Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 11, No. 1, 2023, pp. 143-160
DOI:10.22034/cmde.2022.49491.2060

An ABC algorithm based approach to solve a nonlinear inverse reaction-diffusion problem associate with the ecological invasions

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Abstract

In the present study, we consider an important mathematical model of the spread of two competing species in an ecological system with two species considering the interactions between these species. This model is derived from a system of nonlinear reaction-diffusion equations. We investigate this model as an inverse problem. Using appropriate initial and boundary conditions, the finite difference method in the time variable and the Quartic B-spline collocation method in the spatial variable are used to develop a numerical method. The proposed numerical approach results in an ill-posed linear system of equations and to overcome the ill-posedness, the Tikhonov regularization method is implemented. An effective approach based on the ABC algorithm is established to determine the regularization parameter. To show the robustness and ability of the present approach, for a test case, the results are compared with the results of the L-curve and GCV methods.

Keywords. Inverse problem, Reaction-Diffusion problem, Regularization, ABS algorithm.
2010 Mathematics Subject Classification. 65N32, 65M06, 65M12.

1. Introduction

Nowadays, mathematical modeling of ecological and biological systems as a challenging research field incorporating modern applied mathematics into biotechnology, engineering science, and pharmacy, has received much attention [4, 8, 11, 14, 16]. Using the mathematical models of relevant biological processes and their concerned numerical simulation can reduce complex and costly experimental procedures. Because of the complexity of most ecological and biological systems, very detailed models of these systems may work out to be unsuitable to use; in contrast, too simple models usually can not be able to account for the complexity of such systems. To model and analyze the behavior of ecological and biological systems, partial differential equations (PDEs) are one of the major mathematical tools.

This study deals with a system of nonlinear reaction-diffusion equations associated with ecological invasions. To model population ecology, organisms are considered to have Brownian random motion with a rate that is invariant in space and time. Based on this assumption, the result models can be written as diffusion models. In this study, we investigate an important class of PDE models of the ecological invasions as a nonlinear reaction-diffusion problem. The main model is written as an inverse problem of determination of boundary conditions. The determination of unknown parameters and unknown boundary conditions in such mathematical models of the ecological system has crucial importance. The main motivation of this study is to consider and investigate an important class of ecological invasions as an inverse problem. Investigation of mathematical models of ecological and biological models as direct problems is not a new concept and one may find many studies in the literature [1–5, 8, 11, 14–16]. For example in [3], a class of singularly perturbed parabolic reaction-diffusion initial boundary value problems have been investigated numerically. L. Chen et al. in [4] have reviewed the recent advancements in nonlocal continuous models for migration. M. Dehghan et al. in [5] concerned with a numerical study of a class of stationary states for reaction-diffusion systems with densities having disjoint supports as the rate of interaction between two different species tends to infinity. In

Received: 19 December 2021 ; Accepted: 30 March 2022.
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[14], the RBF method in conjunction with the Crank-Nicolson finite difference approach is conducted to numerically investigate an important class of the ecological invasions reaction-diffusion models. A numerical approach based on the spectral meshless radial point interpolation (SMRPI) technique is established in [15] to find the solution of pattern formation in the form of systems of nonlinear reaction-diffusion equations. Q. J. Tan et al. have investigated a class of two-species invasion models with a free boundary and with cross-diffusion and self-diffusion in [16]. For some reaction-diffusion models resulting from biological systems, the asymptotic behavior of solutions has been investigated in [8].

Motivated by previous studies and the importance of the determination of unknown parameters in the mathematical models of ecological and biological systems, this work concentrates on the numerical solution of an inverse problem that results from ecological invasions. Because of the ill-posedness of most inverse problems, using regularization approaches seems to be necessary [9, 10]. For instance M. Garshasbi et. al. established regularization approaches based on conjugate gradient method and mollification-based marching approaches to investigate some nonlinear reaction-diffusion models of cancer tumors [6, 7]. Y. Wang et al. have investigated an inverse problem of determination of the space-dependent source term in a nonlinear parabolic equation with nonlocal diffusion coefficient depending on the population itself in biology and ecology fields [17].

In the concept of inverse problems, formulation of these problems as variational problems (optimization problems) is a common approach in which for determination of unknown functions and parameters, based on the problem data, a residual functional should be defined and appropriate analytical or numerical methods should be conducted to solve these variational problems. For solving such problems, one may find many approaches but recently there appears a new tool for combinatorial optimization, called the artificial bee colony (ABC) algorithm. Such as the ant colony optimization (ACO) algorithm, the ABC algorithm is a part of Swarm Intelligence (SI), which is a group of algorithms of artificial intelligence, based on the collective behavior of decentralized, self-organized systems of objects [12, 13]. This idea first introduced in the context of cellular robotic systems by Jing Wang and Gerardo Beni in 1989 [18]. Many studies has continued in this manner anyone may find the applications of this idea in different fields in the literature. Most of the present optimization approaches need to fulfill a different number of assumptions about the properties of the optimized function, its variables, or its domain and the classical algorithms usually can be used only for solving a special kind of optimized problem. The algorithms motivated by the nature such as the genetic algorithms or algorithms inspired by the behavior of the insects, like ABC or ACO algorithms prove themselves as effective approaches in optimization problems. The assumption of the existence of the solution is sufficient to use these approaches and the solution can be found with some given precision of course. With regard to these approaches, it is worth noting that the obtained solution should be treated as the best solution at the given moment. Running the algorithm one more time can give a different solution, even better. But it may do not decrease the effectiveness of those algorithms.

In this study after introducing the main problem as an inverse problem, we will conduct a numerical method based on the finite difference and collocation approaches to discretize the problem. Solving the discretized form of the problem deals with solving an ill-conditioned system of linear equations. The Tikhonov regularization method in conjunction ABC algorithm is used to solve this ill-posed problem.

The outline of this paper is as follows:

In section 2, the main problem is briefly introduced. Section 3 deals with the numerical solution of the problem based on time discretization and the Quartic B-spline collocation method. In section 4, the Tikhonov regularization method and the ABC algorithm are discussed. Section 5 contains some numerical results and discussions.

2. Problem definition

In an ecological system with two species considering the interactions between these species, the mathematical modelling of invasions of species in more complex settings may restrict to pairwise interactions. An appropriate model for the spread of two competing species may be written in dimensionless form as follows [11, 14]:

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_u \frac{\partial^2 u}{\partial x^2} + (r_u - \alpha_{uu} u - \alpha_{uv} v)u + F(x,t),\\ \frac{\partial v}{\partial t} &= D_v \frac{\partial^2 v}{\partial x^2} + (r_v \alpha_{vv} v - \alpha_{vu} u)v + G(x,t),\end{aligned} \tag{2.1}$$

where u and v denote the species's densities, D_u and D_v are species-specific diffusion rates, $\alpha_j;\ j = uu, vv, uv, vu$, are the interspecific and intraspecific competition coefficients, and $r_j;\ j = u, v$ represent species-specific intrinsic rates of increase. This system of nonlinear reaction-diffusion equations may be considered in a space-time domain such as $\Omega = \{(x,t) : x \in (0,1), t \in (0,1)\}$, with initial and boundary conditions as

$$u(x,0) = f(x), \tag{2.2}$$
$$v(x,0) = g(x), \tag{2.3}$$
$$u(0,t) = p_1(t), \tag{2.4}$$
$$v(0,t) = p_2(t), \tag{2.5}$$
$$u(1,t) = q_1(t), \tag{2.6}$$
$$v(1,t) = q_2(t), \tag{2.7}$$

where $f(x), g(x)$ are continuous known functions, $p_1(t)$, $q_1(t)$, $\sigma_1(t)$, $\gamma_1(t)$, $\sigma_2(t)$, and $\gamma_2(t)$ are infinitely differentiable known functions and t_f represents the final existence time for the time evolution of the problem. Here we consider that the functions $q_1(t), q_2(t)$ are unknown which remains to be determined. Furthermore we consider following over-determination conditions are at hand

$$u(l_1,t) = \sigma_1(t), \tag{2.8}$$
$$u_x(l_1,t) = \gamma_1(t), \tag{2.9}$$
$$v(l_2,t) = \sigma_2(t), \tag{2.10}$$
$$v_x(l_2,t) = \gamma_2(t), \tag{2.11}$$

where $0 < l_1, l_2 < 1$ are known fixed points and $\sigma_1(t)$, $\sigma_2(t)$ are smooth known functions.

3. Numerical solution

In this section, we establish a numerical approach based on the Crank-Nicolson and Quartic B-spline collocation approaches in conjunction with a regularization method based on the ABC algorithm to solve the inverse problem (2.1)-(2.11).

3.1. Time discretization.

To establish a numerical approach first, we discretize the equations (2.1) by using the forward difference rule for time derivatives and the well known Crank-Nicolson scheme for other terms between successive time levels n and $n+1$. Suppose Δt denotes the time step size, $t_n = t_0 + n\Delta t$, $U^n = u(x,t_n)$, $V^n = v(x,t_n)$, $F^n = F(x,t_n)$ and $G^n = G(x,t_n)$. Discretizing these equations yields

$$\frac{U^{n+1} - U^n}{\Delta t} = D_u \frac{U_{xx}^{n+1} + U_{xx}^n}{2} + r_u \frac{U^{n+1} + U^n}{2} - \alpha_{uu} \frac{(U^2)^{n+1} + (U^2)^n}{2} - \alpha_{uv} \frac{(UV)^{n+1} + (UV)^n}{2} + F^n, \tag{3.1}$$

$$\frac{V^{n+1} - V^n}{\Delta t} = D_v \frac{V_{xx}^{n+1} + V_{xx}^n}{2} + r_u \alpha_{vv} \frac{(V^2)^{n+1} + (V^2)^n}{2} - \alpha_{vu} \frac{(UV)^{n+1} + (UV)^n}{2} + G^n. \tag{3.2}$$

Using the following formula [14]

$$(UV)^{n+1} = U^{n+1}V^n + U^n V^{n+1} - U^n V^n,$$

equations (3.1) and (3.2) may be linearized as rewritten as

$$\begin{aligned} 2U^{n+1} \quad - \quad & D_u \Delta t U_{xx}^{n+1} - r_u \Delta t U^{n+1} + 2\alpha_{uu} U^n U^{n+1} + \alpha_{uv} \Delta t (U^n V^{n+1} + V^n U^{n+1}) \\ & = (2 + r_u \Delta t) U^n + D_u \Delta t U_{xx}^n + F^n, \qquad (3.3) \\ 2V^{n+1} \quad - \quad & D_v \Delta t V_{xx}^{n+1} - 2\Delta t r_v \alpha_{vv} V^n V^{n+1} + \alpha_{uv} \Delta t (U^n V^{n+1} + V^n U^{n+1}) \\ & = 2V^n + D_v \Delta t V_{xx}^n + G^n. \qquad (3.4) \end{aligned}$$

3.2. **Quartic B-spline collocation method.** To solve the system of differential equations (3.3) and (3.4), here we establish an approach based on quartic B-spline collocation method. Note that the boundary functions at $x = 1$ are unknown and the proposed method can not be implied without using over-determination conditions (2.8)- (2.10). First the solution domain $x \in [0, 1]$ is partitioned into a mesh of uniform length $h = x_{i+1} - x_i$ by the knots x_i, $i = 0, 1, ...N-1$, such that $0 = x_0 < x_1 < ... < x_n = 1$ be the partition in $[0, 1]$. B-splines are the unique nonzero splines of smallest compact support with knots at $x_0 < x_1 < ... < x_n$. We define the quartic B-spline $B_i(x)$ for $i = -2, 0, ..., N + 1$ by the following relation [45]

$$B_i(x) = \frac{1}{h^4} \begin{cases} (x - x_{i-2})^4, & x \in [x_{i-2}, x_{i-1}) \\ (x - x_{i-2})^4 + 5(x - x_{i-1})^4, & x \in [x_{i-1}, x_i) \\ (x - x_{i-2})^4 - 5(x - x_{i-1})^4 + 10(x - x_i)^4, & x \in [x_i, x_{i+1}) \\ (x_{i+3} - x)^4 - 5(x_{i+2} - x)^4, & x \in [x_{i+1}, x_{i+2}) \\ (x_{i+3} - x)^4, & x \in [x_{i+2}, x_{i+3}) \\ 0 & o.w \end{cases} \qquad (3.5)$$

It can be easily seen that the set of the quartic B-spline functions as $\Gamma = \{B_{-2}(x), B_{-1}(x), B_0(x), ..., B_{N+1}(x)\}$ is linearly independent on $[0, 1]$, thus $\Theta = span(\Gamma)$ is a subspace of $C^2[0, 1]$ and Θ is $(N + 4)$-dimensional. Let us consider $U_m(x, t) \in \Theta$, $V_m(x, t) \in \Theta$ are the B-spline approximation to the exact solution $U(x, t)$, $V(x, t)$ in the form

$$U_m(x, t) = \sum_{i=-2}^{m+1} c_i(t) B_i(x), \qquad (3.6)$$

$$V_m(x, t) = \sum_{i=-2}^{m+1} \widehat{c}_i(t) B_i(x), \qquad (3.7)$$

where $c_i(t)$ and $\widehat{c}_i(t)$ are time-dependent quantities to be determined from the boundary and over-specified conditions and collocation form of the differential equations.

Using the approximate functions (3.6) and (3.7) and the quartic B-spline (3.5), the approximate values at the knots of $U(x), V(x)$ and their derivatives up to third order are determined in terms of the time-dependent parameters c_m and $\widehat{c}_m$ as

$$\begin{aligned} U_m &= c_{m+1} + 11c_m + 11c_{m-1} + c_{m-2}, \\ U_m' &= (\frac{4}{h})(c_{m+1} + 3c_m - 3c_{m-1} - c_{m-2}), \\ U_m'' &= (\frac{12}{h^2})(c_{m+1} - c_m - c_{m-1} + c_{m-2}), \\ U_m''' &= (\frac{24}{h^3})(c_{m+1} - 3c_m + 3c_{m-1} - c_{m-2}), \qquad (3.8) \end{aligned}$$

and

$$\begin{aligned} V_m &= \widehat{c}_{m+1} + 11\widehat{c}_m + 11\widehat{c}_{m-1} + \widehat{c}_{m-2}, \\ V_m^{'} &= (\frac{4}{h})(\widehat{c}_{m+1} + 3\widehat{c}_m - 3\widehat{c}_{m-1} - \widehat{c}_{m-2}), \\ V_m^{''} &= (\frac{12}{h^2})(\widehat{c}_{m+1} - \widehat{c}_m - \widehat{c}_{m-1} + \widehat{c}_{m-2}), \\ V_m^{'''} &= (\frac{24}{h^3})(\widehat{c}_{m+1} - 3\widehat{c}_m + 3\widehat{c}_{m-1} - \widehat{c}_{m-2}), \end{aligned} \tag{3.9}$$

Substituting the approximate solution (3.6) and (3.7) in (3.3) and (3.4) and using (3.8) and (3.9), one may drive

$$\begin{aligned} a_1 c_{m-2}^{n+1} \quad + &\quad a_2 c_{m-1}^{n+1} + a_3 c_m^{n+1} + a_4 c_{m+1}^{n+1} + g_3(\widehat{c}_{m+1}^{n+1} + 11\widehat{c}_m^{n+1} + 11\widehat{c}_{m-1}^{n+1} + \widehat{c}_{m-2}^{n+1}) \\ &= g_4(c_{m+1}^n + 11c_m^n + 11c_{m-1}^n + c_{m-2}^n) + \frac{12}{h^2}(D_u\Delta t)(c_{m+1}^n - c_m^n - c_{m-1}^n + c_{m-2}^n) + F^n, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} &g_1 = 2 - r_u\Delta t + 2\alpha_{uu}U^n + \alpha_{uv}\Delta t V^n, \\ &g_2 = -D_u\Delta t, \\ &g_3 = \alpha_{uv}\Delta t U^n, \\ &g_4 = 2 + r_u\Delta t, \end{aligned}$$

and

$$a_1 = g_1 + \frac{12}{h^2}g_2, a_2 = 11g_1 - \frac{12}{h^2}g_2, a_3 = 11g_1 - \frac{12}{h^2}g_2, a_4 = g_1 + \frac{12}{h^2}g_2.$$

Now let

$$H(x,t_n) \quad = \quad g_4(c_{m+1}^n + 11c_m^n + 11c_{m-1}^n + c_{m-2}^n) + \frac{12}{h^2}(D_u\Delta t)(c_{m+1}^n - c_m^n - c_{m-1}^n + c_{m-2}^n), \tag{3.11}$$

and

$$\widehat{H}(x,t_n) \quad = \quad 2(\widehat{c}_{m+1}^n + 11\widehat{c}_m^n + 11\widehat{c}_{m-1}^n + \widehat{c}_{m-2}^n) + g_8 12/h^2(\widehat{c}_{m+1}^n - \widehat{c}_m^n - \widehat{c}_{m-1}^n - \widehat{c}_{m-2}^n). \tag{3.12}$$

Then we can write (3.10) as

$$a_1 c_{m-2}^{n+1} + a_2 c_{m-1}^{n+1} + a_3 c_m^{n+1} + a_4 c_{m+1}^{n+1} + g_3(c_{m+1}^{n+1} + 11\widehat{c}_m^{n+1} + 11\widehat{c}_{m-1}^{n+1} + \widehat{c}_{m-2}^{n+1}) = H(x_i,t_n) + F^n, 1 \le i \le N. \tag{3.13}$$

Similarly we can write

$$a_5\widehat{c}_{m-2}^{n+1} + a_6\widehat{c}_{m-1}^{n+1} + a_7\widehat{c}_m^{n+1} + a_8\widehat{c}_{m+1}^{n+1} + g_8(c_{m+1}^{n+1} + 11c_m^{n+1} + 11c_{m-1}^{n+1} + c_{m-2}^{n+1}) = \widehat{H}(x_i,t_n) + G^n, 1 \le i \le N, \tag{3.14}$$

where

$$g_5 = 2 - 2\Delta t r_v \alpha_{vv} V^n + \alpha_{uv}\Delta t U^n, g_6 = -D_v\Delta t, g_7 = \alpha_{uv}\Delta t V^n, g_8 = D_v\Delta t,$$

$$a_5 = g_6 + \frac{12}{h^2}g_7, a_6 = 11g_6 - \frac{12}{h^2}g_7, a_7 = 11g_6 - \frac{12}{h^2}g_7, a_8 = g_6 + \frac{12}{h^2}g_7.$$

The system (3.10) and (3.10) involve $2N+2$ linear equations with $(2N+8)$ unknowns

$$C^{n+1} = (c_{-2}^{n+1}, c_{-1}^{n+1}, c_0^{n+1}, c_1^{n+1}, ...c_N^{n+1}, c_{N+1}^{n+1}, \widehat{c}_{-2}^{n+1}, \widehat{c}_{-1}^{n+1}, \widehat{c}_0^{n+1}, ..., \widehat{c}_N^{n+1}, \widehat{c}_{N+1}^{n+1})^T.$$

To obtain a unique solution of this system, the over-determination conditions (2.8)-(2.11) are required. Expanding U in terms of the approximate quartic B-spline functions at x_s and putting $m=s$, we read

$$\begin{aligned} c_{-2}^{n+1} + 11c_{-1}^{n+1} + 11c_0^{n+1} + c_1^{n+1} &= p_1(t_{n+1}), \\ c_{s+4}^{n+1} + 11c_{s+3}^{n+1} + 11c_{s+2}^{n+1} + c_{s+1}^{n+1} &= \sigma_1(t_{n+1}), \\ (\frac{4}{h})(c_{s+4}^{n+1} + 3c_{s+3}^{n+1} - 3c_{s+2}^{n+1} - c_{s+1}^{n+1}) &= \gamma_1(t_{n+1}). \end{aligned} \tag{3.15}$$

Similarly for V we have

$$\begin{aligned}
\widehat{c}^{n+1}_{-2+N+5} + 11\widehat{c}^{n+1}_{-1+N+5} + 11\widehat{c}^{n+1}_{0+N+5} + \widehat{c}^{n+1}_{1+N+5} &= p_2(t_{n+1}),\\
\widehat{c}^{n+1}_{s+4+N+6} + 11\widehat{c}^{n+1}_{s+3+N+6} + 11\widehat{c}^{n+1}_{s+2+N+6} + \widehat{c}^{n+1}_{s+1+N+6} &= \sigma_2(t_{n+1}),\\
(\frac{4}{h})(\widehat{c}^{n+1}_{s+4+N+7} + 3\widehat{c}^{n+1}_{s+3+N+7} - 3\widehat{c}^{n+1}_{s+2+N+7} - \widehat{c}^{n+1}_{s+1+N+7}) &= \gamma_2(t_{n+1}).
\end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16) with (3.13) and (3.14), we deal with a $(2N+8)\times(2N+8)$ linear system of equations which can be written in the matrix form as $AC^{n+1} = D$, where

$$A = \begin{pmatrix} A_1 & | & A_2 \\ -- & | & -- \\ A_3 & | & A_4 \end{pmatrix}, \tag{3.17}$$

with

$$A_1 = \begin{pmatrix}
1 & 11 & 11 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & 0 & 1 & 11 & 11 & 1 & 0 & \cdots & 0\\
0 & 0 & 0 & 0 & -\frac{4}{h} & -\frac{12}{h} & \frac{12}{h} & \frac{4}{h} & 0 & \cdots & 0\\
a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
 & & & & & \ddots & & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & & a_1 & a_2 & a_3 & a_4
\end{pmatrix},$$

$$A_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
g_3 & 11g_3 & 11g_3 & g_3 & 0 & & \cdots & & & 0\\
 & & & & & & \ddots & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & g_3 & 11g_3 & 11g_3 & g_3
\end{pmatrix},$$

$$A_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
g_8 & 11g_8 & 11g_8 & g_8 & 0 & & \cdots & & & 0\\
 & & & & & & \ddots & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & g_8 & 11g_8 & 11g_8 & g_8
\end{pmatrix},$$

$$A_4 = \begin{pmatrix}
1 & 11 & 11 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & 0 & 1 & 11 & 11 & 1 & 0 & \cdots & 0\\
0 & 0 & 0 & 0 & -\frac{4}{h} & -\frac{12}{h} & \frac{12}{h} & \frac{4}{h} & 0 & \cdots & 0\\
a_5 & a_6 & a_7 & a_8 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
 & & & & & \ddots & & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & & a_5 & a_6 & a_7 & a_8
\end{pmatrix}.$$

and

$$D=(D^n_{-2},D^n_{-1},D^n_0,\cdots,D^n_{N+1},\widehat{D}^n_{-2},\widehat{D}^n_{-1},\widehat{D}^n_0,\cdots,\widehat{D}^n_{N+1})^T,$$

with

$$\begin{aligned}
&D^n_{-2}=p_1(t_{n+1}),\\
&D^n_{-1}=\sigma_1(t_{n+1}),\\
&D^n_0=\gamma_1(t_{n+1}),\\
&D^n_i=H(x_i,t_n)+F^n,\quad 1\le i\le N+1\ ,
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{D}^n_{-2}=p_2(t_{n+1}),\\
&\widehat{D}^n_{-1}=\sigma_2(t_{n+1}),\\
&\widehat{D}^n_0=\gamma_2(t_{n+1}),\\
&\widehat{D}^n_i=H(x_i,t_n)+G^n,\quad 1\le i\le N+1.
\end{aligned}$$

Note that the values of the elements of A and D are dependent to the results of time level t_n for $n\ge 0$. For more details and the convergence and stability results we refer readers to [1, 2].

The initial state: To compute the values of C^0, using the initial conditions (2.2) and (2.3) and the over-determination conditions (2.8)-(2.11) at $t=0$ yields

$$\begin{aligned}
&u(x_i,0)=c^0_{i+1}+11c^0_i+11c^0_{i-1}+c^0_{i-2}=f(x_i),\quad 0\le i\le N,\\
&u(0,0)=c^0_{-2}+11c^0_{-1}+11c^0_0+c^0_1=p_1(0),\\
&u(l_1,0)=c^0_{s+4}+11c^0_{s+3}+11c^0_{s+2}+c^0_{s+1}=\sigma_1(0),\\
&u_x(l_1,0)=(\tfrac{4}{h})(c^0_{s+4}+3c^0_{s+3}-3c^0_{s+2}-c^0_{s+1})=\gamma_1(0),
\end{aligned}$$

$$\begin{aligned}
&v(x_i,0)=\widehat{c}^0_{i+1}+11\widehat{c}^0_i+11\widehat{c}^0_{i-1}+\widehat{c}^0_{i-2}=g(x_i),\quad 0\le i\le N,\\
&v(0,0)=\widehat{c}^0_{-2+N+5}+11\widehat{c}^0_{-1+N+5}+11\widehat{c}^0_{0+N+5}+\widehat{c}^0_{1+N+5}=p_2(0),\\
&v(l_2,0)=\widehat{c}^0_{s+4+N+6}+11\widehat{c}^0_{s+3+N+6}+11\widehat{c}^0_{s+2+N+6}+\widehat{c}^0_{s+1+N+6}=\sigma_2(0),\\
&v_x(l_2,0)=(\tfrac{4}{h})(\widehat{c}^0_{s+4+N+7}+3\widehat{c}^0_{s+3+N+7}-3\widehat{c}^0_{s+2+N+7}-\widehat{c}^0_{s+1+N+7})=\gamma_2(0).
\end{aligned}$$

This system reads as a $(2N+8)\times(2N+8)$ linear system of equations and in matrix form can be written as $A^0C^0=D^0$.

The systems of linear algebraic equations $A^0C^0=D^0$ and $AC^{n+1}=D$ are both ill-posed and using regularization approaches is essential to solve these equations. In fact, due to the large value of the condition numbers of the coefficient matrices A^0 and A at any time levels, the direct and common iterative methods cannot be used. Furthermore increasing the number of collocation points cause a highly instability in the solution. To solve the ill-posed linear systems, many regularization approaches have been proposed in the literature [6, 7, 9, 12, 13, 18]. The Tikonov regularization method is a well-known regularization approach that has been conducted to solve many kinds of ill-posed problems [9, 12, 13, 18]. In the next section, the Tikonov regularization method in conjunction with a new procedure to estimate the regularization parameters is conducted to solve the mentioned system of equations.

4. Tikhonov Regularization Method

For most inverse PDE problems, using numerical approaches such as finite differences and collocation methods result in an ill-condition linear system of equations. Using Tikhonov regularization method deals with the following minimization problem

$$\min\left\{\|AC-D\|^2+\alpha^2\|C\|^2\right\},\tag{4.1}$$

where $\alpha\in R$ denotes the regularization parameter [9, 13]. Based on the Tikhonov regularization method, we find a solution as a minimizer of a weighted combination of the residual norm and a side constraint. Determination of appropriate regularization parameters has crucial importance in the Tikhonov-based regularization approaches. In fact, this parameter controls the weight given to the minimization of the size constraint and the quality of the regularized solution. For many ill-posed problems, the determination of suitable regularization parameters is still a vital problem.

In this paper, we introduce a new method based on the ABC algorithm for selecting the Tikhonov regularization method.

4.1. **The ABC algorithm.** Let us consider the function $F(x)$ defined in the domain $\mathbb{S}$. We do not need to make any assumptions about the function, or its domain. Points of the domain - vectors x- play the role of the sources of nectar. Value of the function in the given point - number x - designates the quality of the source x. Since we are looking for the minimum, the smaller is the value $F(x)$, the better is the source x.

We will proceed according to the following algorithm:

Initialization of the algorithm.

(1) Initial data:
SN- number of the explored sources of nectar (= number of the bees - scouts, = number of the bees - viewers);
D- dimension of the source $x_i, i = 1, ..., SN$,
lim- number of "corrections" of the source position x_i;
MCN- maximal number of cycles.

(2) Initial population - random selection of the initial sources localization,
represented by the $\mathbb{S}$ dimensional vectors x_i, $i = 1, ..., SN$,

(3) Calculation of the values $F(x_i)$, $i = 1, ..., SN$, for the initial population.

The main algorithm.

(1) Modification of the sources localizations by the bees - scouts.

a) Every bee - scout modifies the position x according to the formula:

$$v_i^j = x_i^j + \phi_{ij}(x_i^j - x_k^j), j \in \{1, ..., D\},$$

where $k \in \{1, ..., SN\}, k \neq i, \phi_{ij} \in [-1, 1]$ - randomly selected numbers.

b) If $F(v_i) \leq F(x_i)$, then the position v_i replaces x_i. Otherwise, the position
x_i stays unchanged.

Steps a) and b) are repeated lim times. We take: $lim = SN.D$.

2. Calculation of the probabilities $P_i$for the positions x_i selected in step 1.

We use the formula:

$$P_i = \frac{fit_i}{\sum_{j=1}^{SN} fit_j}, i = 1, ..., SN,$$

where $fit_i = \begin{cases} \frac{1}{1+F(x_i)} & F(x_i) \geq 0, \\ 1 + |F(x_i)| & F(x_i) < 0, \end{cases}$

(1) Every bee - viewer chooses one of the sources $x_i, i = 1, ..., SN$ with the probability P_i. Of course, one source can be chosen by a group of bees.

(2) Every bee - viewer explores the chosen source and modifies its position according to the procedure described in step 1.

(3) Selection of the x_best for the current cycle - the best source among the sources determined by the bees - viewers. If the current x_best is better that the one from the previous cycle, we accept it as the x_best for the whole algorithm.

(4) If in step 1, the bee - scout did not improve the position x_i(x_i did not change), it leaves the source x_i and moves to the new one, according to the formula:

$$x_i^j = x_{\min}^j + \phi_{ij}(x_{\max}^j - x_{\min}^j), j \in \{1, ..., D\},$$

where $\phi_{ij} \in [0, 1]$.

Steps 1-6 are repeated MCN times.

Calculation of fitness

The key of the ABC algorithm approach for selecting regularization parameters lies in the design of its function. This paper selects regularization parameters in accordance with the Morozov deviation principle and therefore regards

$$F(\alpha) = \left\|AC^{\alpha} - D^{\delta}\right\| - \delta^2,$$

as the fitness function, and ε as stop error. When $F(\alpha) < \varepsilon$, we stop the calculation, and the result we get here is considered to be the best regularization parameter. The calculation procedure for fitness function of regularization parameter is as shown in the following algorithm [12, 13]

i: Given observation data error δdeviation function error ε.
ii: Given regularization parameter α.
iii: Solving Euler equation $(A^TA + \alpha I)C^{\alpha} = A^TD$.
iv: Solving deviation equation

$$F(\alpha) = \left|\left\|AC^{\alpha} - D^{\delta}\right\| - \delta^2\right|.$$

5. Numerical discussions

In this section, we illustrate the validity of the proposed numerical approach. To this end, we conduct two test problems using the proposed numerical approach.

Example 5.1. In the problem (2.1)-(2.11), consider

$$\begin{aligned}
F(x,t) &= sech^3(t-x)(-\cosh(t-x) + \sinh(t-x))((-9 + 7\cosh 2(t-x)) + 10\sinh 2(t-x)),\\
G(x,t) &= (1 + 2\tanh(t-x))(4 + sech^2(t-x) + 2\tanh(t-x)),\\
f(x) &= 2(1 + \tanh(x)),\ g(x) = 2 - \tanh(x),\\
p_1(t) &= 2(1 - \tanh(t)),\ p_2(t) = 2 + \tanh(t),\\
\sigma_1(t) &= 2(1 - \tanh(t - x_s)),\ \sigma_2(t) = 2 + \tanh(t - x_s),\\
\gamma_1(t) &= 2 - 2\tanh^2(t - x_s),\ \gamma_2(t) = \tanh^2(t - x_s) - 1.\\
Du &= 4, Dv = 1, ru = rv = 2, \alpha_{uu} = \alpha_{vv} = \alpha_{uv} = \alpha_{vu} = 1,\\
l_1 &= l_2 = 0.7.
\end{aligned}$$

With these assumptions, one can find the exact solution as [14]

$$u(x,t) = 2(1 - \tanh(t-x)), v(x,t) = 2 + \tanh(t-x).$$

Furthermore in the ABC algorithm, we consider

$$\begin{aligned}
&SN = 20 : \text{The number of bees}\\
&D = 1 : \text{The dimension of the source}\\
&lim = SN.D = 60 :\ \text{The number of "corrections" of the source position}\\
&MCN = 50 :\ \text{The maximal number of cycles}
\end{aligned}$$

.

In Figures 1 and 2, the exact and approximate results for $u(1,t)$ and $v(1,t)$ and the L_2 error norms between them are demonstrated respectively. The numerical results show a good agreement between the approximate and exact solutions when the ABC algorithm is used to determine the regularization parameters at the noise level $\delta = 0.01$.

Figures 3 and 4 show the comparison between the exact and approximate results for $u(1,t)$ and $v(1,t)$ and their L_2 error norm when we use the L-curve approach for finding the regularization parameter α at the noise level $\delta = 0.01$. In addition, these results are shown in Figures 5 and 6 using the GCV method for determining the regularization parameter. For the implementation of the numerical approach based on the L-curve and GCV methods, we use the packages gained from the website:

http://www.mathworks.com/matlabcentral/fileexchange/52-regtools.

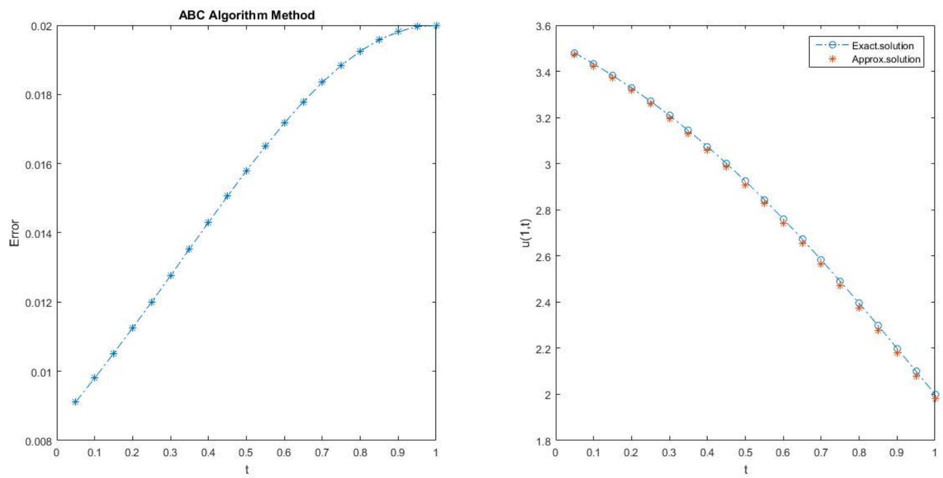

FIGURE 1. The Comparison between the exact and approximate solutions for $u(1,t)$ at the noise level $\delta = 0.01$ based on the ABC algorithm for Example 5.1.

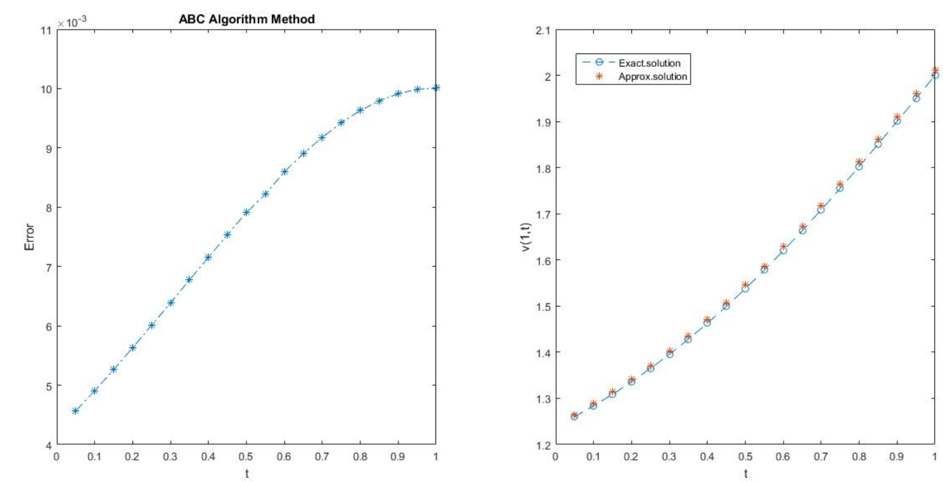

FIGURE 2. The Comparison between the exact and approximate solutions for $v(1,t)$ at the noise level $\delta = 0.01$ based on the ABC algorithm for Example 5.1.

To demonstrate the effect of regularization parameter on the numerical results, the results of GCV and L-curve approaches are compared with the results of ABC algorithm in Tables 1, 2 and 3 at three noise levels $\delta = 0.01$, $\delta = 0.001$, and $\delta = 0.0001$, respectively. The numerical results for $u(1,t)$ and $v(1,t)$ show that in comparison with the exact solution, the results of the ABC algorithm are reasonable.

The behaviors of L_2 error norm for u and v against the noise level δ for ABC algorithm are demonstrated in Figure 7.

As another test problem, we investigate the following problem.

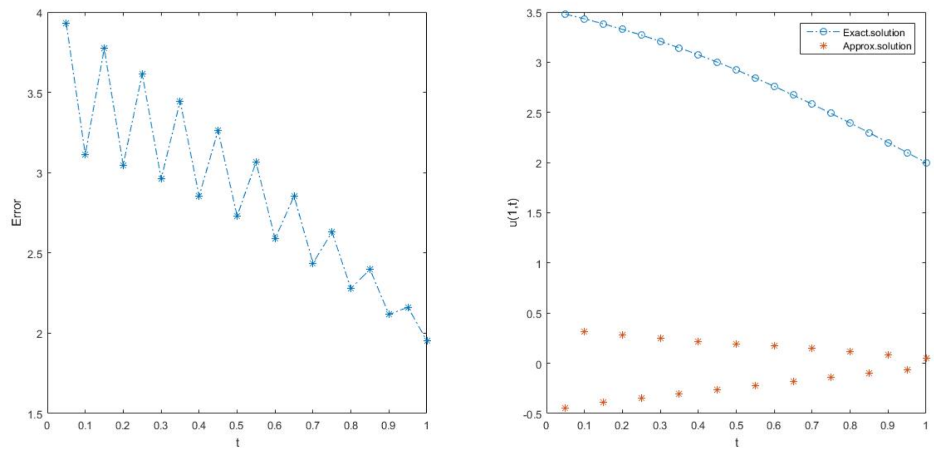

FIGURE 3. The Comparison between the exact and approximate solutions for $u(1,t)$ at the noise level $\delta = 0.01$ based on the L-curve algorithm for Example 5.1.

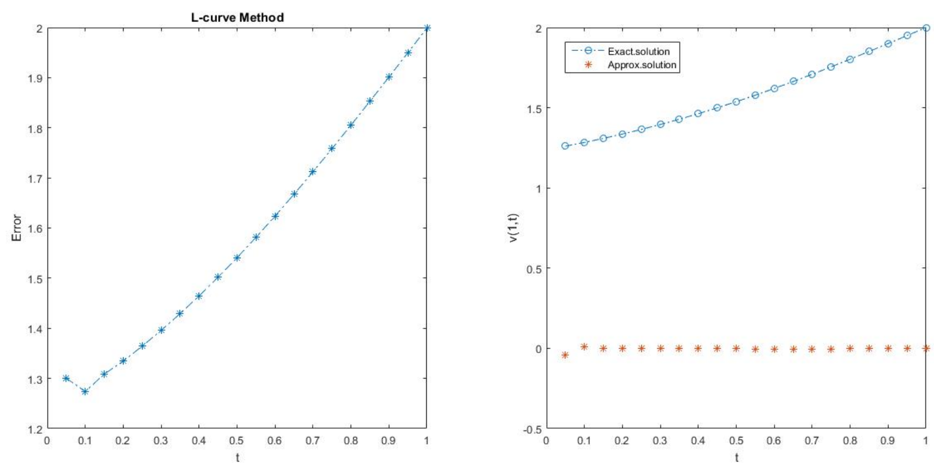

FIGURE 4. The Comparison between the exact and approximate solutions for $v(1,t)$ at the noise level $\delta = 0.01$ based on the L-curve algorithm for Example 5.1.

Example 5.2. For the problem (2.1)-(2.11), using the following data

$$
\begin{aligned}
F(x,t) &= -1.001e^{-t-x},\\
G(x,t) &= e^{-2(t+x)}(2-2.001e^{t+x}),\\
f(x) &= e^{-x},\ g(x) = 1-e^{-x},\\
p_1(t) &= e^{-t},\ p_2(t) = 1-e^{-t},\\
\sigma_1(t) &= e^{-t-x_s},\ \sigma_2(t) = 1-e^{-t-x_s},\\
\gamma_1(t) &= -e^{-t-x_s},\ \gamma_1(t) = e^{-t-x_s}.\\
Du &= 0.01, Dv = 0.01, ru = rv = 2,\\
\alpha_{uu} &= \alpha_{vv} = \alpha_{uv} = \alpha_{vu} = 1, N = 100, h = 0.01, t = 0.01,\\
l_1 &= l_2 = 0.7.
\end{aligned}
$$

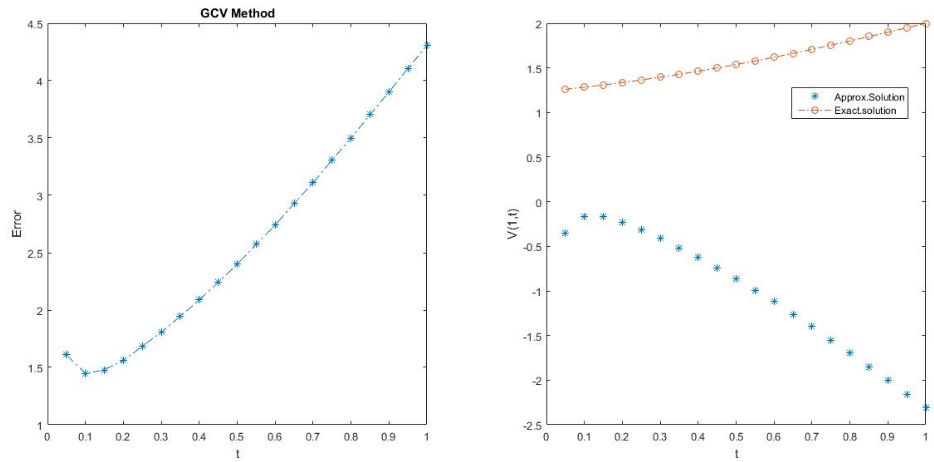

FIGURE 5. The Comparison between the exact and approximate solutions for $u(1,t)$ at the noise level $\delta = 0.01$ based on the GCV method for Example 5.1.

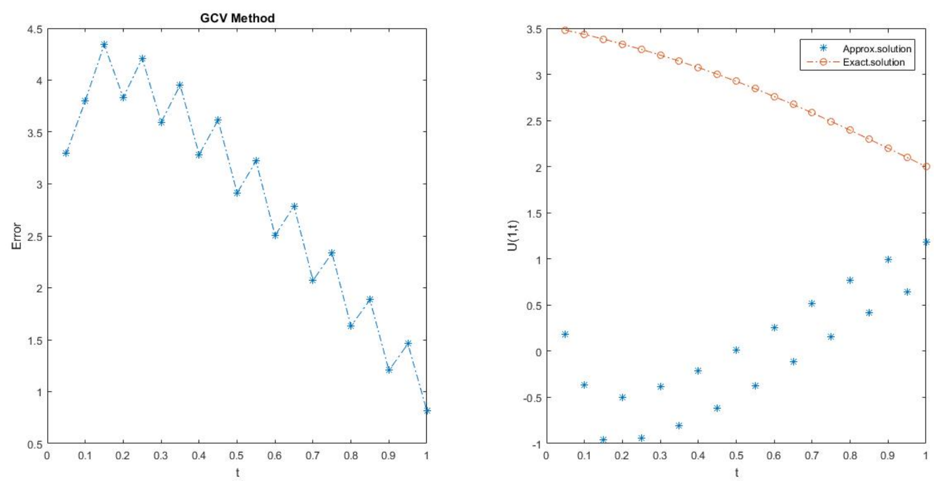

FIGURE 6. The Comparison between the exact and approximate solutions for $v(1,t)$ at the noise level $\delta = 0.01$ based on the GCV method for Example 5.1.

Based on these data, we can drive the exact solution as

$$u(x,t) = e^{-t-x}, v(x,t) = 1 - e^{-t-x}.$$

In this example, we consider the parameters in the ABC algorithm similar to the Example 5.1. In Figures 8 and 9, the exact and approximate solutions for $u(1,t)$ and $v(1,t)$ are compared when we use the ABC algorithm to determine the regularization parameters at the noise level $\delta = 0.01$, respectively. Furthermore, in these figures, the L_2 error norms between the exact and numerical results are demonstrated. As we expected conducting the ABC algorithm in this example yields reasonable results.

The L-curve and GCV approaches are conducted to determine the regularization parameters at noise level $\delta = 0.01$ and the comparison between the exact and numerical solutions for $u(1,t)$ and $v(1,t)$ based on the L-curve and GCV

Table 1. A comparison between the exact solution and the numerical results using the GCV, L-Curve, and ABC algorithms at the noise level $\delta = 0.01$ for Example 5.1.

	t	GCV	L-curve	ABC	Exact
u(1,t)	0. 2	-0.5181	-0.5933	3.3171	3.3280
	0. 4	0.0084	-0.6136	3.0691	3.0740
	0. 6	-0.2210	-0.4682	2.7499	2.7598
	0. 8	0.8941	0.4244	2.3850	2.3947
	1	0.7592	-0.460	1.9819	2
L_2 Error	–	3.8462	3.9214	0.0181	–
v(1,t)	0. 2	-0.7460	-0.0524	1.3454	1.3359
	0. 4	-0.6486	-0.4941	1.4740	1.4629
	0. 6	-1.0671	-0.0398	1.6340	1.6200
	0. 8	-1.5771	-0.0585	1.8174	1.8026
	1	-2.1523	-0.0728	2.0154	2
L_2 Error	–	4.1523	2.0728	0.0154	–

Table 2. A comparison between the exact solution and the numerical results using the GCV, L-Curve and ABC algorithms at the noise level $\delta = 0.001$ for Example 5.1.

	t	GCV	L-curve	ABC	Exact
u(1,t)	0.2	-0.5111	-0.5709	3.3174	3.3280
	0.4	0.0043	-0.6455	3.0606	3.0740
	0.6	-0.2158	-0.4473	2.7430	2.7598
	0.8	0.8910	0.4063	2.3755	2.3947
	1	0.7637	-0.4428	1.9801	2
L_2 Error	–	3.8392	3.8990	0.0198	–
v(1,t)	0.2	-0.7427	-0.0516	1.3423	1.3359
	0.4	-0.6478	-0.4945	1.4707	1.4629
	0.6	-1.0661	-0.0376	1.6291	1.6200
	0.8	-1.5760	-0.0556	1.8128	1.8026
	1	-2.1506	-0.06928	2.0107	2
L_2 Error	–	4.1506	2.0693	0.0107	–

methods are shown in Figures 10,11,12 and 13. The results clearly show that in this example, using the ABC algorithm yields reasonable solutions rather than the L-curve and GCV approaches.

At the noise levels $\delta = 0.001$ and $\delta = 0.0001$, the GCV, L-Curve, ABC and exact solutions are compared at some points in Tables 4 and 5. It is clear that using the ABC algorithm result in acceptable results.

Based on the numerical results, we found that at the noise levels $\delta = 0.001$ and $\delta = 0.0001$, the numerical results obtained based on the ABC algorithm do not noticeable variations.

6. Conclusion

In this study, we investigate a nonlinear parabolic type inverse problem resulting from a mathematical model of the spread of two competing species in an ecological system with two species considering the interactions between these species. Based on appropriate initial and boundary conditions, the finite difference approach in conjunction with the Quartic B-spline collocation method are used to solve the proposed inverse problem. Solution of our interest inverse problem deals solving an ill-posed system of equations. To this end, the Tikhonov regularization method is used. To determine the regularization parameter, the ABC algorithm is used. The comparison between the numerical results of this algorithm with the results of the L-curve and GCV approaches shows that this algorithm can be very useful.

TABLE 3. A comparison between the exact solution and the numerical results using the GCV, L-Curve and ABC algorithms at the noise level $\delta = 0.0001$ for Example 5.1.

	t	GCV	L-curve	ABC	Exact
u(1,t)	0.2	-0.5095	-0.5723	3.3168	1.3359
	0.4	0.0030	-0.6433	3.0598	1.4629
	0.6	-0.2144	-0.4486	2.7427	1.6200
	0.8	0.8898	0.4075	2.3755	1.8026
	1	0.7650	-0.4439	1.9800	2
L_2 Error	–	3.8376	3.9004	0.0199	–
v(1,t)	0.2	-0.7428	-0.0520	1.3416	1.3359
	0.4	-0.6476	-0.4943	1.4701	1.4629
	0.6	-1.0660	-0.0377	1.6286	1.6200
	0.8	-1.5759	-0.0557	1.8123	1.8026
	1	-2.1505	-0.0694	2.0100	2
L_2 Error	–	4.1505	2.0695	0.0100	–

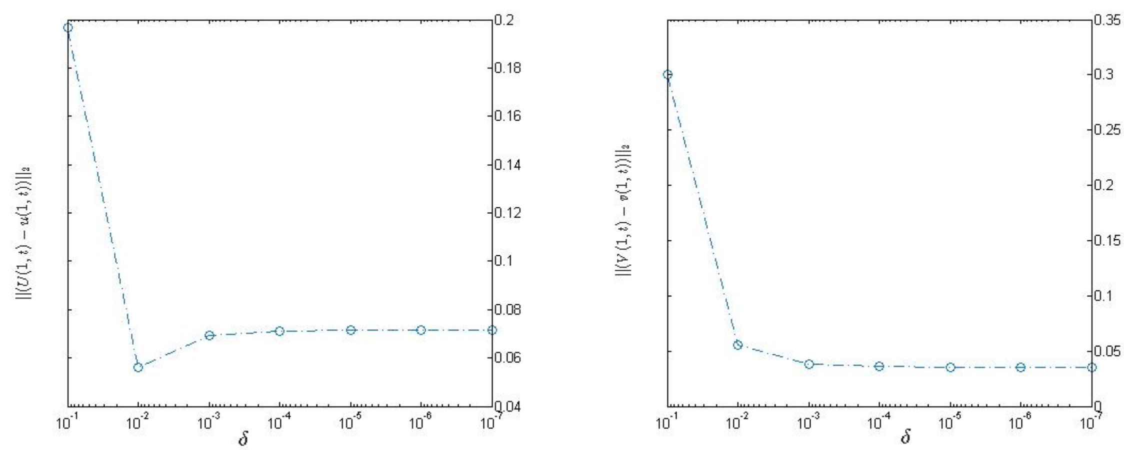

FIGURE 7. The behavior of l_2 error norm for $u(1,t)$ and $v(1,t)$ verses the noise level δ based on the ABC algorithm for Example 5.1.

ACKNOWLEDGMENT

It is a pleasure to thank the anonymous referees for their valuable suggestions and helpful comments which have materially improved the presentation of the paper.

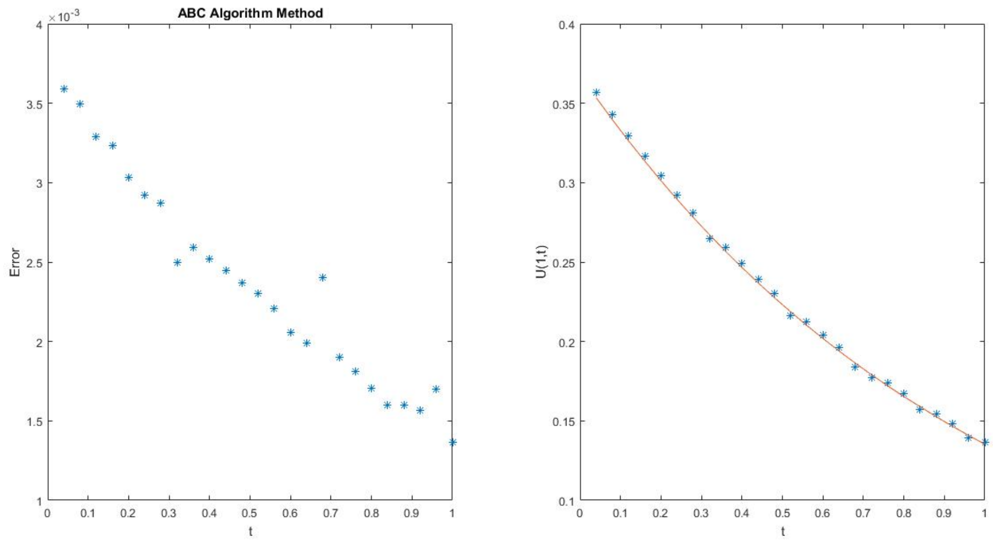

FIGURE 8. The Comparison between the exact and approximate solutions for $u(1,t)$ at the noise level $\delta = 0.001$ based on the ABC algorithm for Example 5.2.

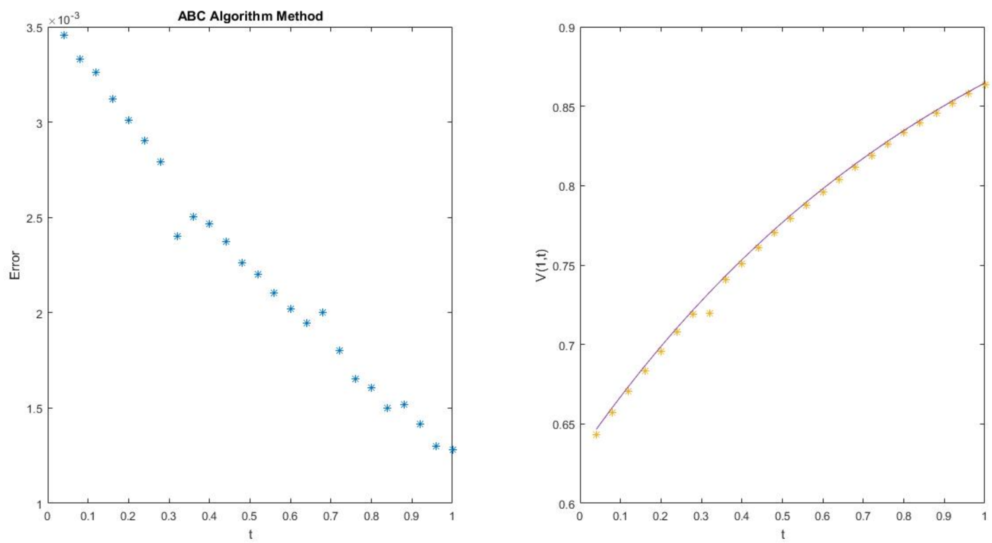

FIGURE 9. The Comparison between the exact and approximate solutions for $v(1,t)$ at the noise level $\delta = 0.001$ based on the ABC algorithm for Example 5.2.

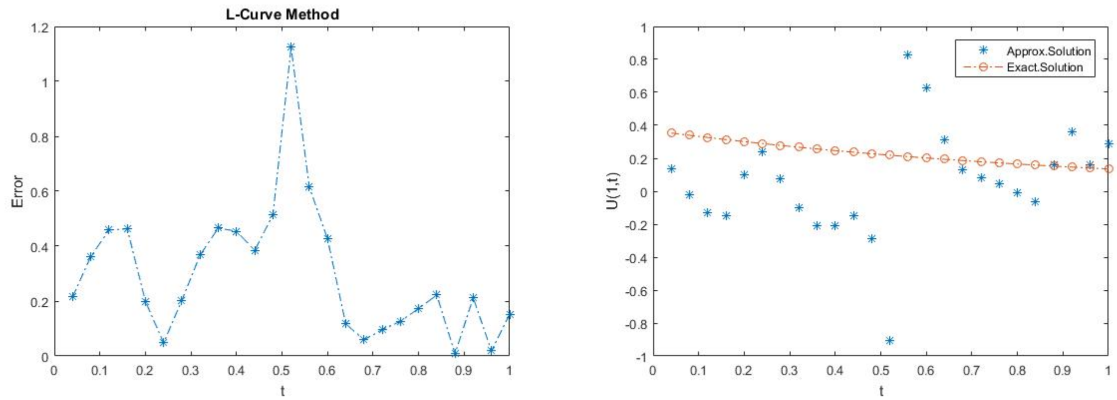

FIGURE 10. The Comparison between the exact and approximate solutions for $u(1,t)$ at the noise level $\delta = 0.01$ based on the L-curve algorithm for Example 5.2.

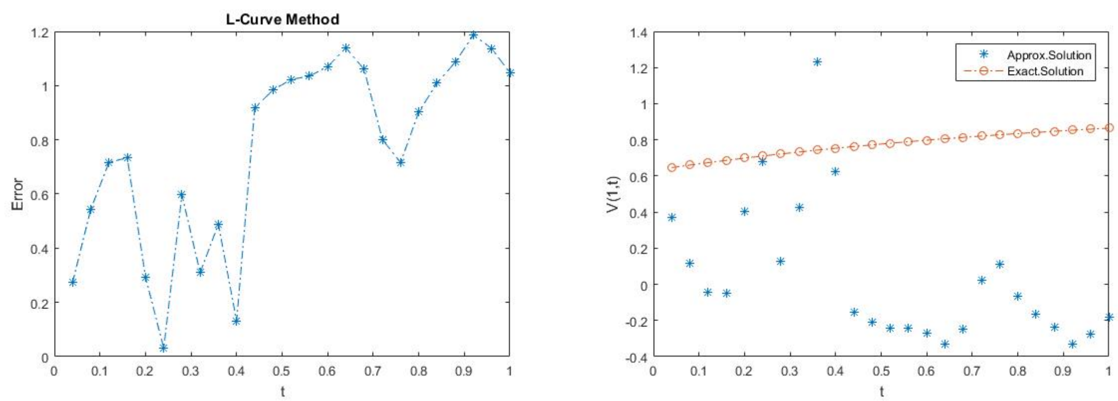

FIGURE 11. The Comparison between the exact and approximate solutions for $v(1,t)$ at the noise level $\delta = 0.01$ based on the L-curve algorithm for Example 5.2.

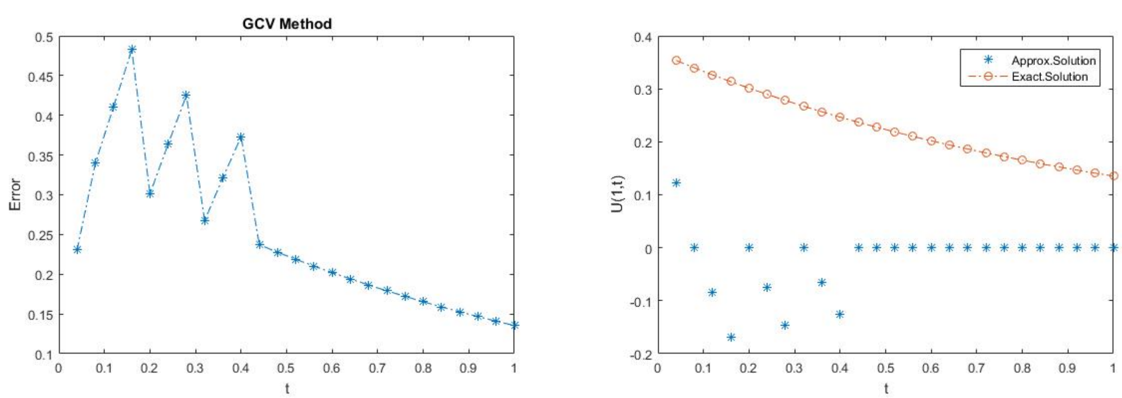

FIGURE 12. The Comparison between the exact and approximate solutions for $u(1,t)$ at the noise level $\delta = 0.01$ based on the GCV method for Example 5.2.

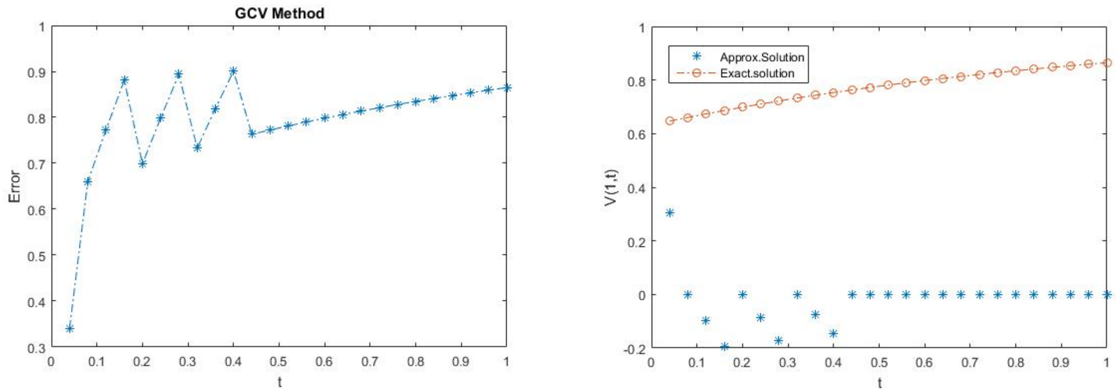

FIGURE 13. The Comparison between the exact and approximate solutions for $v(1,t)$ at the noise level $\delta = 0.01$ based on the GCV method for Example 5.2.

TABLE 4. A comparison between the exact solution and the numerical results using the GCV, L-Curve and ABC algorithms at the noise level $\delta = 0.001$ for Example 5.2

	t	GCV	L-curve	ABC	Exact
u(1,t)	0.2	-0.00017	-0.1203	0.3051	0.3012
	0.4	-0.1261	0.1995	0.2499	0.2466
	0.6	-0.000063	0.2040	0.2047	0.2019
	0.8	-0.000052	0.2398	0.1679	0.1653
	1	-0.000042	0.0374	0.1372	0.1353
L_2 Error	–	0.4822	0.9368	0.0039	–
v(1,t)	0.2	-0.00025	0.0754	0.6959	0.6988
	0.4	-0.1472	-0.0501	0.7516	0.7534
	0.6	-0.000065	0.7549	0.7963	0.7981
	0.8	-0.000045	0.6049	0.8339	0.8347
	1	-0.000029	0.6578	0.8641	0.8647
L_2 Error	–	0.9006	1.0616	0.0029	–

TABLE 5. A comparison between the exact solution and the numerical results using the GCV, L-Curve and ABC algorithms at the noise level $\delta = 0.0001$ for Example 5.2

	t	GCV	L-curve	ABC	Exact
u(1,t)	0. 2	-0.00073	0.1023	0.3042	0.3012
	0. 4	-0.1264	0.8260	0.2466	0.2491
	0. 6	-0.0000662	0.5341	0.2040	0.2019
	0. 8	-0.000052	0.0060	0.1670	0.1653
	1	-0.000043	-0.2171	0.1367	0.1353
L_2 Error	–	0.4826	0.5572	0.0036	–
v(1,t)	0. 2	-0.00025	0.4050	0.6958	0.6988
	0. 4	-0.1476	0.6225	0.7509	0.7534
	0. 6	-0.000067	0.2724	0.7961	0.7981
	0. 8	-0.00045	-0.0679	0.8331	0.8347
	1	-0.000027	-0.1827	0.8634	0.8647
L_2 Error	–	0.9010	1.1864	0.0035	–

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