phi_{L,M}(\iota),\\ &\nu(0,\xi) = \Phi^T_{T,N}(\xi) A \Phi_{L,M}(0),\\ &\nu(L,\xi) = \Phi^T_{T,N}(\xi) A \Phi_{L,M}(L). \end{aligned} \tag{3.3}$$

By employing Eqs. (3.2)-(3.3), the Eqs. (1.1)-(1.2) may be written as follows

$$\begin{aligned} &a_1(\iota,\xi)\Phi^T_{T,N}(\xi) {D^{(\alpha)}_{\xi}}^T A \Phi_{L,M}(\iota) - a_2(\iota,\xi)\Phi^T_{T,N}(\xi) A D^{(\beta_1)}_{\iota}\Phi_{L,M}(\iota)\\ &-a_3(\iota,\xi)\Phi^T_{T,N}(\xi) A D^{(\beta_2)}_{\iota}\Phi_{L,M}(\iota) - a_4(\iota,\xi)\Phi^T_{T,N}(\xi) A \Phi_{L,M}(\iota) = a_5(\iota,\xi), \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\Phi^T_{T,N}(0) A \Phi_{L,M}(\iota) = \nu_0(\iota),\\ &\Phi^T_{T,N}(\xi) A \Phi_{L,M}(0) = \nu_0(\xi),\\ &\Phi^T_{T,N}(\xi) A \Phi_{L,M}(L) = \nu_L(\xi). \end{aligned} \tag{3.5}$$

A collocation method is employed at the points (ι_i,ξ_j) for Eqs. (3.4)-(3.5). To have suitable collocation points, we apply the shifted Jacobi nodes $\iota_i(0\le i\le M-1), \xi_j(0\le j\le N-1)$. So, the Eq. (3.4) can be rewritten as follows

$$\begin{aligned} &a_1(\iota_i,\xi_j)\Phi^T_{T,N}(\xi_j) {D^{(\alpha)}_{\xi}}^T A \Phi_{L,M}(\iota_i) - a_2(\iota_i,\xi_j)\Phi^T_{T,N}(\xi_j) A D^{(\beta_1)}_{\iota}\Phi_{L,M}(\iota_i)\\ &-a_3(\iota_i,\xi_j)\Phi^T_{T,N}(\xi_j) A D^{(\beta_2)}_{\iota}\Phi_{L,M}(\iota_i) - a_4(\iota_i,\xi_j)\Phi^T_{T,N}(\xi_j) A \Phi_{L,M}(\iota_i) = a_5(\iota_i,\xi_j),\\ &\qquad\qquad 0\le i\le M-1, 0\le j\le N-1, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\Phi^T_{T,N}(0) A \Phi_{L,M}(\iota_i) = \nu_0(\iota_i), && 0\le i\le M,\\ &\Phi^T_{T,N}(\xi_j) A \Phi_{L,M}(0) = \nu_0(\xi_j), && 0\le j\le N-1,\\ &\Phi^T_{T,N}(\xi_j) A \Phi_{L,M}(L) = \nu_L(\xi_j), && 0\le j\le N-1. \end{aligned} \tag{3.7}$$

Table 1. Maximum absolute errors at $L = 1, T = 1$ and $L = 1, T = 0.8$ for $N = M = 5, 7$ for problem 1

	$L = 1, T = 1$		$L = 1, T = 0.8$	
(θ, η)	$N = 5$	$N = 7$	$N = 5$	$N = 7$
$(1, \frac{1}{2})$	5.4×10^{-3}	7.6×10^{-4}	3.5×10^{-3}	4.7×10^{-4}
$(\frac{1}{2}, \frac{1}{2})$	4.5×10^{-3}	9.2×10^{-4}	2.5×10^{-3}	5.4×10^{-4}

By applying the collocation procedure, we get the system of linear algebraic equations. Such that, equations (3.6)-(3.7) obtain a $(N+1) \times (M+1)$ linear algebraic system of equations with $(N+1) \times (M+1)$ unknowns $a_{ij}, i = 0, 1, 2, ..., M, j = 0, 1, 2, ..., N$. We solved this system of equations applying MAPLE program. Consequently, $\nu_{N,M}(\iota, \zeta)$ given in Eq. (3.1) can be evaluated.

4. Illustrative and Examples

In this section, several numerical results are illustrated to demonstrate the effectiveness of the suggested scheme. To propound the efficiency for the method in the present paper, the absolute error, $e_{N,M}(\iota, \xi)$ or L_∞, Maximum absolute error (MAE) is specified as

$$\begin{aligned} &e_{N,M}(\iota, \xi) = \mid \nu(\iota, \xi) - \nu_{N,M}(\iota, \xi) \mid, \\ &MAE = L_\infty = \max \ \ e_{N,M}(\iota, \xi). \end{aligned} \tag{4.1}$$

Example 1: Consider the following space-time fractional advection diffusion equation [28],

$$\frac{\partial^\alpha \nu(\iota, \xi)}{\partial \xi^\alpha} = a_2(\iota, \xi) \frac{\partial^{\beta_1} \nu(\iota, \xi)}{\partial \iota^{\beta_1}} - a_3(\iota, \xi) \frac{\partial^{\beta_2} \nu(\iota, \xi)}{\partial \iota^{\beta 2}} + a_5(\iota, \xi), \qquad (\iota, \xi) \in [0, 1] \times [0, 1], \tag{4.2}$$

where $\alpha \in (0, 1], \beta_1 \in (1, 2], \beta_2 \in (0, 1]$, and

$$\begin{aligned} a_2(\iota, \xi) &= \frac{5\Gamma(3 - \beta_1)}{\Gamma(3 - \alpha)} \iota^{\beta_1} \xi^{2-\alpha}, \\ a_3(\iota, \xi) &= \frac{\Gamma(3 - \beta_2)}{\Gamma(3 - \alpha)} \iota^{\beta_2} \xi^{2-\alpha}, \\ a_5(\iota, \xi) &= \frac{4}{\Gamma(3 - \alpha)} \iota^2 \xi^{2-\alpha} (2 - 2(1 + 4\xi^2)), \end{aligned} \tag{4.3}$$

with boundary conditions

$$\begin{aligned} &\nu(0, \xi) = 0, \qquad 0 < \xi \leq T, \\ &\nu(1, \xi) = 4\xi^2 + 1, \end{aligned} \tag{4.4}$$

and the initial condition

$$\nu(\iota, 0) = \iota^2, \qquad 0 < \iota \leq L. \tag{4.5}$$

The exact solution is given by $\nu(\iota, \xi) = (4\xi^2 + 1)\iota^2$.

We consider $\alpha = 0.6, \beta_1 = 1.6, \beta_2 = 0.6$. Maximum absolute errors at $N = M = 5, 7$ are plotted in Figure 1 and the numerical results are obtained in Table 1 for Example 1. Moreover, Table 2 illustrates our results and a comparison of the method in [28] at $N = M = 5$ and $\theta = 1, \eta = \frac{1}{2}$ for $L = T = 0.1, 0.05$ and it is clear that our method is accurate. Values of $\| \nu_{N+1,M+1}(\iota, \xi) - \nu_{N,M}(\iota, \xi) \|_\infty$, which illustrates the numerical convergence in the lack of the exact solution at $L = T = 1$ for several choices of $N(N = M)$, are calculated in Table 3 for Example 1.

TABLE 2. Comparison of maximum absolute errors for the proposed method with the method in [28] at $N = M = 5$ and $\theta = 1, \eta = \frac{1}{2}$ for problem 1

L,T	Proposed scheme	scheme in [28]
L=T=0.05	3.3×10^{-4}	1.9×10^{-2}
L=T=0.1	2.5×10^{-3}	3.9×10^{-2}

TABLE 3. Values of $\parallel \nu_{N+1,M+1}(\iota,\xi) - \nu_{N,M}(\iota,\xi) \parallel_\infty$ at $L = T = 1$ for several choices of $N(N = M)$ for problem 1

(θ, η)	N=3	N=5	N=7
$(1, \frac{1}{2})$	2.4×10^{-3}	5.4×10^{-4}	6.1×10^{-4}

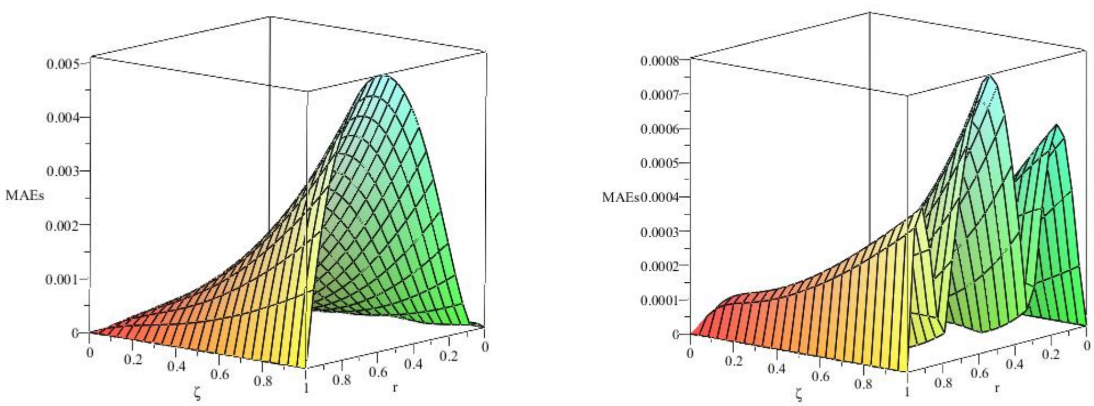

FIGURE 1. Plot of absolute errors for $\theta = 1, \eta = \frac{1}{2}$ and $N = 5$ (left), $N = 7$ (right) at $T = L = 1$ for problem 1

Example 2: Consider the following equation [22]:

$$\frac{\partial^\alpha \nu(\iota,\xi)}{\partial \xi^\alpha} = \frac{\partial^{\beta_1} \nu(\iota,\xi)}{\partial \iota^{\beta_1}} + a_5(\iota,\xi), \qquad (\iota,\xi) \in [0,1] \times [0,1], \tag{4.6}$$

where $\alpha \in (0,1], \beta_1 \in (1,2], \beta_2 = 0$, and

$$a_5(\iota,\xi) = \frac{\Gamma(q+1)}{\Gamma(q+1-\alpha)} \iota^p \xi^{q-\alpha} - \frac{\Gamma(p+1)}{\Gamma(p+1-\beta_1)} \xi^q \iota^{p-\beta_1}, \tag{4.7}$$

with boundary conditions

$$\begin{aligned} &\nu(0,\xi) = 0, \qquad 0 < \xi \leq T, \\ &\nu(1,\xi) = \xi^q, \end{aligned} \tag{4.8}$$

TABLE 4. Maximum absolute errors for s $N = M = 5, 7$, at $L = 1, T = 1$ and $L = 1, T = 0.8$ for case 1 for problem 2

	$L = 1, T = 1$		$L = 1, T = 0.8$	
(θ, η)	$N = 5$	$N = 7$	$N = 5$	$N = 7$
$(1, \frac{1}{2})$	9.1×10^{-4}	2.5×10^{-5}	5.2×10^{-5}	2.08×10^{-6}
$(\frac{1}{2}, \frac{1}{2})$	6.5×10^{-5}	25.2×10^{-6}	4.1×10^{-5}	1.6×10^{-6}

TABLE 5. Maximum absolute errors for $N = M = 5, 7$, at $L = 1, T = 1$ and $L = 1, T = 0.8$ for case 2 for problem 2

	$L = 1, T = 1$		$L = 1, T = 0.8$	
(θ, η)	$N = 5$	$N = 7$	$N = 5$	$N = 7$
$(1, \frac{1}{2})$	2.02×10^{-4}	2.5×10^{-5}	1.6×10^{-4}	3.1×10^{-5}
$(\frac{1}{2}, \frac{1}{2})$	2.08×10^{-4}	3.4×10^{-5}	1.5×10^{-4}	2.02×10^{-5}

TABLE 6. Maximum absolute errors for $N = M = 5, 7$, at $L = 1, T = 1$ and $L = 1, T = 0.8$ for case 3 for problem 2

	$L = 1, T = 1$		$L = 1, T = 0.8$	
(θ, η)	$N = 5$	$N = 7$	$N = 5$	$N = 7$
$(1, \frac{1}{2})$	20.8×10^{-4}	6.5×10^{-5}	1.6×10^{-4}	3.8×10^{-5}
$(\frac{1}{2}, \frac{1}{2})$	20.8×10^{-4}	6.5×10^{-5}	10.5×10^{-4}	3.7×10^{-5}

and the initial condition

$$\nu(\iota, 0) = 0, \qquad 0 < \iota \leq L. \tag{4.9}$$

The analytical solution of the problem is presented as $\nu(\iota, \xi) = \xi^q \iota^p$.

Case 1: $p = 3, q = 2$

For $\alpha = 0.7, \beta_1 = 1.6$, the absolute errors are plotted in Figure 2 and numerical results are shown in Table 4. Values of $\| \nu_{N+1,M+1}(\iota, \xi) - \nu_{N,M}(\iota, \xi) \|_\infty$, which illustrates the numerical convergence in the lack of the exact solution at $L = T = 1$ for several choices of $N (N = M)$, are calculated in Table 7 for Example 2.

Case 2: $p = noninteger, q = 2$

For $\alpha = 0.7, \beta_1 = 1.6$ and $p = 3.2$, the absolute errors are plotted in Figure 3 and numerical results are shown in Table 5.

Case 3: $p = noninteger, q = noninteger$

For $\alpha = 0.7, \beta_1 = 1.6$ and $p = 3.2, q = 2.5$, the absolute errors are plotted in Figure 4 and numerical results are shown in Table 6.

TABLE 7. Values of $\| \nu_{N+1,M+1}(\iota,\xi) - \nu_{N,M}(\iota,\xi) \|_{\infty}$ at $L = T = 1$ for several choices of $N(N = M)$ for case 1 at problem 2

(θ,η)	N=3	N=5	N=7
$(1,\frac{1}{2})$	8.1×10^{-4}	5.4×10^{-4}	10.2×10^{-5}

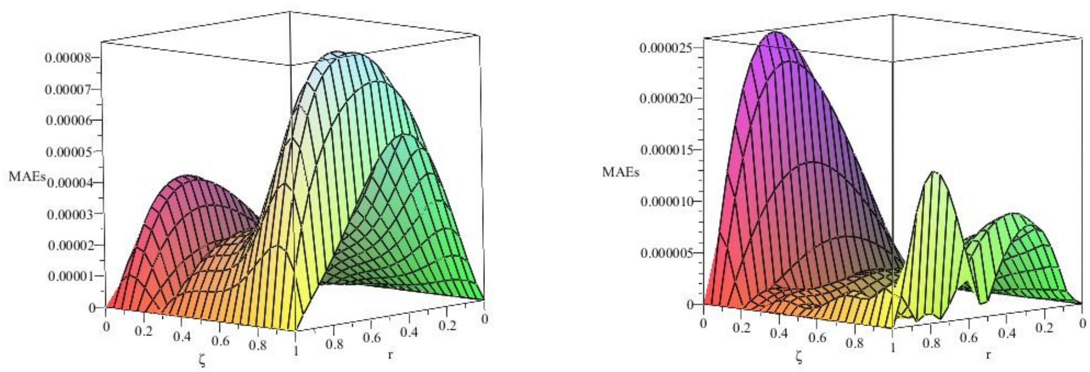

FIGURE 2. Plot of absolute errors for $\theta = 1, \eta = \frac{1}{2}$ and $N = 5$ (left), $N = 8$ (right) at $T = L = 1$ for case 1 for problem 2

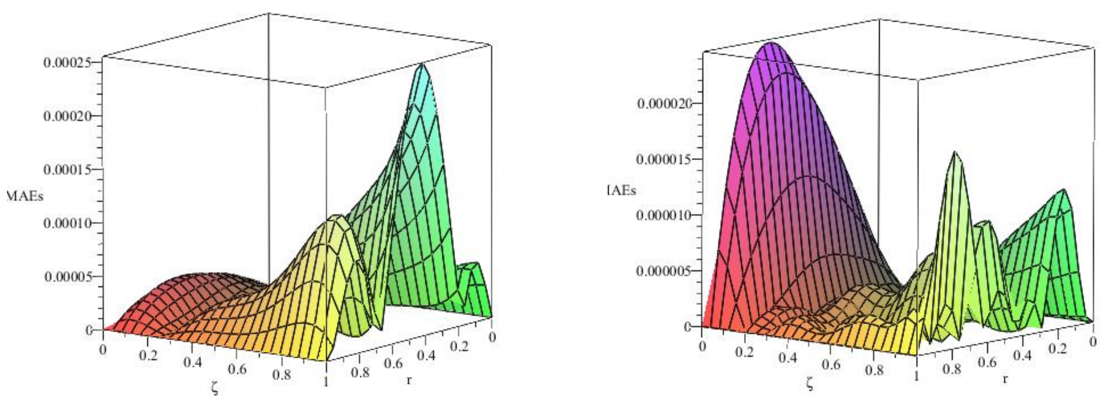

FIGURE 3. Plot of absolute errors for $\theta = 1, \eta = \frac{1}{2}$ and $N = 5$ (left), $N = 8$ (right) at $T = L = 1$ for case 2 for problem 2

Example 3: Consider the following equation [24]:

$$\frac{\partial \nu(\iota,\xi)}{\partial \xi} = a_2(\iota,\xi)\frac{\partial^{\beta_1}\nu(\iota,\xi)}{\partial \iota^{\beta_1}} + a_5(\iota,\xi), \qquad (\iota,\xi) \in [0,1] \times [0,1], \tag{4.10}$$

where $\alpha = 1, \beta_1 \in (1,2], \beta_2 = 0$, and $a_5(\iota,\xi) = 3\iota^2(2\iota - 1)e^{-t}$.

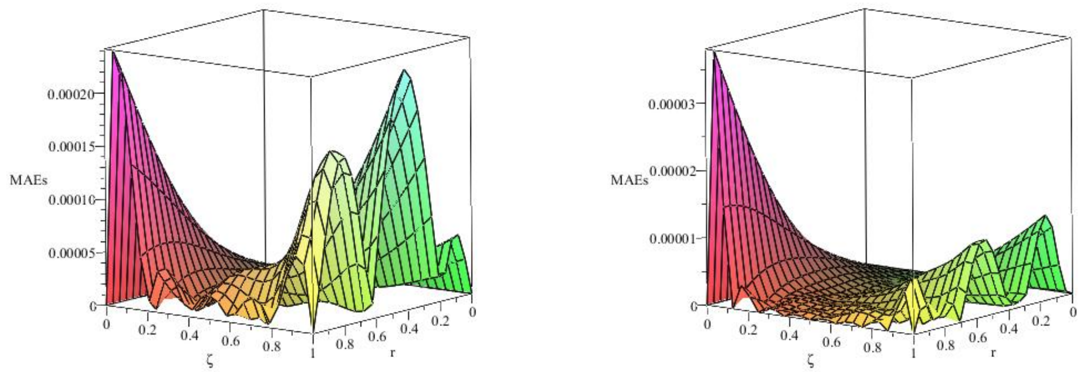

FIGURE 4. Plot of absolute errors for $\theta = 1, \eta = \frac{1}{2}$ and $N = 5$ (left), $N = 8$ (right) at $T = L = 1$ for case 3 for problem 2

TABLE 8. Maximum absolute errors for different values of $N, M(N = M), \theta$ and η at $T = L = 1$ for problem 3

(θ, η)	$N = 3$	$N = 5$	$N = 7$
$(\frac{1}{2}, \frac{1}{2})$	8.5×10^{-4}	4.4×10^{-4}	9.2×10^{-5}
$(\frac{1}{2}, \frac{3}{2})$	2.5×10^{-4}	1.2×10^{-4}	6.4×10^{-5}
$(1, 0)$	5.2×10^{-4}	10.5×10^{-4}	25.6×10^{-5}

TABLE 9. Values of $\parallel \nu_{N+1,M+1}(\iota, \xi) - \nu_{N,M}(\iota, \xi) \parallel_\infty$ at $L = T = 1$ for several choices of $N(N = M)$ for problem 3

(θ, η)	N=3	N=5	N=7
$(\frac{1}{3}, \frac{1}{2})$	6.5×10^{-4}	4.3×10^{-4}	15.6×10^{-5}

With boundary conditions

$$\begin{aligned} &\nu(0, \xi) = 0, \qquad && 0 < \xi \leq T, \\ &\nu(1, \xi) = 0, && \end{aligned} \tag{4.11}$$

and the initial condition

$$\nu(\iota, 0) = \iota^2(1 - \iota), \qquad 0 < \iota \leq L. \tag{4.12}$$

The exact solution of the problem is $\nu(\iota, \xi) = \iota^2(1 - \iota)e^{-t}$.

We consider $\beta_1 = 1.8$. Maximum absolute errors at $N = M = 5, 7$ for Example 3 are plotted in figure 5. Numerical results for several values of $N, M(N = M)$ and θ, η at $T = L = 1$ are obtained in Table 8. Values of $\parallel \nu_{N+1,M+1}(\iota, \xi) - \nu_{N,M}(\iota, \xi) \parallel_\infty$, which illustrates the numerical convergence in the lack of the exact solution at $L = T = 1$ for several choices of $N(N = M)$, are calculated for Example 3.

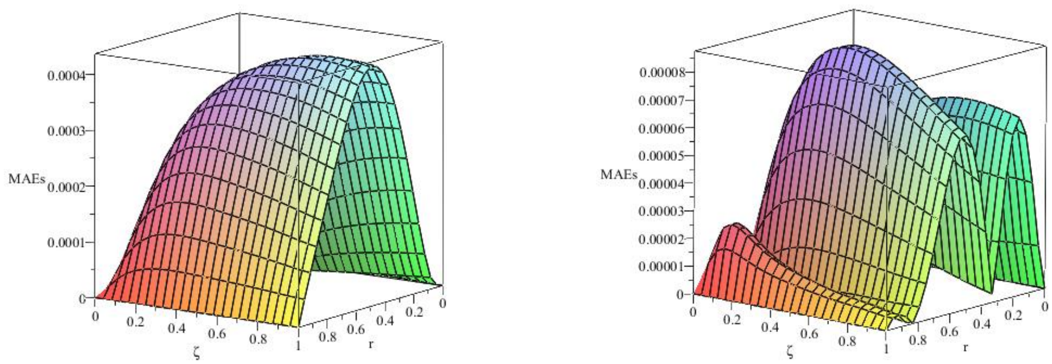

FIGURE 5. Plot of absolute errors for $\theta = \frac{1}{3}, \eta = \frac{1}{2}$ and $N = 5$ (left), $N = 7$ (right) at $T = L = 1$ for problem 3

5. Conclusion

This paper introduced a new numerical procedure based on shifted Jacobi orthogonal polynomials in conjunction with operational matrix of Caputo fractional derivative. we employed this procedure for solving space-time fractional PDEs. The suggested method obtained an easy way to solve numerically. Since, the propounded problem is reduced to a system of algebraic equations to provide the approximate solution and the system can be solved by iteration methods. According to the numerical results, the presented method is accurate. It may be extended to solve different types of fractional PDEs with variable coefficients and the suggested scheme is applicable to all types of boundary conditions.

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