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Hybrid collocation method for some classes of second-kind nonlinear weakly singular integral equations

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Abstract

In the current study, a fast, accurate and reliable numerical scheme for approximating second-kind nonlinear Fredholm, Volterra, and Fredholm-Volterra integral equations with a weakly singular kernel and invertible nonlinearity is presented. The computational approach is based upon function especially hybrid one. Hybrid functions give us the opportunity to achieve an appropriate solution by adjusting a suitable order for polynomials' degrees and block-pulse functions. The basic idea of this method is based on using the invertibility of the nonlinear function as a benefit to reduce the total error and simplify the procedure. The scheme reduces these types of equations to nonlinear systems of algebraic equations. Convergence analysis of the method under the infinity norm is well-studied. Numerical results indicate the superiority of the present method compared with other existing method in the literature.

Keywords. Hybrid functions, Collocation scheme, Weakly singular kernel, Nonlinear Fredholm-Volterra integral equation, Convergence analysis.
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1. Introduction

Nonlinear singular integral equations with logarithmic or algebraic kernel have frequently appeared in various problems, such as mechanic, physics, engineering and chemical reactions [9, 14, 22, 26, 36, 37]. The object of this study is to approximate the solution of the following nonlinear weakly singular integral equations of the second kind:

$$u(t) = g(t) + \lambda \int_D k(t,x)\rho_\alpha(t-x)\psi(u(x))dx,\ 0 \le \alpha \le 1,\ D = [a,b], \tag{1.1}$$

where

$$\rho_\alpha(v) = \begin{cases} v^{-\alpha}, & 0 \le \alpha < 1, \\ ln|v|, & \alpha = 1. \end{cases}$$

Moreover, g, k and ψ are known functions chosen from suitable function spaces defined in Section 2. If the parameter b is a constant then the integral equation is named as Fredholm, otherwise, known as Volterra.

In the recent literature, several algorithms have been proposed to approximate solution of the integral equations with singular kernels. To present an appropriate numerical scheme for Eq. (1.1), knowing the behavior of the solution is importance. These equations have singular solutions which behave like $x^{1-\alpha}$ near $x = 0$. Therefore, it seems the local view points may be helpful. Hybrid collocation schemes are intensively employed as powerful tools in the sense of accuracy to approximate the solution of mathematical models such as differential equations, integral equations, optimal control and etc [5, 8, 18, 20, 30]. In literature, Brunner et al. have widely considered the pseudo-spectral collocation method with the global error estimation and convergence analysis [11, 12, 24, 25, 33]. Li et al. [28] have

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proposed a pseudo-spectral Galerkin method to solve these equations. Also, Maleknejad et al. [32] have described Sinc-collocation method for Fredholm types of these equations. Other recent methods include the SCW method [40], wavelet Galerkin method [35], operational matrix approach [3, 38, 39], Tau method [2], and fractional-order Legendre collocation method [19]. Recently, the $hp-$collocation approach accompanied by Jacobi polynomials was described for nonlinear Abel integral equations [17].
In spite of lots of research to approximate these equations, a few numerical schemes have been worked on when the kernel is both logarithmic and Abel's type, especially in a nonlinear case and $k(t,x) \neq 1$. Furthermore, numerical schemes for nonlinear weakly singular Fredholm-Volterra integral equations are rarely investigated in the literature. Regarding the applicability of these equations in real-world problems, so choosing the appropriate schemes with high accuracy is significant.
The main purpose of the paper is to describe an efficient pseudo-spectral collocation method for Fredholm, Volterra, and mixed Fredholm-Volterra integral equations with Abel and a logarithmic kernel. The standard basis function is used to approximate these equations which give the opportunity to choose an arbitrary basis function. Furthermore, this work is concerned with the uniform convergence analysis of the scheme.
The outline of this study is as follows: section 2 is devoted to some basic concepts and denotations. The description of the present method is stated in section 3. The convergence analysis under the infinity norm is presented in section 4. In section 5, some experimental examples and their comparison results with other existing methods are provided.

2. Background material

Regularity properties of the solution. In order to introduce a numerical method for Eq. (1.1), knowledge of the smoothness properties of the exact solution is needful. In the terminology of [11, 23, 29], the regularity properties of the solution under their mentioned assumptions was obtained that u belongs to $C^{m,v}(a,b]$ which can be extended into a continuous function on $[a,b]$. In other words, $u \in C^m(a,b] \cap C[a,b]$.
Here, $C^{m,v}(D)$ denotes the class of $v-$Holder $m-$times continuously differentiable functions on D and m indicates the smoothness of the kernel $k(t,x)$ and the known function $g(t)$. For comprehensive details, see Theorem A in [23] and Theorem 2.1 in [11].
To approximate a continuous function $u \in C(D)$, we assign an arbitrary polynomial interpolant $P_N u : D \to \mathbb{R}$ with its following property at $N+1$ distinct points $t_i,\ i = 1, ..., N+1$:

$$P_N u(t_i) = u(t_i), \qquad t_i \in D.$$

To be sure that $\|P_N u - u\|$ converges to zero, the Lebesgue constant should be finite for any interpolant operator P_N [21]. Between all choices of P_N, one can assign complete polynomial basis functions in $L^2(D)$. This choice is valid since these interpolants have a finite Lebesgue constant and also $C(D) \subset L^2(D)$. Here, for a sake of good comparison, we employ the well-known Chebyshev and Legendre polynomials as basis functions $\{\phi_i\}_{i=0}^{\infty}$ which form complete sets in $L^2[a,b]$, i.e., a function $u(t) \in L^2[a,b]$ can be represented as

$$u(t) = \sum_{i=0}^{\infty} u_i \phi_i(t) = U^T \Phi(t). \tag{2.1}$$

Consequently, each function $u \in C(D)$ may be approximated in terms of $\Phi_N(t)$ as

$$u(t) \simeq P_N u(t) = u_N(t) = \sum_{i=0}^{N} u_i \phi_i(t) = U^T \Phi_N(t), \tag{2.2}$$

where $U = [u_0, u_1, ..., u_N]$ is the unknown vector .
Suppose $X = C[a,b]$ and $X_N = span\{\phi_i\}_{i=0}^{N} \subset X$. Therefore, there exist a unique best approximation of X_N for each u belongs to X as:

$$\|u - \mathcal{P}_N(u)\| = \inf_{g \in X_N} \|u - g\|, \tag{2.3}$$

and

$$\mathcal{P}_{_N}(u) = \sum_{i=0}^{N} \overline{u}_i \phi_i(t) = \overline{U}^T \Phi_{_N}(t),$$

where $\overline{U} = [\overline{u}_0, ..., \overline{u}_{_N}]^T$. For brevity, let $\overline{u}_{_N}(t) := \mathcal{P}_{_N}(u)$ as the best approximation of $u(t)$.

3. Description of the numerical method

Here, we apply a pseudo-spectral collocation scheme for nonlinear second-kind weakly singular Volterra, Fredholm and Volterra-Fredholm integral equations.
First, assume $L_{_N}(t) := [1, t, t^2, ..., t^N]^T$ as the standard polynomial basis functions, then obviously each polynomial basis $\Phi(t) = [\phi_0(t), \phi_1(t), ..., \phi_{_N}(t)]^T$ can satisfy the following formula

$$\Phi_{_N}(t) = AL_{_N}(t), \tag{3.1}$$

where A is an invertible matrix.
Consider the Eq. (1.1) as

$$u(t) - g(t) - \lambda \int_D k(t, x)\rho_{_\alpha}(t - x)\psi(u(x))dx = 0,$$

or in operator form

$$u - g - \lambda \mathcal{K}\mathcal{G}u = 0,$$

where $\mathcal{K}\mathcal{G}u(t) := \int_D k(t, x)\rho_{_\alpha}(t - x)\psi(u(x))dx$. Now, we rewrite the Eq. (1.1) as

$$Z(t) = \psi(g(t) + \lambda \int_D k(t, x)\rho_{_\alpha}(t - x)Z(x)dx), \tag{3.2}$$

where $\psi(u(t)) = Z(t)$. The approximation of $Z(t)$ using the definition (2.2) is as

$$Z_{_N}(t) := \psi(u_{_N}(t)) = \sum_{i=0}^{N} z_i \phi_i(t) = \mathbf{Z}^T \Phi_{_N}(t).$$

Here, we seek $Z_{_N}(t)$ instead of $u_{_N}(t)$ which satisfies in Eq. (3.2) at the collocation points t_i,

$$\left[Z_{_N}(t) - \psi(g(t) + \lambda \int_D k(t, x)\rho_{_\alpha}(t - x)Z_{_N}(x)dx)\right]_{t=t_i} = 0, \qquad i = 0, ..., N, \tag{3.3}$$

or, regarding the relation (3.1)

$$\left[\Phi_{_N}^T(t)\mathbf{Z} - \psi\big(g(t) + \lambda(\int_D k(t, x)\rho_{_\alpha}(t - x)L_{_N}^T(x)dx)A^T\mathbf{Z}\big)\right]_{t=t_i} = 0, \tag{3.4}$$

where the collocation points are equi-distance points in this domain without using the start-up point. It should be noted that the Eq. (1.1) may have a solution with weak singularity at $t = a$.
For simplification, let us define $\mathcal{K}_{_N} = \int_D k(t, x)\rho_{_\alpha}(t - x)L_{_N}^T(x)dx$, so the above equation simplify as

$$[\Phi_{_N}^T(t)\mathbf{Z} - \psi(g(t) + \lambda\mathcal{K}_{_N} A^T\mathbf{Z})]_{t=t_i} = 0. \tag{3.5}$$

The above nonlinear system may solve using Newton's iteration method to obtain the unknown vector U and therefore the unknown function $Z_{_N}(t)$. If ψ is an invertible function, then one can derive the approximate solution $u_{_N}(t) = U\Phi_{_N}(t) = \psi^{-1}(Z_{_N}(t))$. In general case,

$$u_{_N}(t) = g(t) + \lambda \int_D k(t, x)\rho_{_\alpha}(t - x)Z_{_N}(t)dx. \tag{3.6}$$

In the present scheme, the integral operator is evaluated at first and then we have a simple nonlinear system to obtain the approximate solution. This treatment helps us to have low computational complexity and CPU time.

Remark 3.1. The key idea of the present method is to use the invertibility of the nonlinear function ψ as a benefit which simplifies the procedure. Moreover, if $u(t)$ belongs to $C^0 \setminus C^1$, then approximating $\psi(u(t)) \in C^1(R)$ by polynomials is more accurate than approximating $u(t)$. For instance, when $u(t) = t^{0.5}$ and $\psi(u(t)) = u^2(t) = t$. As it can be seen, $u(t)$ has a weak singularity at $t = 0$ but $\psi(u(t))$ is smooth enough. The numerical examples 1-4 verify this claim.

Remark 3.2. The integrals $\int_D k(t_i, x)\rho_\alpha(t_i - x)L_N^T(x)dx$ can be evaluated once and then used for any polynomial basis functions. In addition, the values of $\int_D k(t_i, x)\rho_\alpha(t_i - x)L_N^T(x)dx$ can solve quicker than the values of $\int_D k(t_i, x)\rho_\alpha(t_i - x)\Phi_N^T(x)dx$, which yield lower computational cost.

3.1. **Approximation of $\mathbf{u} \in \mathbf{C^0} \setminus \mathbf{C^1}$.** As mentioned in Remark 3.1, if $\psi(u(t)) \in C^1(R)$ then the described method works well, otherwise the following discussion may be useful.
As noted about the exact solution of $t^\nu-$type singularity of the Eq. (1.1), the continuous functions like polynomials may not be a good interpolant. Since these solutions have a weak singularity at startup points whereas the polynomials are smooth enough in the whole interval D. In the above discussion, we use global polynomials to solve numerically the Eq. (1.1). For better result, the polynomial basis functions can generalize into the piecewise forms using block-pulse functions. These hybrid functions have local view point to approximate more precisely the main problem at each subinterval of D.
Consider a sample of these hybrid polynomials and block pulse functions.

Definition 3.3. ([16]) Block-pulse functions $b_n(t), n = 1, ..., N$ on the interval $[0, 1]$ are defined as

$$b_n(t) = \begin{cases} 1, & t \in \left[\frac{n-1}{N}, \frac{n}{N}\right], \\ 0, & otherwise. \end{cases}$$

Definition 3.4. ([16]) Hybrid block-pulse functions and Legendre polynomials $H_{nm}(t)$ with two adjustable parameters define as follow:

$$H_{nm}(t) = \begin{cases} P_m(2Nt - 2n + 1), & t \in \left[\frac{n-1}{N}, \frac{n}{N}\right], \qquad 1 \le n \le N, \\ 0, & otherwise. \end{cases}$$

in which m, n denote the degree of Legendre polynomials $P_m(t)$, $m = 0, 1, ..., M - 1$ and block-pulse functions, respectively.

Hybrid functions $H_{nm}(t)$ form an orthogonal complete set in $L^2[0, 1]$, i.e., a function $u(t) \in L^2[0, 1]$ can be represented as

$$u(t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} u_{nm}H_{nm}(t).$$

Thus, each function $u(t) \in L^2[0, 1]$ may be approximated as

$$u_{NM}(t) \simeq \sum_{m=0}^{M-1}\sum_{n=1}^{N} u_{nm}H_{nm}(t) = U^T\mathbf{H}(t).$$

In order to implement the same method described in the previous subsection, first we define the following standard hybrid polynomials function instead of $L(t) = [1, t, t^2, ..., t^n]$:

Definition 3.5. Hybrid basis functions with block-pulse functions are defined as

$$\mathbf{L}(t) := [l_{10}(t), ..., l_{1M-1}(t), l_{20}(t), ..., l_{2M-1}(t), ..., l_{N0}(t), ..., l_{NM-1}]^T,$$

where

$$l_{ni}(t) = \begin{cases} t^i, & t \in \left[\frac{n-1}{N}, \frac{n}{N}\right], \\ 0, & otherwise, \end{cases}$$

where $1 \le n \le N$ and $0 \le i \le M - 1$.

Similarly to the relation (3.1), there exists a matrix $A_{NM\times NM}$ such that

$$\mathbf{H}(t) = A\mathbf{L}(t). \tag{3.7}$$

Consequently, all the relations (3.2-3.5) are held for $\mathbf{H}(t)$.

Remark 3.6. The approach can generalize to the following nonlinear weakly singular Volterra-Fredholm integral equations:

$$u(t) = g(t) + \lambda_1 \int_{D_1} k_1(t,x)\rho_{\alpha_1}(t-x)Z_1(x)dx + \lambda_2 \int_{D_2} k_2(t,x)\rho_{\alpha_2}(t-x)Z_2(x)dx, \tag{3.8}$$

where $0 \le \alpha_1, \alpha_2 \le 1$, $D_1 = [a,b]$, $D_2 = [c,t]$ and

$$\rho_\alpha(v) = \begin{cases} v^{-\alpha}, & 0 \le \alpha < 1, \\ \\ ln|v|, & \alpha = 1, \end{cases}$$

Moreover, g, k_1, k_2, ψ_1 and ψ_2 are known functions chosen from suitable function spaces defined in section 2. In such cases, we have two systems like Eq. (3.5) with two nonlinear unknown vectors $Z_1 = \psi_1(u(t))$ and $Z_2 = \psi_2(u(t))$. In this regard, we have two nonlinear systems as follows

$$\left[Z_{1N}(t) - \psi_1\left(g(t) + \lambda_1 \int_{D_1} k_1(t,x)\rho_{\alpha_1}(t-x)Z_{1N}(x)dx\right) + \lambda_2 \int_{D_2} k_2(t,x)\rho_{\alpha_2}(t-x)Z_{2N}(x)dx)\right]_{t=t_i} = 0,$$

$$\left[Z_{2N}(t) - \psi_2\left(g(t) + \lambda_1 \int_{D_1} k_1(t,x)\rho_{\alpha_1}(t-x)Z_{1N}(x)dx\right) + \lambda_2 \int_{D_2} k_2(t,x)\rho_{\alpha_2}(t-x)Z_{2N}(x)dx)\right]_{t=t_i} = 0.$$

4. Convergence analysis

Now, we are going to provide the uniform convergence analysis for Eq. (1.1). As noted in the introduction, we will assume that the solution of Eq. (1.1) exist under some conditions stated in [4, 6, 10, 15, 17, 23, 34]. For convenience, we consider this equation in Volterra type and $\|.\|$ as infinity norm.

Definition 4.1. ([13]) A function $g : [0,1] \to \mathcal{R}$ belongs to Sobolev space $W^{r,p}[0,1]$, if all distributional derivative of g, $g^{(i)}$, lies into $L^p[0,1]$ for all $0 \le i \le r$ with the norm

$$\|g\|_{W^{r,p}} = \left(\sum_{i=0}^{r} \|g^{(i)}\|_{L^p}\right)^{\frac{1}{2}},$$

where $\|.\|_{L^p}$ denotes the Lebesgue norm.

Theorem 4.2. *([13]) If $g(t) \in W^{r,\infty}[0,1]$(Sobolev space) and $g_N(t) = \sum_{i=0}^{N} c_i\Phi_i(t) = \mathbf{C}^T\Phi(t)$ be the best approximation polynomial of $g(t)$, then*

$$\|g(t) - g_N(t)\| \le CC_0N^{-r} = \gamma N^{-r},$$

where C is a constant, $C_0 = \|g\|_{W^{r,\infty}}$ and $\gamma := CC_0$.

Theorem 4.3. *Suppose that the integral Eq. (1.1) has a unique solution. Under the hypothesis of Theorem 4.2 for $u(t)$ and assuming that $\overline{k} := sup\{|k(x,t) : 0 \le x \le t \le 1\}$ and $|\psi'(t)| \le M$, the following error estimate can be derived as*

$$\|u(t) - u_N(t)\| \le \gamma N^{-m} + C\|A^{-1}\|N^{-m+1}, \tag{4.1}$$

where

$$A = [\phi_j(t_i) - \int_0^{t_i} k(t_i,x)\rho_\alpha(t_i - x)\psi'(\xi)\phi_j(x)dx]_{i,j},\ i,j = 0,1,...,N,$$

assumed to be a nonsingular matrix.

Proof. In the proposed method, $u_N(t) = U\Phi_N(t)$ approximates $u(t)$ by collocating the following equation:

$$f(t) = u_N(t) - \int_0^t k(t,x)\rho_\alpha(t-x)\psi(u_N(x))dx. \tag{4.2}$$

Besides, the best approximation of $u(t)$, $\bar{u}_N(t)$ satisfies the following relation:

$$\overline{f}(t) = \overline{u}_N(t) - \int_0^t k(t,x)\rho_\alpha(t-x)\psi(\overline{u}_N(t))dx, \tag{4.3}$$

where $\overline{u}_N(t) = P_N(u(t)) = \overline{U}\Phi_N(t)$ defined in (2.3). Besides, with triangular inequality,

$$\begin{aligned} \|u(t) - u_N(t)\| &\leq \|u(t) - \overline{u}_N(t)\| + \|\overline{u}_N(t) - u_N(t)\| \\ &\leq E_1 + E_2, \end{aligned} \tag{4.4}$$

where E_1 can easily be obtained by Theorem 4.2, since $u \in C^m(a,b] \cap C[a,b]$. To obtain E_2, first subtracting Eq. (4.2) from Eq. (4.3),

$$f(t) - \overline{f}(t) = u_N(t) - \overline{u}_N(t) - \int_0^t k(t,x)\rho_\alpha(t-x)(\psi(u_N(x)) - \psi(\overline{u}_N(x))dx, \tag{4.5}$$

or, regarding to mean value theorem,

$$f(t) - \overline{f}(t) = u_N(t) - \overline{u}_N(t) - \int_0^t k(t,x)\rho_\alpha(t-x)\psi'(\xi)(u_N(x) - \overline{u}_N(x))dx, \tag{4.6}$$

in which $\xi \in (min(u_N, \overline{u}_N(t)), max(u_N, \overline{u}_N(t))$.
Now collocating the above equation with $N+1$ distinct points t_i,

$$\begin{aligned} f(t_i) - \overline{f}(t_i) &= \sum_{j=0}^{N}(u_j - \overline{u}_j)\phi_j(t_i) - \int_0^{t_i} k(t_i,x)\rho_\alpha(t_i - x)\psi'(\xi)\sum_{j=0}^{N}(u_j - \overline{u}_j)\phi_j(x)dx \\ &= (U - \overline{U})\Phi_N(t_i) - \int_0^{t_i} k(t_i,x)\rho_\alpha(t_i - x)\psi'(\xi)(U - \overline{U})\Phi_N(x)dx. \end{aligned} \tag{4.7}$$

The operator form of the above equation with interpolatory projection operator is

$$P_N(f - \overline{f}) = (I - P_N Q)(u_N - \overline{u}_N), \tag{4.8}$$

where the operator $Q = \int_0^t k(t,x)\rho_\alpha(t-x)\psi'(\xi)dt$ and

$$P_N Q(g) = \int_0^{t_i} k(t_i,x)\rho_\alpha(t_i - x)\psi'(\xi)g(x)dx, \tag{4.9}$$

and the matrix form of Eq. (4.7) is

$$(F - \overline{F}) = A(U - \overline{U}), \tag{4.10}$$

where $F = [f(t_i)]_i,\ \overline{F} = [\overline{f}(t_i)]_i$ and

$$A = [\phi_j(t_i) - \int_0^{t_i} k(t_i,x)\rho_\alpha(t_i - x)\psi'(\xi)\phi_j(x)dx]_{i,j},\ i,j = 0,1,...,N. \tag{4.11}$$

Now using the Eq. (4.10), the following error bound obtain

$$\|U - \overline{U}\| \leq \|A^{-1}\|\|F - \overline{F}\|. \tag{4.12}$$

An upper bound for $\|F-\overline{F}\|$ obtain as follows

$$\|F-\overline{F}\| = \max_{0\le i\le N} |f(t_i)-\overline{f}(t_i)| \le \max_{t\in[0,1]} |f(t)-\overline{f}(t)| = \|f(t)-\overline{f}(t)\|. \tag{4.13}$$

Now, the subtraction of Eq. (1.1) from Eq. (4.3) and also utilizing mean value theorem, one can conclude that

$$f(t)-\overline{f}(t) = u(t)-\overline{u}_N(t) - \int_0^t k(t,x)\rho_\alpha(t-x)\psi'(\xi)(u(x)-\overline{u}_N(x))dx, \tag{4.14}$$

where $\xi \in (min(u,u_N), max(u,u_N))$. Define

$$\varrho_\alpha(v) = \int_D \rho_\alpha(t-x)v(x)dx,\ t\in D,\ v\in C(D). \tag{4.15}$$

As described in [7, p. 8]), $\varrho_\alpha : C(D)\to C(D)$ is bounded compact operator and $\|\varrho_\alpha\| = \max_{t\in D}\int_D |\rho_\alpha(t-x)|dx$. It can be checked that there exist a constant $\eta_\alpha > 0$ such that $\|\varrho_\alpha\| \le \eta_\alpha,\ 0\le\alpha\le 1$.
Also, using triangular inequality we get

$$\begin{aligned} \|f-\overline{f}\| &\le \|u-\overline{u}_N\| + \overline{k}M\|\varrho_\alpha(u-\overline{u}_N)\| \\ &\le \|u-\overline{u}_N\|(1+\overline{k}\eta_\alpha M), \end{aligned} \tag{4.16}$$

in which $\overline{k} := sup\{|k(x,t) : 0\le x\le t\le 1\}$, $|\psi'(x)|\le M$. Therefore, using Theorem 4.2 led to following relation:

$$\|f(t)-\overline{f}(t)\| \le (1+\overline{k}\eta_\alpha M)c_0 N^{-m}, \tag{4.17}$$

where $c_0 > 0$ is constant. Finally, from Eq. (4.13) obtain an upper bound for $\|F-\overline{F}\|$,

$$\|F-\overline{F}\| \le (1+\overline{k}\eta_\alpha M)c_0 N^{-m}. \tag{4.18}$$

Substituting the above Eq. back into Eq. (4.12), we deduce

$$\|U-\overline{U}\| \le c_0(1+\overline{k}\eta_\alpha M)\|A^{-1}\|N^{-m}. \tag{4.19}$$

Hence,

$$\begin{aligned} \|u_N(t)-\overline{u}_N(t)\| &= \max_{t\in D}|u_N(t)-\overline{u}_N(t)| = \max_{t\in D}\sum_{i=0}^{N}|u_i(t)-\overline{u}_i(t)| \le \max_{t\in D}\sum_{i=0}^{N}|u_i-c_i||\phi_i(t)| \\ &\le c_1(N+1)\|U-\overline{U}\| \le C\|A^{-1}\|N^{-m+1}, \end{aligned} \tag{4.20}$$

where $c_1 := \max_{t\in D, 0\le i\le N}|\phi_i(t)|$ and $C := 2c_0c_1(1+\overline{k}\eta_\alpha M)$.
It is worthwhile noting that if $\overline{k}\eta_\alpha M < 1$, then the uniqueness of the solution is provided [23].

Consequently, the error estimation regarding to Eqs. (4.4) and (4.20), can be derived as

$$\begin{aligned} \|u(t)-u_N(t)\| &\le E_1+E_2 \\ &\le \gamma N^{-m} + C\|A^{-1}\|N^{-m+1} \\ &= O(N^{-m+1}). \end{aligned} \tag{4.21}$$

□

This obtained rate of convergence indicate that the higher smoothness of kernel and known function f leads to a better order of convergence. .
To discuss about the invertibility of matrix $A = (I - P_N Q)$ defined in Eqs. (4.8-4.11), we need some assumptions.

Theorem 4.4. *([7, pp. 55-58]) Assume X be a Banach space, $Q : X \to X$ be bounded and compact and P_N be a bounded family of projections on X with $P_N g \to g$ as $N \to \infty$. Further assume $I - Q$ is nonsingular. Hence, an operator $(I - P_N Q)^{-1}$, for all sufficiently large N exists as a bounded operator on X.*

It is worthwhile to note that the operator Q is both bounded and compact if $Q(t, x)$ is continuous [7, p. 8]. We remark that all above discussion in this section can also apply for Fredholm integral equation and hybrid functions. For hybrid functions, Theorem 1 changes as follows:

Theorem 4.5. *([31]) Suppose that $g(t) \in C^M[0,1]$ and $\overline{g}_{NM} = \sum_{m=0}^{M-1}\sum_{n=1}^{N} c_{nm}H_{nm}(t) = \mathbf{C}^T\mathbf{H}(t)$, be the best approximate solution by hybrid polynomials of $g(t)$, then*

$$\|g(t) - \overline{g}_{NM}(t)\| \leq \frac{\gamma}{N^{M-1}M!},\ \gamma = \max_{t\in[0,1]} |g^{(M)}(t)|. \tag{4.22}$$

5. Numerical experiment

Some examples demonstrate the accuracy and high performance of the scheme. For convenience, let L denote the number of all used basis functions. Here, we use Chebyshev and Legendre polynomials as basis polynomial functions that do not differ practically in terms of accuracy. All calculations are supported with Mathematica 9.

5.1. Nonsmooth solution with $\psi(\mathbf{u(t)}) \in \mathbf{C^1}$.

Example 5.1. Consider the nonlinear second-kind weakly singular VIE

$$u(t) = g(t) + \int_0^t |t - x|^{\frac{-1}{2}} u^2(x)dx,\ 0 \leq t \leq 1, \tag{5.1}$$

where $g(t) = t^{\frac{1}{2}}(1 - \frac{4}{3}t)$. The exact solution is $u(t) = t^{\frac{1}{2}}$.

The exact solution $u(t)$ has a weak singularity at $t = 0$. Brunner et al. solved this example with piecewise polynomial collocation methods on graded grids [11]. The best results of their method has the accuracy of order $O(10^{-5})$ when employ 256 collocation points (the number of unknown coefficients). The present method has the accuracy of order $O(10^{-16})$ with using only 2 terms. It takes only 0.7 seconds.

Example 5.2. As a test problem, consider the nonlinear second kind VIE with logarithmic kernel and nonsmooth solution as follows:

$$u(t) = g(t) + 2\int_0^t (t^2 + x)\ln|t - x|u^2(x)dx,\ 0 \leq t \leq 1, \tag{5.2}$$

in which $g(t)$ is obtained such that the exact solution is $u(t) = |t - 0.5|$.

The exact solution belongs to the class of $C^0 \setminus C^1$. The present approach gives has the accuracy of order $O(10^{-16})$ with just three terms ($N = 2$) in 61 seconds. This result for this nonsmooth solution verifies the efficiency of our scheme to overcome the difficulties of these singular equations with a few basis functions. Figure 1 depicts both exact and approximate solution when $N = 2$.

Example 5.3. Consider the second kind weakly singular Fredholm integral equation

$$u(t) = g(t) + \int_0^1 |t - x|^{\frac{-1}{2}} u^2(x)dx,\ 0 \leq t \leq 1, \tag{5.3}$$

in which the known function $g(t)$ with the exact solution $u(t) = (t(1 - t))^{\frac{1}{2}}$ can be derived.

This example has been studied in [1, 27, 32] by Sinc collocation, Legendre wavelet and Haar wavelet methods, respectively. The scheme has the accuracy order $O(10^{-17})$ when using only 3 basis functions with low CPU time (8.8 seconds). Table 1 illustrates the superiority of our scheme comparing to other methods in literature.

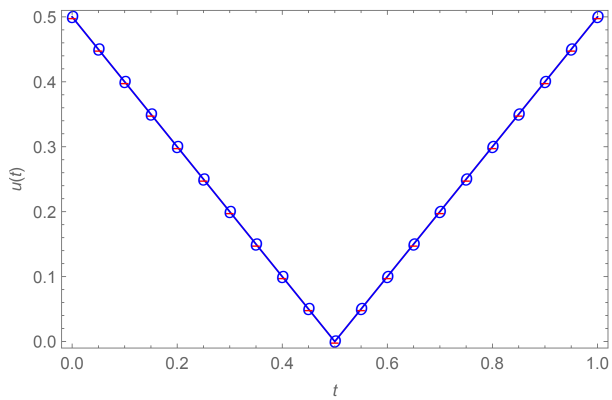

FIGURE 1. The exact and approximate solution with $L = 3$ for Ex. 5.2.

TABLE 1. Comparison of our method with those results in [1, 27, 32]

t	Haar wavelet method $L = 32$	Legendre wavelet method $L = 24$	DE Sinc-collocation method $L = 71$	Present method $L = 3$
0.1	$1.18E-02$	$1.26E-03$	$1.01E-13$	$6.94E-17$
0.3	$1.48E-03$	$1.99E-04$	$5.60E-14$	0.0
0.5	$3.26E-04$	$7.68E-06$	$8.10E-15$	0.0
0.7	$1.48E-03$	$1.23E-06$	$5.65E-14$	0.0
0.9	$1.18E-02$	$1.31E-03$	$1.01E-13$	$1.39E-17$

5.1.1. *Long time calculation.*

Example 5.4. As a test problem, we study the Fredholm integral equation with logarithmic kernel

$$u(t) = g(t) + 0.1\int_0^T x^3|t-x|^{\frac{-1}{3}}\sqrt{u(x)}dx,\ 0 \le t \le T, \tag{5.4}$$

where $g(t)$ is obtained such that the exact solution is $u(t) = (t+5)^2$.

This approach gives the exact solution with three terms ($N = 2$). Figure 2 indicates the absolute error for various time T. This result again indicates the efficiency of our method with low storage requirements and CPU time.

5.1.2. **Nonsmooth solution with $\psi(\mathbf{u(t)}) \in \mathbf{C^0} \setminus \mathbf{C^1}$.**

Example 5.5. As a test problem, we study the following weakly Fredholm integral equation

$$u(t) = f(t) + \int_0^1 \frac{ln(u(x))}{(t-x)^{0.4}}dx,\ 0 \le t \le 1, \tag{5.5}$$

where $f(t)$ can obtion such that the exact solution be

$$u(t) = exp(|t - \frac{1}{2}|).$$

In this example, the exact solution belongs to the class $C^0 \setminus C^1$ and $\psi(u(t)) = |t - \frac{1}{2}|$. As noted in subsection 3.1, the continuous polynomials functions can not achieve a good accuracy. Hybrid block-pulse functions and Legendre polynomials works well for this equation. As an advantages of hybrid functions is that the degree of polynomials and block pulse functions can be adjusted to achieve better approximate solution. Here, the approach obtains the absolute error of order $O(10^{-14})$ when we apply $N = 2$ and $M = 2$, or $L = 4$ basis functions. Figure 3 it indicates that the

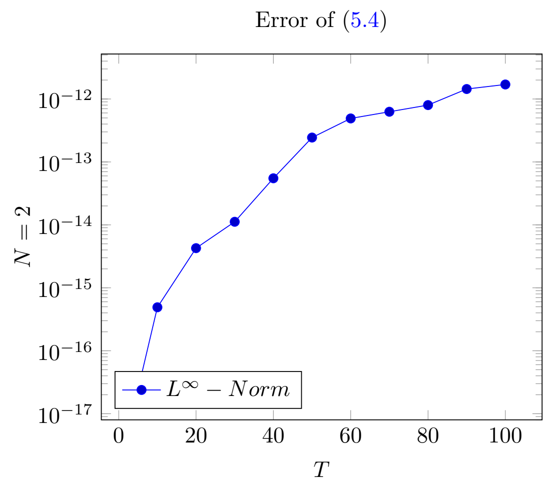

FIGURE 2. Plots of the error in logarithmic scale of Ex. 5.4

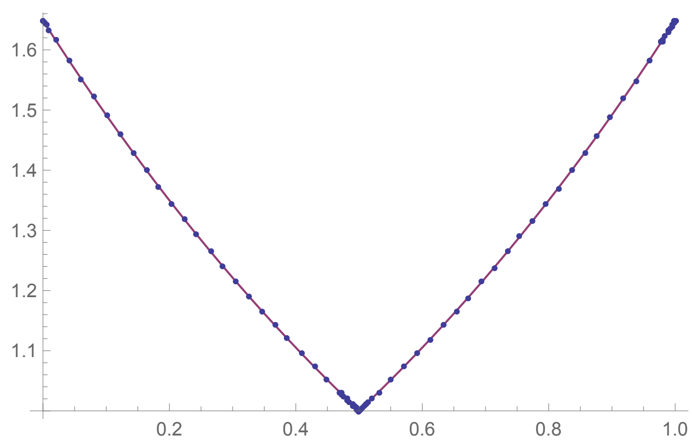

FIGURE 3. Results of Ex. 5.5 when $u_{22}(t)$ approximates $u(t)$.

proposed procedure is conducive for non-sufficiently smooth solution and also more appropriate to have high accuracy with low computational cost.

5.1.3. *Comparison with other existing scheme.*

Example 5.6. Consider the nonlinear second kind weakly singular VIE

$$u(t) = g(t) + \int_0^t xt|t-x|^{\frac{-1}{2}} u^2(x)dx, \ 0 \le t \le 1, \tag{5.6}$$

where $g(t) = t^3 - \frac{4096}{6435} t^{\frac{17}{2}}$. The solution is $u(t) = t^3$.

The proposed method gives has the accuracy order $O(10^{-15})$ when we employ $N = 8$. This computational method takes only 3.4 seconds. Recently, Zhu and Wang have studied this problem by using SCW method [40]. Table 2 provides the comparison between our scheme and SCW method when $k = 5$, $M = 2$ and so $L = 2^{k-1}M = 32$.

Table 2. Comparison of the method with those result in [40] for Ex. 5.6.

t	SCW scheme $L=32$	Present scheme $L=9$
0.1	$5.90E-05$	$5.55E-17$
0.3	$9.18E-05$	$2.77E-17$
0.5	$1.12E-03$	0.0
0.7	$2.39E-06$	$2.91E-16$
0.9	$2.99E-03$	$1.89E-15$

Example 5.7. The following nonlinear second kind weakly singular VIE is considered

$$u(t) = g(t) + \int_0^t |t-x|^{\frac{-1}{2}} \exp(u(x))dx,\ 0 \le t \le 1, \tag{5.7}$$

where $g(t) = \ln(t+1) + 2\sqrt{t} + \frac{4}{3}t^{\frac{3}{2}}$. The exact solution is $u(t) = \ln(1+t)$.

Recently, this problem is solved by Tau method [2]. Table 3 reports the comparison of Tau approximation and the present method. It can be easily observed that our scheme has the absolute error of $O(10^{-16})$ when $L=2$ and takes only 1.7 seconds.

Table 3. Results of our method and [2] for Ex. 5.7.

L	Tau method	Present scheme
2	$----$	$2.25E-16$
3	$5.0E-03$	$2.21E-16$
6	$3.0E-03$	$----$
8	$2.0E-05$	$----$
10	$9.0E-06$	$----$

Example 5.8. As an another test problem, consider the following nonlinear second kind weakly singular VIE

$$u(t) = g(t) + \int_0^t |t-x|^{-0.6}\ sin(u(x))dx,\ 0 \le t \le 1, \tag{5.8}$$

where $g(t) = t - 1.78571\ t^{1.4}\ F(1;\ 1.7,\ 1.2;\ -0.25\ t^2)$. $F(a;b;c)$ is a Hypergeometric function and $u(t) = t$.

Figure 4 illustrates the results of the present approach and higher accuracy by increasing parameter L.

5.2. **Weakly singular Fredholm-Volterra integral equation.**

Example 5.9. As the final test problem, consider the following nonlinear weakly singular Fredholm-Volterra integral equation as follows

$$u(t) = g(t) + 3\int_0^t (x^2+t)\ |t-x|^{-0.23} u^2(x)dx + \int_0^1 t\ Ln|t-x|\ u^3(x)dx,\ 0 \le t \le 1 \tag{5.9}$$

$g(t)$ can easily be computed by knowing $u(t) = t$.

The absolute error of the suggested method with $L=4$ used basis functions is the order of $O(10^{-15})$. We obtained this result in 26 seconds.

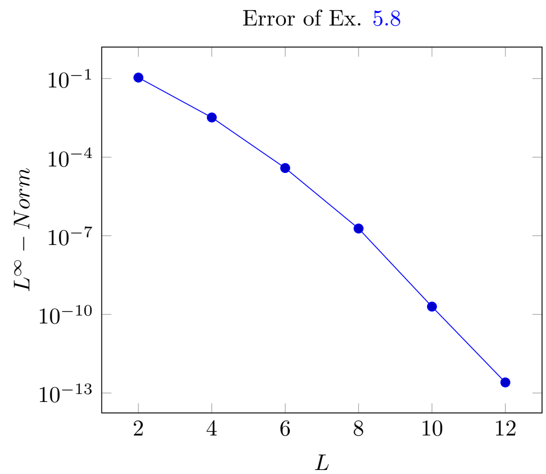

FIGURE 4. Plots of the error in logarithmic scale of Ex. 5.8

6. CONCLUSION

This work was concerned with the development of an efficient pseudo-spectral method to approximate nonlinear Fredholm, Volterra, and Fredholm-Volterra integral equations with a singular kernel. The approach is based upon an arbitrary orthogonal polynomial basis function along with the collocation method. The main benefits of the method are simplicity of performance, high-order accuracy, and low computational cost.

Theoretically, convergence analysis indicates that the present scheme is precise and of order $O(N^{-m+1})$, where m comes from the smoothness degree of the solution, and N shows the quantity of all considered subintervals used for local approximation. Furthermore, hybrid functions are assigned to approximate some non-smooth solutions with property $\psi(u(t)) \in C^0 \setminus C^1$. The main characteristic of these piecewise polynomials is the capability of adjusting the degree of polynomials and block pulse function to achieve a suitable approximate solution. The efficiency and superiority of the scheme for a smooth or non-smooth solution have been shown through the experimental examples.

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