Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-16
DOI:10.22034/cmde.2022.47106.1974

On the numerical scheme for solving non-linear two-dimensional Hammerstein integral equations

Manochehr Kazemi[1,*] **and Mohammad Reza Doostdar**[2]
[1]Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran.
[2]Department of Mathematics, Zarandieh Branch, Islamic Azad University, Zarandieh, Iran.

Abstract

In this work, solving non-linear two-dimensional Hammerstein integral equations is considered by an iterative method of successive approximation. The method is an efficient approach based on combination of the quadrature formula and the successive approximations method. Also, the convergence analysis and the numerical stability of the suggested method are studied. Finally, to survey the accuracy of the present method, some numerical experiments are given.

Keywords. Fixed point theorem, Hammerstein integral equations, Quadrature formula, Iterative method.
2010 Mathematics Subject Classification. 47H09, 47H10.

1. Introduction

Non-linear two-dimensional Fredholm integral equations of the second kind appear in the mathematical modeling of various physical phenomena (such as electro-magnetic fluid dynamics), engineering problems and applied sciences [8, 18, 21, 31, 32, 34–36]. The numerical methods elaborated for two-dimensional non-linear integral equations involve various techniques: the Galerkin and collocations methods [13, 17], successive approximations and other iterative techniques [7, 12, 24, 25, 30, 38, 39], the Nyström type methods [16], the piecewise approximation by Chebyshev polynomials [15], the operational matrix operational matrix [11, 26, 33], wavelet method [9], the rationalized Haar functions [10], triangular and block-pulse functions [29], multi-step methods[37], Runge-Kutta method [27], neural network method [20], expansion method [28] and the regularization-homotopy method [3].
For our purpose here, we consider non-linear two-dimensional Hammerstein Fredholm integral equation of the second kind (2D-NHFIE) as follows

$$X(s,t) = r(s,t) + \lambda \int_c^d \int_a^b K(s,t,x,y)\psi(x,y,X(x,y))dxdy, \ (s,t) \in I, \tag{1.1}$$

and focus on a numerical iterative algorithm for solving it, where $a, b, c, d \in \mathbb{R}$, $r : I = [a,b] \times [c,d] \to \mathbb{R}, K : I \times I \to \mathbb{R}$ and r, K are continuous. Having a strong physical background, the Hammerstein equations arise from the electro-magnetic fluid dynamics. In particular, Eq. (1.1) arises from various reformulations of an elliptic partial differential equation with non-linear boundary conditions [4, 6]. Also, one-dimensional analogues of Eq. (1.1) are a reformulation of two-point boundary value problems with a certain non-linear boundary condition, [5, 14]. It should be noted that as a special case of the Hammerstein integral equations, the Fredholm integral equations of the second kind can be considered.
Here, by the combination the fixed point technique and the Simpson's rule, a numerical method is presented for solving Eq. (1.1). Our method is an iterative procedure using successive approximations method.
Some results about the existence and uniqueness of the solution of non-linear 2D integral equations can be studied in [1, 2, 19, 22, 23]. In this paper, the Banach fixed point theorem is used to prove the existence and uniqueness of the solution of Eq. (1.1).
We arranged the remainder of the paper as follows: In Sect. 2, some required definitions and theorems are provided.

Received: 21 July 2021 ; Accepted: 24 January 2022.
∗ Corresponding author. Email: univer_ka@yahoo.com.

In Sect. 3 an iterative procedure based on quadrature formula is presented. Also, the numerical stability and the convergence are investigated. Numerical results are presented in Sect. 4. Moreover, a comparison is made between the method [24] and our method, to show the efficiency of our method. Some conclusions drawn in Sect. 5.

2. Preliminaries

In current section, some basic definitions and required theorems that will be needed are expressed.

Definition 2.1. For $L \geq 0$, the function $f : I \to \mathbb{R}$ is L-Lipschitz if

$$|f(x,y) - f(x',y')| \leq L(|x - x'| + |y - y'|), \quad \forall x, x' \in [a,b] \, and \, y, y' \in [c,d].$$

Theorem 2.2. *([24]) Let $f : [a,b] \times [c,d] \to \mathbb{R}$, be a L-Lipschitz function. Then, for any divisions $a = x_0 < x_1 < \ldots < x_n = b$, $c = y_0 < y_1 < \ldots < y_n = d$ and any points $\xi_i \in [x_{i-1}, x_i]$, $\eta_j \in [y_{j-1}, y_j]$, we have*

$$\begin{aligned} &\left| \int_c^d \int_a^b f(s,t)dsdt - \sum_{j=1}^n \sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1}) f(\xi_i, \eta_j) \right| \\ \leq \; & \frac{L}{2} \sum_{j=1}^n \sum_{i=1}^n (y_j - y_{j-1})(x_i - x_{i-1})^2 + \frac{L}{2} \sum_{j=1}^n \sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})^2. \end{aligned}$$

Corollary 2.3. *Let f be the function defined in Theorem 2.2. Then, we have*

$$\begin{aligned} \Big| \int_c^d \int_a^b f(s,t)dsdt \; & - \; ((\gamma - \theta)(\alpha - a) f(u,p) + (\theta - c)(\alpha - a) f(u,r) \\ & - \; (d - \gamma)(\alpha - a) f(u,s) + (\gamma - \theta)(\beta - \alpha) f(v,p) - (\beta - \alpha)(\theta - c) f(v,r) \\ & + \; (\beta - \alpha)(d - \gamma) f(v,s) - (b - \beta)((\gamma - \theta) f(w,p) \\ & + \; (\theta - c) f(w,r) - f(w,s)(d - \gamma))) \Big| \\ & \leq \; \frac{L(d-c)}{2} ((\beta - v)^2 + (b - \beta)^2 + (v - \alpha)^2 + (\alpha - a)^2) \\ & + \; \frac{L(b-a)}{2} ((\theta - c)^2 + (p - \theta)^2 + (\gamma - p)^2 + (d - \gamma)^2) \end{aligned}$$

for any $\alpha, \beta, \gamma, \theta, r, s, p, u, v, w$ where $a \leq u < \alpha < v < \beta < w \leq b$ and $c \leq r < \theta < p < \gamma < s \leq d$.

Proof. In theorem 2.2, take $n = 4$ and

$$\begin{aligned} & x_0 = a, x_1 = \alpha, x_2 = v, x_3 = \beta, x_4 = b, \xi_1 = u, \xi_2 = \xi_3 = v, \xi_4 = w, \\ & y_0 = a_2, y_1 = \theta, y_2 = p, y_3 = \gamma, y_4 = d, \eta_1 = r, \eta_2 = \eta_3 = p, \eta_4 = s. \end{aligned}$$

□

Corollary 2.4. *For the same function f, we have*

$$\begin{aligned} \Big| \int_c^d \int_a^b f(s,t)dsdt \; & - \; \frac{1}{36}(b-a)(d-c) \Big(f(a,c) + f(a,d) + f(b,d) + f(b,c) \\ & + \; 4(f(\frac{a+b}{2}, c) + f(\frac{a+b}{2}, d) + f(a, \frac{c+d}{2}) + f(b, \frac{c+d}{2})) \\ & + \; 16 f(\frac{a+b}{2}, \frac{c+d}{2}) \Big) \Big| \\ & \leq \; \frac{5}{36} L(b-a)(d-c)(b-a+d-c) \end{aligned}$$

for all $x \in [a,b], u \in [a,x], v \in [x,b], y \in [c,d], \alpha \in [c,y]$ and $\beta \in [y,d]$.

Proof. In Corollary 2.3, take

$$\alpha=\frac{5a+b}{6}, v=\frac{a+b}{2}, \beta=\frac{a+5b}{6}, u=a, w=b,$$
$$\theta=\frac{5c+d}{6}, p=\frac{c+d}{2}, \gamma=\frac{c+5d}{6}, r=c, s=d.$$

□

In fact, the recent corollary is the classical Simpson's rule with a new error bound. Also, The Corollary 2.4, is extended for uniform partitions in the following Corollary.

Corollary 2.5. *Let $D_x : a = s_0 < s_1 < s_2 < ... < s_{2n-1} < s_{2n} = b, D_y : c = t_0 < t_1 < t_2 < ... < t_{2n-1} < t_{2n} = d$, with $s_i = a + ih_x$, $t_j = c + jh_y$, where $h_x = \frac{b-a}{2n}$, $h_y = \frac{d-c}{2n}$, $i, j = 0, 1, 2, ..., 2n$, then*

$$\Big|\int_c^d\int_a^b f(s,t)dsdt - S(f)\Big| \le \frac{5L}{18}(b-a)(d-c)(h_x+h_y)$$

where

$$\begin{aligned} S(f) &= \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{1}{36}(s_{2i-2}-s_{2i})(t_{2j-2}-t_{2j}) \\ &\times \Bigg[\Big(f(s_{2i-2},t_{2j-2})+f(s_{2i-2},t_{2j})+f(s_{2i},t_{2j})+f(s_{2i},t_{2j-2})\Big) \\ &+ 4\Big(f(s_{2i-1},t_{2j-2})+f(s_{2i-1},t_{2j})+f(s_{2i-2},t_{2j-1})+f(s_{2i},t_{2j-1})\Big) \\ &+ 16f(s_{2i-1},t_{2j-1})\Bigg] \end{aligned}$$

Proof. By previous corollary, we have

$$\begin{aligned} \Big|\int_{t_{2j-2}}^{t_{2j}}\int_{s_{2i-2}}^{s_{2i}} f(s,t)dsdt &- \frac{1}{36}(s_{2i}-s_{2i-2})(t_{2j}-t_{2j-2}) \\ &\times \Bigg[\Big(f(s_{2i-2},t_{2j-2})+f(s_{2i-2},t_{2j})+f(s_{2i},t_{2j})+f(s_{2i},t_{2j-2})\Big) \\ &+ 4\Big(f(s_{2i-1},t_{2j-2})+f(s_{2i-1},t_{2j})+f(s_{2i-2},t_{2j-1})+f(s_{2i},t_{2j-1})\Big) \\ &+ 16f(s_{2i-1},t_{2j-1})\Bigg]\Big| \\ &\le \frac{5L}{36}(s_{2i}-s_{2i-2})(t_{2j}-t_{2j-2})\big((s_{2i}-s_{2i-2})+(t_{2j}-t_{2j-2})\big) \end{aligned}$$

Where L is the Lipschitz constant of f. So, we have

$$
\begin{aligned}
|\int_c^d \int_a^b f(s,t)dsdt - S(f)| &= |\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{t_{2j-2}}^{t_{2j}}\int_{s_{2i-2}}^{s_{2i}} f(s,t)dsdt - S(f)| \\
&\leq \sum_{j=1}^{n}\sum_{i=1}^{n}\Big|\int_{t_{2j-2}}^{t_{2j}}\int_{s_{2i-2}}^{s_{2i}} f(s,t)dsdt - \frac{1}{36}(s_{2i}-s_{2i-2})(t_{2j}-t_{2j-2}) \\
&\times \Big[\Big(f(s_{2i-2},t_{2j-2}) + f(s_{2i-2},t_{2j}) + f(s_{2i},t_{2j}) + f(s_{2i},t_{2j-2})\Big) \\
&+ 4\Big(f(s_{2i-1},t_{2j-2}) + f(s_{2i-1},t_{2j}) + f(s_{2i-2},t_{2j-1}) + f(s_{2i},t_{2j-1})\Big) \\
&+ 16 f(s_{2i-1},t_{2j-1})\Big]\Big| \\
&\leq \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{5L}{36}(s_{2i}-s_{2i-2})(t_{2j}-t_{2j-2})\big((s_{2i}-s_{2i-2})+(t_{2j}-t_{2j-2})\big) \\
&\leq \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{5L}{36}(\frac{b-a}{n})(\frac{d-c}{n})(\frac{b-a}{n}+\frac{d-c}{n}) \\
&\leq \frac{5L}{18}(b-a)(d-c)(h_x+h_y).
\end{aligned}
$$

□

3. Main results

3.1. The existence problem.

Theorem 3.1. *([24]) Under the conditions*

(i) $r \in C(I,\mathbb{R}), K \in C(I\times I,\mathbb{R}), \psi \in C(I\times\mathbb{R},\mathbb{R})$ *where* $C(I,\mathbb{R}) = \{f : I \to \mathbb{R}; f \text{ is continuous}\}$,

(ii) *there exist* $\alpha, \beta > 0$, *such that*
$|\psi(x,y,u) - \psi(x',y',u')| \leq \alpha(|x-x'| + |y-y'|) + \beta\,|u-u'|,\ \forall (x,y) \in I, \forall u,u' \in \mathbb{R}.$

(iii) $\Theta = \beta\lambda M_k(b-a)(d-c) < 1$, *where* $M_k > 0$ *is such that* $|K(s,t,x,y)| \leq M_k,\ \forall s,x \in [a,b]\ ,t,y \in [c,d]$, *according continuity of* K,

the Eq. (1.1) has a unique solution $X^* \in \boldsymbol{X}$, *and the following sequence of successive approximations*

$$X_0(s,t) = r(s,t),$$

$$X_m(s,t) = r(s,t) + \lambda\int_c^d\int_a^b K(s,t,x,y)\psi(x,y,X_{m-1}(x,y))dxdy,\ m \geq 1, \tag{3.1}$$

converges to the solution X^*. *Also, the following error estimates hold.*

$$\|X^* - X_m\| \leq \frac{\Theta^m}{1-\Theta}\|X_0 - X_1\|, \tag{3.2}$$

$$\|X^* - X_m\| \leq \frac{\Theta}{1-\Theta}\|X_{m-1} - X_m\|, \tag{3.3}$$

and choosing $X_0 \in \boldsymbol{X}, X_0 = r$, *the inequality (3.2) becomes*

$$\|X^* - X_m\| \leq \frac{\Theta^{m+1}}{\beta(1-\Theta)}M_0, \tag{3.4}$$

where M_0 *is given in (3.7).*

Now, for implementation of the method, we consider Eq. (1.1) where $K(s,t,x,y)$ is a continuous kernel defined on $[a,b]\times[c,d]\times[a,b]\times[c,d]$. Also, let

$$\begin{aligned} &D_x : a = s_0 < s_1 < s_2 < ... < s_{2n-1} < s_{2n} = b,\\ &D_y : c = t_0 < t_1 < t_2 < ... < t_{2n-1} < t_{2n} = d, \end{aligned} \tag{3.5}$$

are uniform partitions with $s_i = a + ih_x$, $t_j = c + jh_y$, where $h_x = \frac{b-a}{2n}$, $h_y = \frac{d-c}{2n}$, $i,j = 0,1,2,...,2n$ or $i,j = \overline{0,2n}$. Then, the approximate solution of Eq. (1.1) in (s,t) is obtained by the following iterative procedure.

$$\begin{aligned} \overline{X}_0(s,t) &= r(s,t)\\ \overline{X}_m(s,t) &= r(s,t) + \frac{\lambda h_x h_y}{9}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l_2=0}^{2}\sum_{l_1=0}^{2}(C_2^{l_2})^2(C_2^{l_1})^2\\ &\quad K(s,t,s_{2i-2+l_1},t_{2j-2+l_2})\psi(s_{2i-2+l_1},t_{2j-2+l_2},\overline{X}_{m-1}(s_{2i-2+l_1},t_{2j-2+l_2})). \end{aligned} \tag{3.6}$$

where

$$C_2^l = \binom{2}{l} = \frac{2!}{l!(2-l)!}, \quad l = 0,1,2$$

Proposition 3.2. *Under the assumptions (i)-(iii) of Theorem (3.1), the sequence of successive approximations (3.1) is uniformly bounded. Moreover, let* $\Psi_m(x,y) = \psi(x,y,X_m(x,y)), m \in N$, *and suppose that*

(i) *there exist* η, $\zeta > 0$, *such that*
$|(K(s_1,t_1,x_1,y_1) - K(s_2,t_2,x_2,y_2)| \le \zeta(|s_1 - s_2| + |t_1 - t_2|) + \eta(|x_1 - x_2| + |y_1 - y_2|)$,
$\forall (s_1,t_1),(s_2,t_2),(x_1,y_1),(x_2,y_2) \in I$.

(ii) *there exists* $\theta > 0$, *such that*
$|r(s_1,t_1) - r(s_2,t_2)| \le \theta(|s_1 - s_2| + |t_1 - t_2|)$, $\forall (s_1,t_1),(s_2,t_2) \in I$.

then the functions $\Psi_m, m \in N$ *is uniformly Lipschitz with constant* $L' = \alpha + \beta(\theta + \lambda(b-a)(d-c)M\zeta)$, *where* M *is given in (3.8).*

Proof. Let $\psi_0 : [a,b]\times[c,d] \to \mathbb{R}$ and $\psi_0(x,y) = \psi(x,y,r(x,y))$. Since ψ, r are continuous, it can be concluded that ψ_0 is continuous on the compact set $[a,b]\times[c,d]$. Hence, $M_0 \ge 0$ exists such that

$$|\psi_0(x,y)| \le M_0 \qquad \forall (x,y) \in [a,b]\times[c,d]. \tag{3.7}$$

For $(s,t) \in [a,b]\times[c,d]$, we have

$$\begin{aligned} \left|X_m(s,t) - X_{m-1}(s,t)\right| &\le \lambda\left|K(s,t,x,y)\right|\int_c^d\int_a^b \left|\psi(x,y,X_{m-1}(x,y)) - \psi(x,y,X_{m-2}(x,y))\right|dxdy\\ &\le \lambda M_k \int_c^d\int_a^b \left|\psi(x,y,X_{m-1}(x,y)) - \psi(x,y,X_{m-2}(x,y))\right|dxdy\\ &\le \beta\lambda M_k(b-a)(d-c)\max_{\substack{a\le x\le b\\ c\le y\le d}}|X_{m-1}(x,y) - X_{m-2}(x,y)| = \Theta\left\|X_{m-1} - X_{m-2}\right\|. \end{aligned}$$

and by induction,

$$\left|X_m(s,t) - X_{m-1}(s,t)\right| \le \Theta^{m-1}\left\|X_1 - X_0\right\|.$$

So,

$$\begin{aligned} \left|X_m(s,t) - X_0(s,t)\right| &\le \left|X_m(s,t) - X_{m-1}(s,t)\right| + ... + \left|X_1(s,t) - X_0(s,t)\right|\\ &\le (\Theta^{m-1} + \Theta^{m-2} + ... + \Theta + 1)\left\|X_1 - X_0\right\|\\ &= \frac{1-\Theta^m}{1-\Theta}.\left\|X_1 - X_0\right\| \le \frac{\Theta M_0}{\beta(1-\Theta)} \qquad \forall (s,t) \in [a,b]\times[c,d]. \end{aligned}$$

Let $M_r \ge 0$ such that $|r(s,t)| \le M_r$ for all $(s,t) \in [a,b]\times[c,d]$. Then

$$\left|X_m(s,t)\right| \le \left|X_m(s,t) - X_0(s,t)\right| + \left|X_0(s,t)\right| \le \frac{\Theta M_0}{\beta(1-\Theta)} + M_r = L.$$

Moreover, by considering

$$M = max\big(M_0, max\{|\psi(s,t,u)| : (s,t) \in [a,b]\times[c,d], u \in [-L,L]\}\big) \tag{3.8}$$

we get

$$|\Psi_m(s,t)| = |\psi(s,t,X_m(s,t))| \le M,$$

for all $(s,t) \in [a,b]\times[c,d]$ and $m \in \mathbf{N}$. Let $(s_1,t_1),(s_2,t_2) \in [a,b]\times[c,d]$, we obtain

$$\big|X_0(s_1,t_1) - X_0(s_2,t_2)\big| \le \theta(\big|s_1-s_2\big| + \big|t_1-t_2\big|)$$

$$\begin{aligned}\big|X_m(s_1,t_1) - X_m(s_2,t_2)\big| \le{}& \big|r(s_1,t_1) - r(s_2,t_2)\big| \\ &+ \lambda\int_c^d\int_a^b \mid K(s_1,t_1,x,y) - K(s_2,t_2,x,y)) \mid\mid \psi\big(x,y,X_{m-1}(x,y)\big) \mid dxdy \\ &\le \theta(\big|s_1-s_2\big| + \big|t_1-t_2\big|) + \lambda(b-a)(d-c)M\zeta(\big|s_1-s_2\big| + \big|t_1-t_2\big|) \\ &= L_0(\big|s_1-s_2\big| + \big|t_1-t_2\big|)\end{aligned}$$

with $L_0 = \theta + \lambda(b-a)(d-c)M\zeta$ and

$$\begin{aligned}\big|\Psi_0(s_1,t_1) - \Psi_0(s_2,t_2)\big| &\le \alpha(\big|s_1-s_2\big| + \big|t_1-t_2\big|) + \beta\big|X_0(s_1,t_1) - X_0(s_2,t_2)\big| \\ &\le (\alpha+\beta\theta)(\big|s_1-s_2\big| + \big|t_1-t_2\big|)\end{aligned}$$

$$\begin{aligned}\big|\Psi_m(s_1,t_1) - \Psi_m(s_2,t_2)\big| &\le \alpha(\big|s_1-s_2\big| + \big|t_1-t_2\big|) + \beta\big|X_m(s_1,t_1) - X_m(s_2,t_2)\big| \\ &\le \alpha(\big|s_1-s_2\big| + \big|t_1-t_2\big|) + \beta L_0(\big|s_1-s_2\big| + \big|t_1-t_2\big|) \\ &= (\alpha+\beta L_0)(\big|s_1-s_2\big| + \big|t_1-t_2\big|).\end{aligned}$$

□

Corollary 3.3. *The functions* $K(s_p,t_q,x,y)\psi(x,y,X_m(x,y)), p = \overline{0,2n}, q = \overline{0,2n}, m \in \mathbb{N}$ *are uniformly Lipschitz with constant*

$$L = \eta M + M_k\big(\alpha + \beta(\theta + \lambda(b-a)(d-c)M\zeta)\big)$$

Proof. Let $(s_1,t_1),(s_2,t_2) \in [a,b]\times[c,d]$. Define the function $\Psi_{m,p,q} : [a,b]\times[c,d] \to \mathbb{R}$,

$$\Psi_{m,p,q}(x,y) = K(s_p,t_q,x,y)\psi(x,y,X_m(x,y)), \quad p = \overline{0,2n}\ q = \overline{0,2n}.$$

Then,

$$\begin{aligned}\big|\Psi_{m,p,q}(x_1,y_1) - \Psi_{m,p,q}(x_2,y_2)\big| \le{}& \big|K(s_p,t_q,x_1,y_1)\psi(x_1,y_1,X_m(x_1,y_1)) \\ &\quad - K(s_p,t_q,x_2,y_2)\psi(x_1,y_1,X_m(x_1,y_1))\big| \\ &+ \big|K(s_p,t_q,x_2,y_2)\psi(x_1,y_1,X_m(x_1,y_1)) \\ &\quad - K(s_p,t_q,x_2,y_2)\psi(x_2,y_2,X_m(x_2,y_2))\big| \\ &\le M\big|K(s_p,t_q,x_1,y_1) - K(s_p,t_q,x_2,y_2)\big| \\ &+ M_k\big|\psi(x_1,y_1,X_m(x_1,y_1)) - \psi(x_2,y_2,X_m(x_2,y_2))\big| \\ &\le M\eta(\big|x_1-x_2\big| + \big|y_1-y_2\big|) + M_k(\alpha+\beta L_0)(\big|x_1-x_2\big| + \big|y_1-y_2\big|) \\ &\le L(\big|x_1-x_2\big| + \big|y_1-y_2\big|).\end{aligned} \tag{3.9}$$

for $m \in \mathbf{N}$. □

3.2. **Algorithm of the approach.** Consider the uniform partitions 3.5 with $s_p = a + p\frac{b-a}{2n}, p = \overline{0, 2n}$ and $t_q = c + q\frac{d-c}{2n}, q = \overline{0, 2n}$, on these knots, relation (3.1) becomes:

$$X_0(s_p, t_q) = r(s_p, t_q),$$
$$X_m(s_p, t_q) = r(s_p, t_q) + \lambda \int_c^d \int_a^b K(s_p, t_q, x, y)\psi(x, y, X_{m-1}(x, y))dxdy,\ p = \overline{0, 2n}\ q = \overline{0, 2n},\ m \geq 1 \tag{3.10}$$

and applying the quadrature (2.1) to relation (3.10), we obtain:

$$X_0(s_p, t_q) = r(s_p, t_q)$$
$$X_m(s_p, t_q) = r(s_p, t_q) + \frac{\lambda h_x h_y}{9} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l_2=0}^{2} \sum_{l_1=0}^{2} (C_2^{l_2})^2 (C_2^{l_1})^2$$
$$K(s_p, t_q, s_{2i-2+l_1}, t_{2j-2+l_2})\psi(s_{2i-2+l_1}, t_{2j-2+l_2}, X_{m-1}(s_{2i-2+l_1}, t_{2j-2+l_2})) + R_{m,p,q} \tag{3.11}$$

with the following remainder estimate

$$|R_{m,p,q}| \leq \frac{5L(b-a)(d-c)}{18}(h_x + h_y), \quad \forall p, q = \overline{0, 2n}, \forall m \in \mathbb{N}, \tag{3.12}$$

according (3.9) and (2.1). It obtains the following iterative algorithm:

Step 1: Choose the functions r, K, ψ and the values $a, b, c, d, \lambda, \varepsilon', n$.
Step 2: Set $h_x = \frac{b-a}{2n}$ and $h_y = \frac{d-c}{2n}$.
Step 3: Choose $\varepsilon' > 0$. For $p = \overline{0, 2n}$, $q = \overline{0, 2n}$, set $\overline{X}_0(s_p, t_q) = r(s_p, t_q)$.
Step 4: For $m = 1$, and for all $p = \overline{0, 2n}$, $q = \overline{0, 2n}$, Compute

$$\overline{X}_1(s_p, t_q) = r(s_p, t_q) + \frac{\lambda h_x h_y}{9} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l_2=0}^{2} \sum_{l_1=0}^{2} (C_2^{l_2})^2 (C_2^{l_1})^2$$
$$K(s_p, t_q, s_{2i-2+l_1}, t_{2j-2+l_2})\psi(s_{2i-2+l_1}, t_{2j-2+l_2}, \overline{X}_0(s_{2i-2+l_1}, t_{2j-2+l_2})). \tag{3.13}$$

Step 5: For $m \geq 2$, and for all $p = \overline{0, 2n}$, $q = \overline{0, 2n}$, Compute

$$\overline{X}_m(s_p, t_q) = r(s_p, t_q) + \frac{\lambda h_x h_y}{9} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l_2=0}^{2} \sum_{l_1=0}^{2} (C_2^{l_2})^2 (C_2^{l_1})^2$$
$$K(s_p, t_q, s_{2i-2+l_1}, t_{2j-2+l_2})\psi(s_{2i-2+l_1}, t_{2j-2+l_2}, \overline{X}_{m-1}(s_{2i-2+l_1}, t_{2j-2+l_2})). \tag{3.14}$$

Step 6: Compute $\mid \overline{X}_m(s_p, t_q) - \overline{X}_{m-1}(s_p, t_q) \mid$, for $p = \overline{0, 2n}$, $q = \overline{0, 2n}$.
Step 7: If $\mid \overline{X}_m(s_p, t_q) - \overline{X}_{m-1}(s_p, t_q) \mid < \varepsilon'$, print m and print $\overline{X}_m(s_p, t_q)$, for all $p = \overline{0, 2n}$, $q = \overline{0, 2n}$; if not set $m = m + 1$ and go to Step 5.

A practical criterion for this algorithm is presented below in Remark (3.7).

3.3. **The convergence analysis.** In this section, an upper bound of the error for the present method is obtained.

Theorem 3.4. *Consider the assumptions of Theorem 3.1. Then, the iterative procedure (3.6) converges to X^* (the unique solution of Eq. (1.1)). Also, the upper bound of the error is as follows*

$$d(X^*, X_m) \leq \frac{\Theta^{m+1}}{\beta(1-\Theta)} M_0 + \frac{5L(b-a)(d-c)}{1-\Theta}(h_x + h_y),$$

Proof. Using (3.4) we have

$$d(X^*, \overline{X}_m) \leq d(X^*, X_m) + d(X_m, \overline{X}_m) \leq \frac{\Theta^{m+1}}{\beta(1-\Theta)} M_0 + d(X_m, \overline{X}_m) \tag{3.15}$$

Therefore, it is necessary to estimate $\mid X_m(s,t,r) - \overline{X}_m(s,t,r) \mid$.
Form (3.10) for $m = 1$, (3.13) and (3.12) we obtain

$$\left|X_1(s_p,t_q) - \overline{X}_1(s_p,t_q)\right| \leq |R_{1,p,q}| \leq \frac{5L(b-a)(d-c)}{18}(h_x + h_y)$$

Using (3.11) and (3.14) we get

$$\begin{aligned}\left|X_m(s_p,t_q) - \overline{X}_m(s_p,t_q)\right| &\leq |R_{m,p,q}| + \frac{\lambda h_x h_y}{9}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l_2=0}^{2}\sum_{l_1=0}^{2}(C_2^{l_2})^2(C_2^{l_1})^2|K(s_p,t_q,s_{2i-2+l_1},t_{2j-2+l_2})| \\ &|\psi(s_{2i-2},t_{2j-2},X_{m-1}(s_{2i-2+l_1},t_{2j-2+l_2})) - \psi(s_{2i-2},t_{2j-2},\overline{X}_{m-1}(s_{2i-2+l_1},t_{2j-2+l_2}))| \\ &\leq \frac{5L(b-a)(d-c)}{18}(h_x+h_y) + \frac{5M_k\beta L(b-a)(d-c)}{36}\sum_{l_2=0}^{2}\sum_{l_1=0}^{2}(C_2^{l_2})^2(C_2^{l_1})^2||X_{m-1} - \overline{X}_{m-1}||\end{aligned} \tag{3.16}$$

Now, from (3.16) for $m = 2$ it follows that

$$\begin{aligned}\left|X_2(s_p,t_q) - \overline{X}_2(s_p,t_q)\right| &\leq \frac{5L(b-a)(d-c)}{18}(h_x+h_y) \\ &+ \frac{5\lambda M_k\beta L(b-a)(d-c)}{36}\sum_{l_2=0}^{2}\sum_{l_1=0}^{2}(C_2^{l_2})^2(C_2^{l_1})^2||X_1 - \overline{X}_1|| \\ &\leq \frac{5L(b-a)(d-c)}{18}(h_x+h_y) + 5\lambda M_k\beta L(b-a)(d-c)||X_1 - \overline{X}_1|| \\ &\leq (1 + \lambda M_k\beta L(b-a)(d-c))\frac{5L(b-a)(d-c)}{18}(h_x+h_y)\end{aligned}$$

For $m \in N$, $m \geq 3$ and by using induction, we get

$$\begin{aligned}&\mid X_m(s_p,t_q) - \overline{X}_m(s_p,t_q) \mid \\ &\leq [1 + \lambda\beta M_K(b-a)(d-c) + ... + (\lambda\beta M_K(b-a)(d-c))^{m-1}]\frac{5L(b-a)(d-c)}{18}(h_x+h_y) \\ &= \frac{1 - (\lambda\beta M_K(b-a)(d-c))^m}{1 - \lambda\beta M_K(b-a)(d-c)}\frac{5L(b-a)(d-c)}{18}(h_x+h_y) \\ &\leq \frac{1}{1 - \lambda\beta M_K(b-a)(d-c)}\frac{5L(b-a)(d-c)}{18}(h_x+h_y) \\ &= \frac{5L(b-a)(d-c)}{18(1-\Theta)}(h_x+h_y).\end{aligned}$$

Therefore,

$$d(X_m, \overline{X}_m) \leq \frac{5L(b-a)(d-c)}{18(1-\Theta)}(h_x+h_y). \tag{3.17}$$

Hence, from (3.4) and (3.17) we conclude that

$$d(X^*, \overline{X}_m) \leq \left(\frac{\Theta^{m+1}}{\beta(1-\Theta)}\right)M_0 + \frac{5L(b-a)(d-c)}{18(1-\Theta)}(h_x+h_y)$$

Since $\Theta < 1$, it is easy to see that

$$\lim_{\substack{m\to\infty \\ h_x,h_y\to 0}} d(X^*, \overline{X}_m) = 0,$$

This means that the proposed method is convergent.

3.4. **The stability analysis.** Here, the numerical stability of the suggested method is studied. To do this, we consider the term $X_0(s,t) = f(s,t) \in C([a,b]\times[c,d], R)$ such that for an $\varepsilon > 0$ we have $\left|r(s,t) - f(s,t)\right| < \varepsilon, \forall (s,t) \in [a,b]\times[c,d]$. Consider θ' and M'_0 such that

$$| f(s,t) - f(s',t') | \leq \theta'(| s - s' | + | t - t' |) \ \forall (s,t),(s',t') \in I,$$
$$M'_0 = max\{|\psi(x,y,r(x,y))| : (s,t) \in I\}$$

and L' the Lipschitz contant obtained similar as in Proposition (3.2) and Corollary (3.3).
By replacing $r(s,t) = f(s,t)$ in Eq. (1.1) and applying the iterative method to the obtained equation, we have

$$Y(s,t) = f(s,t) + \lambda \textstyle\int_c^d \int_a^b K(s,t,x,y)\psi(x,y,Y(x,y))dxdy, \ (s,t) \in I, \tag{3.18}$$

The following sequence of successive approximations on the knots $s_p = a + p\frac{b-a}{2n}, p = \overline{0,2n}$ and $t_q = c + q\frac{d-c}{2n}, q = \overline{0,2n}$ is obtained.

$$Y_0(s_p,t_p) = f(s_p,t_p) \tag{3.19}$$

$$Y_m(s_p,t_p) = f(s_p,t_p) + \lambda \int_c^d \int_a^b K(s_p,t_p,x,y)\psi(x,y,Y_{m-1}(x,y))dxdy, \ m \geq 1 \tag{3.20}$$

Now, by using the iterative procedure (3.6), the following values are obtained.

$$\overline{Y}_0(s,t) = f(s,t)$$

$$\overline{Y}_m(s,t) = f(s,t) + \frac{\lambda h_x h_y}{9} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l_2=0}^{2}\sum_{l_1=0}^{2}(C_2^{l_2})^2(C_2^{l_1})^2$$
$$K(s,t,s_{2i-2+l_1},t_{2j-2+l_2})\psi(s_{2i-2+l_1},t_{2j-2+l_2},\overline{Y}_{m-1}(s_{2i-2+l_1},t_{2j-2+l_2})). \tag{3.21}$$

Theorem 3.5. *Under assumptions of theorem 3.4 and with respect to the first iteration, the iterative procedure (3.6) is numerically stable.*

Proof. We have

$$d(Y_m, \overline{Y}_m) \leq \frac{5L'(b-a)(d-c)}{36(1-\Theta)}(h_x + h_y).$$

Clearly, we have

$$\begin{aligned}\|\overline{X}_m - \overline{Y}_m\| &\leq \|\overline{X}_m - X_m\| + \|X_m - Y_m\| + \|Y_m - \overline{Y}_m\| \\ &\leq \frac{5L(b-a)(d-c)}{36(1-\Theta)}(h_x + h_y) \\ &+ \frac{5L'(b-a)(d-c)}{36(1-\Theta)}(h_x + h_y) + \|X_m - Y_m\| \end{aligned}$$

.

Now, we can write
$| X_0(s,t) - Y_0(s,t) | < \varepsilon$, for all $(s,t) \in I$,
and

$$\begin{aligned}| X_1(s,t) - Y_1(s,t) | &\leq | X_0(s,t) + \lambda \int_c^d \int_a^b K(s,t,x,y)\Psi\big(x,y,X_0(x,y)\big)dxdy \\ &- Y_0(s,t) - \lambda \int_c^d \int_a^b K(s,t,x,y)\Psi\big(x,y,Y_0(x,y)\big)dxdy | \\ &\leq \varepsilon + \beta\lambda M_k \int_c^d \int_a^b | X_0(s,t) - Y_0(s,t) | \, dxdy \\ &\leq \varepsilon + \beta\lambda M_k(b-a)(d-c)\varepsilon,\end{aligned}$$

By considering the condition $\beta\lambda M_k(b-a)(d-c) < 1$, and using induction for $m \geqslant 2$, we obtain

$$\|X_m - Y_m\| \leq \frac{1}{1-\Theta}\varepsilon,$$

for $m \geq 0$. Therefore, we can write

$$\|X_m - Y_m\| \leq \frac{1}{1-\Theta}\varepsilon + \frac{5L(b-a)(d-c)}{18(1-\Theta)}(h_x + h_y) + \frac{5L'(b-a)(d-c)}{36(1-\Theta)}(h_x + h_y)$$

□

Remark 3.6. Since $\Theta < 1$, it can be concluded that

$$\lim_{h_x, h_y, \varepsilon \to 0} \|X_m - Y_m\| = 0.$$

Remark 3.7. The relation (3.3) can be used to get the stopping criterion as follows:
Consider previously chosen $\varepsilon' > 0$. The first integer positive number m is determined such that

$$| \overline{X}_m(s_p, t_q) - \overline{X}_{m-1}(s_p, t_q) | < \varepsilon'.$$

We can write

$$\begin{aligned}\|X^* - \overline{X}_m\| &\leq \|X^* - X_m\| + \|X_m - \overline{X}_m\| \\ &\leq \frac{\Theta}{1-\Theta}\|X_m - X_{m-1}\| + \frac{5L(b-a)(d-c)}{18(1-\Theta)}(h_x + h_y)\end{aligned}$$

and

$$\begin{aligned}\|X_m - X_{m-1}\| &\leq \|X_m - \overline{X}_m\| + \|\overline{X}_m - \overline{X}_{m-1}\| + \|\overline{X}_{m-1} - X_{m-1}\| \\ &\leq \frac{5L(b-a)(d-c)}{9(1-\Theta)}(h_x + h_y) + \|\overline{X}_m - \overline{X}_{m-1}\|\end{aligned}$$

So,

$$\|X^* - \overline{X}_m\| \leq \frac{\Theta}{1-\Theta}\|\overline{X}_m - \overline{X}_{m-1}\| + \frac{5\Theta L(b-a)(d-c)}{9(1-\Theta)^2}(h_x + h_y) + \frac{5L(b-a)(d-c)}{18(1-\Theta)}(h_x + h_y)$$

In order to get $| X^*(s,t) - \overline{X}_m(s,t) | < \varepsilon$, it is necessary that

$$\frac{5L(b-a)(d-c)(1+\Theta)}{18(1-\Theta)^2}(h_x + h_y) < \frac{\varepsilon}{2} \tag{3.22}$$

and

$$\frac{\Theta}{1-\Theta}\|\overline{X}_m - \overline{X}_{m-1}\| < \frac{\varepsilon}{2}.$$

Now, to hold inequality (3.22), the smallest integer positive number n can be selected. Finally, the smallest number $m \in \mathbb{N}$ (as the last iterative step) is obtained and

$$\|\overline{X}_m - \overline{X}_{m-1}\| < \frac{\varepsilon}{2}.\frac{1-\Theta}{\Theta} = \varepsilon'.$$

With these, the inequality $| \overline{X}_m(s_p, t_q) - \overline{X}_{m-1}(s_p, t_q) | < \varepsilon'$ leads to $| X^*(s_p, t_q) - \overline{X}_m(s_p, t_q) | < \varepsilon$.

4. Numerical examples

In this section, some examples are studied to show the applicability, efficiency and accuracy of the proposed method. We introduce the notations

$$E_{p,q} := |X^*(s_p, t_q) - \overline{X}_m(s_p, t_q)|, \tag{4.1}$$

and

$$\|E_n\|_\infty := \max\{E_{p,q}|p, q = 0, 1, ..., n\} \tag{4.2}$$

where X^* and $\overline{X}_m$ are the exact and approximate solutions of the integral equation, respectively. The numerical implementation is carried out in Maple 17.

Example 1. As the first example, a 2D non-linear Fredholm integral equation is considered as

$$X(s,t) = s^2 + t^2 + 1 + \frac{0.00568783}{(s+1)(t+3)} + \int_0^1 \int_0^1 \frac{xy}{(s+1)(t+3)} \sin(x^2 + y + X(x,y))dxdy, \ (s,t) \in [0,1]^2 \tag{4.3}$$

with the exact solution

$$X(s,t) = s^2 + t^2 + 1.$$

In order to ensure that there is an unique solution for equation (4.3) with above r and k and ψ, we test conditions of Theorem (3.1).
It is obvious that $r \in C([0,1]^2, \mathbb{R}), K \in C([0,1]^4, \mathbb{R}), \psi \in C([0,1]^2 \times \mathbb{R}, \mathbb{R})$. We show that Conditions (ii) and (iii) also holds. Since $x, x', y, y' \in [0,1]$ and $X, X' : [0,1]^2 \to [1,3]$, we have

$$\begin{aligned} |\psi(x,y,X) - \psi(x',y',X')| &= |\sin(x^2 + y + X) - \sin(x'^2 + y' + X')|, \\ &\leq |x^2 - x'^2| + |y - y'| + |X - X'|, \\ &\leq 2(|x - x'| + |y - y'|) + |X - X'|. \end{aligned}$$

So, $\alpha = 2, \beta = 1$. Also $|k(s,t,x,y)| \leq \frac{1}{3} = M_k$ and $\lambda = 1$, therefore we have $\Theta = (1)(1)(\frac{1}{3})(1-0)(1-0) < 1$. Now, by using the algorithm for $n = 10$, $\varepsilon' = 10^{-15}$, we obtain $m = 9$ (m is the number of iterations). For more details, see Table 1. For testing the numerical stability, we put $\varepsilon = 0.1$. The computed values $D_{p,q} = |\overline{X}_9(s_p, t_q) - \overline{Y}_9(s_p, t_q)|, p, q = \overline{0, 10}$ in Table 1 show that the algorithm has the numerical stability.
In order to study the convergence, by taking $n = 100$, $\varepsilon' = 10^{-15}$, we have $m = 11$. Now, one can see how $E_{p,q}, p, q = \overline{0, n}$ decreases when h_x and h_y decrease. For this case, the results are presented in Table 2. Also, for $n = 1000, \varepsilon' = 10^{-15}$ with $m = 15$, the results are listed in 3. The values $\|E_n\|_\infty$ for $\varepsilon' = 10^{-15}$ and $n \in \{10, 100, 1000\}$ are $8.6950 \times 10^{-7}, 6.8423 \times 10^{-11}$ and 4.5911×10^{-14}, respectively. The reported results in Tables 1-3 show that the numerical method is convergent, that is $E_{p,q} \to 0$ as $h_x, h_y \to 0$.

Table 1. Numerical results for Example 1 for $n = 10$

(s_p, t_q)	$X^*(s_p, t_q)$	$\overline{X}_9(s_p, t_q)$	$E_{p,q}$	$D_{p,q}$
(0.0,0.0)	1.00	1.0000008198280153642	8.1982801×10^{-7}	0.108
(0.1,0.1)	1.02	1.0200007212563185023	7.2125632×10^{-7}	0.106
(0.2,0.2)	1.08	1.0800006404906370033	6.4049064×10^{-7}	0.104
(0.3,0.3)	1.18	1.1800005733063044505	5.7330630×10^{-7}	0.103
(0.4,0.4)	1.32	1.3200005166983290111	5.1669832×10^{-7}	0.102
(0.5,0.5)	1.50	1.5000004684731516367	4.6847315×10^{-7}	0.101
(0.6,0.6)	1.72	1.7200004269937580022	4.2699376×10^{-7}	0.101
(0.7,0.7)	1.98	1.9800003910149516841	3.9101495×10^{-7}	0.101
(0.8,0.8)	2.28	2.2800003595736909492	3.5957369×10^{-7}	0.100
(0.9,0.9)	2.62	2.6200003319141762608	3.3191418×10^{-7}	0.100
(1.0,1.0)	3.00	3.0000003074355057616	3.0743551×10^{-7}	0.100

TABLE 2. Numerical results for Example 1 for $n = 100$

(s_p, t_q)	$X^*(s_p, t_q)$	$\overline{X}_{11}(s_p, t_q)$	$E_{p,q}$
(0.0,0.0)	1.00	1.0000000000823828075	$8.2382807 \times 10^{-11}$
(0.1,0.1)	1.02	1.0200000000724775433	$7.2477543 \times 10^{-11}$
(0.2,0.2)	1.08	1.0800000000643615684	$6.4361568 \times 10^{-11}$
(0.3,0.3)	1.18	1.1800000000576103549	$5.7610355 \times 10^{-11}$
(0.4,0.4)	1.32	1.3200000000519219375	$5.1921937 \times 10^{-11}$
(0.5,0.5)	1.50	1.5000000000349954371	$4.7075890 \times 10^{-11}$
(0.6,0.6)	1.72	1.7200000000429077122	$4.2907712 \times 10^{-11}$
(0.7,0.7)	1.98	1.9800000000292092282	$3.9292277 \times 10^{-11}$
(0.8,0.8)	2.28	2.2800000000361328103	$3.6132810 \times 10^{-11}$
(0.9,0.9)	2.62	2.6200000000333533634	$3.3353363 \times 10^{-11}$
(1.0,1.0)	3.00	3.0000000000308935528	$3.0893553 \times 10^{-11}$

TABLE 3. Numerical results for Example 1 for $n = 1000$

(s_p, t_q)	$X^*(s_p, t_q)$	$\overline{X}_{15}(s_p, t_q)$	$E_{p,q}$
(0.0,0.0)	1.00	1.00000000000004010262	$4.0102061 \times 10^{-14}$
(0.1,0.1)	1.02	1.02000000000003528040	$3.5280405 \times 10^{-14}$
(0.2,0.2)	1.08	1.08000000000003132973	$3.1329735 \times 10^{-14}$
(0.3,0.3)	1.18	1.18000000000002804340	$2.8043399 \times 10^{-14}$
(0.4,0.4)	1.32	1.32000000000002527441	$2.5274408 \times 10^{-14}$
(0.5,0.5)	1.50	1.50000000000002291546	$2.2915463 \times 10^{-14}$
(0.6,0.6)	1.72	1.72000000000002088649	$2.0886490 \times 10^{-14}$
(0.7,0.7)	1.98	1.98000000000001912658	$1.9126579 \times 10^{-14}$
(0.8,0.8)	2.28	2.28000000000001758862	$1.7588623 \times 10^{-14}$
(0.9,0.9)	2.62	2.62000000000001623565	$1.6235652 \times 10^{-14}$
(1.0,1.0)	3.00	3.00000000000001503827	$1.5038273 \times 10^{-14}$

Example 2. Here, we consider the following 2D non-linear Fredholm integral equation [24]

$$X(s,t) = r(s,t) + \int_0^1 \int_0^1 K(s,t,x,y)\Psi(X(x,y))dxdy, \ \ (s,t) \in [0,1] \times [0,1], \tag{4.4}$$

where

$$r(s,t) = \sin(t) - \frac{1}{18}st^2\big(1 - \cos(1)(\frac{1}{2}\sin^2(1) + 1)\big),$$
$$K(s,t,x,y) = \frac{1}{6}xst^2,$$
$$\Psi(X) = X^3,$$

with the exact solution

$$X(s,t) = \sin(t).$$

In this example, we have $a = c = 0, b = d = 1, \lambda = 1, M_k = \frac{1}{6}$. Also

$$|\psi(x,y,X) - \psi(x',y',X')| = |X^3 - X'^3| \leq 3|X - X'|,$$

for $X, X' : [0,1]^2 \to [0,1]$. Therefore $\beta = 3$ and we have $\Theta = (1)(3)(\frac{1}{6})(1-0)(1-0) = \frac{1}{2} < 1$. Because of all conditions hold, Theorem (3.1) implies the existence of unique solution for equation.

Comparisons between the obtained errors by the present method and the method in [24] are shown in Tables 4-6.

TABLE 4. Comparison of errors of numerical results for Example 2 for $n = 10$

(s_p, t_q)	$X^*(s_p, t_q)$	Present method	Method of [24]
(0.0,0.0)	0.	0.	0.
(0.1,0.1)	0.09983341664682815230	3.899410×10^{-11}	$8.203942576330200 \times 10^{-8}$
(0.2,0.2)	0.19866933079506121546	3.119528×10^{-10}	$6.563154061064100 \times 10^{-7}$
(0.3,0.3)	0.29552020666133957511	1.052848×10^{-9}	$2.215064495609150 \times 10^{-6}$
(0.4,0.4)	0.38941834230865049167	2.495623×10^{-9}	$5.250523248851320 \times 10^{-6}$
(0.5,0.5)	0.47942553860420300027	4.874263×10^{-9}	$1.025492822041272 \times 10^{-5}$
(0.6,0.6)	0.56464247339503535720	8.422726×10^{-9}	$1.772051596487319 \times 10^{-5}$
(0.7,0.7)	0.64421768723769105367	1.337498×10^{-8}	$2.813952303681252 \times 10^{-5}$
(0.8,0.8)	0.71735609089952276163	1.996498×10^{-8}	$4.200418599081052 \times 10^{-5}$
(0.9,0.9)	0.78332690962748338846	2.842670×10^{-8}	$5.980674138144701 \times 10^{-5}$
(1.0,1.0)	0.84147098480789650665	3.899410×10^{-8}	$8.203942576330180 \times 10^{-5}$

TABLE 5. Comparison of errors of numerical results for Example 2 for $n = 100$

(s_p, t_q)	$X^*(s_p, t_q)$	Present method	Method of [24]
(0.0,0.0)	0.	0.	0.
(0.1,0.1)	0.09983341664682815230	3.888832×10^{-15}	$8.1846966058500 \times 10^{-10}$
(0.2,0.2)	0.19866933079506121546	3.111065×10^{-14}	$6.5477572846800 \times 10^{-9}$
(0.3,0.3)	0.29552020666133957511	1.049985×10^{-13}	$2.2098680835790 \times 10^{-8}$
(0.4,0.4)	0.38941834230865049167	2.488852×10^{-13}	$5.2382058277440 \times 10^{-8}$
(0.5,0.5)	0.47942553860420300027	4.861039×10^{-13}	$1.0230870757312 \times 10^{-7}$
(0.6,0.6)	0.56464247339503535720	8.399876×10^{-13}	$1.7678944668635 \times 10^{-7}$
(0.7,0.7)	0.64421768723769105367	1.333869×10^{-12}	$2.8073509358064 \times 10^{-7}$
(0.8,0.8)	0.71735609089952276163	1.991082×10^{-12}	$4.1905646621950 \times 10^{-7}$
(0.9,0.9)	0.78332690962748338846	2.834958×10^{-12}	$5.9666438256644 \times 10^{-7}$
(1.0,1.0)	0.84147098480789650665	3.888832×10^{-12}	$8.1846966058496 \times 10^{-7}$

TABLE 6. Comparison of errors of numerical results for Example 2 for $n = 1000$

(s_p, t_q)	$X^*(s_p, t_q)$	Present method	Method of [24]
(0.0,0.0)	0.	0.	0.
(0.1,0.1)	0.09983341664682815230	$6.2226423 \times 10^{-18}$	$9.093915900300 \times 10^{-12}$
(0.2,0.2)	0.19866933079506121546	$4.9781138 \times 10^{-17}$	$7.275132720200 \times 10^{-11}$
(0.3,0.3)	0.29552020666133957511	$1.6801134 \times 10^{-16}$	$2.455357293070 \times 10^{-10}$
(0.4,0.4)	0.38941834230865049167	$3.9824911 \times 10^{-16}$	$5.820106176170 \times 10^{-10}$
(0.5,0.5)	0.47942553860420300027	$7.7783029 \times 10^{-16}$	$1.136739487534 \times 10^{-9}$
(0.6,0.6)	0.56464247339503535720	$1.3440907 \times 10^{-15}$	$1.964285834459 \times 10^{-9}$
(0.7,0.7)	0.64421768723769105367	$2.1343663 \times 10^{-15}$	$3.119213153794 \times 10^{-9}$
(0.8,0.8)	0.71735609089952276163	$3.1859928 \times 10^{-15}$	$4.656084940940 \times 10^{-9}$
(0.9,0.9)	0.78332690962748338846	$4.5363062 \times 10^{-15}$	$6.629464691299 \times 10^{-9}$
(1.0,1.0)	0.84147098480789650665	$6.2226423 \times 10^{-15}$	$9.093915900273 \times 10^{-9}$

Example 3. [17, 24] As the last example, we consider the integral equation (4.3) with

$$r(s,t) = -\frac{s}{6(1+t)} + \frac{1}{(1+s+t)^2}$$

$$K(s,t,x,y) = \frac{s}{1+t}(1+y+x),$$

$$\Psi(X) = X^2,$$

Table 7. Comparison of errors of the numerical results for Example 3 for $n = 10, n = 100, n = 1000$

	Method of [24]			Present Method		
(s , t)	n=10	n=100	n=1000	n=10	n=100	n=1000
(0.0,0.0)	0	0	0	0	0	0
(0.1,0.1)	1.381×10^{-6}	1.380×10^{-8}	2.839×10^{-10}	1.071×10^{-7}	1.078×10^{-11}	8.302×10^{-16}
(0.2,0.2)	5.525×10^{-6}	5.520×10^{-8}	6.389×10^{-10}	1.963×10^{-7}	1.977×10^{-11}	1.530×10^{-15}
(0.3,0.3)	1.243×10^{-5}	1.242×10^{-7}	1.775×10^{-9}	2.718×10^{-7}	2.737×10^{-11}	2.112×10^{-15}
(0.4,0.4)	2.210×10^{-5}	2.208×10^{-7}	2.555×10^{-9}	3.365×10^{-7}	3.388×10^{-11}	2.624×10^{-15}
(0.4,0.5)	3.453×10^{-5}	3.450×10^{-7}	3.478×10^{-9}	3.925×10^{-7}	3.953×10^{-11}	3.061×10^{-15}
(0.6,0.6)	4.972×10^{-5}	4.968×10^{-7}	4.543×10^{-9}	4.416×10^{-7}	4.447×10^{-11}	3.436×10^{-15}
(0.7,0.7)	6.768×10^{-5}	6.762×10^{-7}	5.750×10^{-9}	4.850×10^{-7}	4.883×10^{-11}	3.771×10^{-15}
(0.8,0.8)	8.836×10^{-5}	8.832×10^{-7}	7.099×10^{-9}	5.234×10^{-7}	5.280×10^{-11}	4.077×10^{-15}
(0.9,0.9)	1.119×10^{-4}	1.118×10^{-6}	1.147×10^{-8}	5.578×10^{-7}	5.618×10^{-11}	4.343×10^{-15}
(1.0,1.0)	1.381×10^{-4}	1.380×10^{-6}	1.394×10^{-8}	5.888×10^{-7}	5.930×10^{-11}	4.582×10^{-15}

where the exact solution is

$$X(s,t) = \frac{1}{(1+s+t)^2}.$$

Table 7 shows comparisons between the obtained errors by the present method and the method in [24]. According to the reported results, our proposed method has a reasonable convergence rate. Also, the absolute errors in the solutions by our method are accurate in comparison with [24].

5. Conclusions

This paper applied an iterative method of successive approximation for solving the 2D non-linear Hammerstein integral equations. The basic results of this paper are presented in Theorems 3.1, 3.4, and 3.5 which survey the numerical stability and convergence of the proposed method. These two aspects of the method were studied and confirmed by three numerical examples. The numerical results show that to obtain a good approximate solution a small number of iteration is required.

Acknowledgment

The authors are grateful to anonymous referees for their constructive comments and suggestions.

References

[1] M. Abdou, A. Badr and M. Soliman, *On a method for solving a two-dimensional nonlinear integral equation of the second kind*, J. comput. appl. math., *235* (2011) 3589–3598.

[2] R. P. Agarwal, N. Hussain and M.-A. Taoudi, *Fixed point theorems in ordered banach spaces and applications to nonlinear integral equations*, Abst. Appl. Anal., volume 2012, Hindawi Publishing Corporation, 2012.

[3] A. Altürk, *The regularization-homotopy method for the two-dimensional fredholm integral equations of the first kind*, Math. Comput. Appl., *21* (2016), P 9.

[4] K.E. Atkinson, *The numerical solution of a nonlinear boundary integral equation on smooth surfaces*, IMA J. Numer. Anal., *14* (1994) 461–483.

[5] K.E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, Cambridge, 1997.

[6] K.E. Atkinson and G. Chandler, *BIE methods for solving Laplace's equation with nonlinear boundary conditions: The smooth boundary case*, Math. Comp., *55* (1990) 455–472.

[7] K. Atkinson and W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework*, Springer Science+ Business Media LLC, NewYork, 2007.

[8] K. Atkinson and F. A. Potra, *Projection and iterated projection methods for nonlinear integral equations*, SIAM J. Numer. Anal., *24* (1987) 1352–1373.

[9] I. Aziz, F. Khan and et al., *A new method based on haar wavelet for the numerical solution of two-dimensional nonlinear integral equations*, J. Comput. Appl. Math., *272* (2014) 70–80.

[10] E. Babolian, S. Bazm and P. Lima, *Numerical solution of nonlinear two-dimensional integral equations using rationalized Haar functions*, Commun. Nonlinear Sci. Numer. Simul., *16* (2011)1164–1175.

[11] S. Bazm and A. Hosseini, *Bernoulli operational matrix method for the numerical solution of nonlinear two-dimensional VolterraFredholm integral equations of Hammerstein type*, Comput. Appl. Math., *39* (2020), p 49.

[12] A. M. Bica , M. Curila and S. Curila, *About a numerical method of successive interpolations for functional Hammerstein integral equations*, J. Comput. Appl. Math., *236* (2012) 2005–2024.

[13] H. Brunner, *Collocation methods for Volterra integral and related functional differential equations*, volume 15, Cambridge University Press, 2004.

[14] L. M. Delves and J. L. Mohamed, *Computational Methods for Integral Equations*, Cambridge University Press, Cambridge, 1985.

[15] S. Fazeli, G. Hojjati and H. Kheiri, *A piecewise approximation for linear two-dimensional volterra integral equation by chebyshev polynomials*, Int. J. Nonlinear Sci., *16* (2013) 255–261.

[16] H. Guoqiang and W. Jiong, *Extrapolation of nyström solution for two-dimensional nonlinear fredholm integral equations*, J. Comput. Appl. Math., *134* (2001) 259–268.

[17] G. Han and R. Wang, *Richardson extrapolation of iterated discrete galerkin solution for two-dimensional fredholm integral equations*, J. Comput. Appl. Math., *139* (2002) 49–63.

[18] R. J. Hanson and J. L. Phillips, *Numerical solution of two-dimensional integral equations using linear elements*, SIAM J. Numer. Anal., *15* (1978) 113–121.

[19] S. A. Hosseini, S. Shahmorad and A. Tari, *Existence of an Lp-solution for two dimensional integral equations of the Hammerstein type*, Bull. Iran. Math. Soc., *40* (2014) 851–862.

[20] A. Jafarian and S. M. Nia, *Utilizing feed-back neural network approach for solving linear fredholm integral equations system*, Appl. Math. Model., *37* (2013) 5027–5038.

[21] A. Jerri, *Introduction to integral equations with applications*, John Wiley & Sons, 1999.

[22] M. Kazemi and R. Ezzati, *Existence of solution for some nonlinear two-dimensional volterra integral equations via measures of noncompactness*, Appl. Math. Comput., *275* (2016) 165–171.

[23] M. Kazemi and R. Ezzati, *Existence of Solutions for some Nonlinear Volterra Integral Equations via Petryshyn's Fixed Point Theorem*, Int. J. Nonlinear Anal. Appl., *9* (2018) 1–12.

[24] M. Kazemi, H. M. Golshan, R. Ezzati and M. Sadatrasoul, *New approach to solve two-dimensional Fredholm integral equations*, J. Comput. Appl. Math., *354* (2019) 66–79.

[25] M. Kazemi, V. Torkashvand and R. Ezzati, *On a method based on Bernstein operators for 2D nonlinear Fredholm-Hammerstein integral equations*, U.P.B. Sci. Bull., Series A, *83* (2021) 178–198.

[26] A. Khajehnasiri, *Numerical solution of nonlinear 2D VolterraFredholm integro-differential equations by two-dimensional triangular function*, Int. J. Appl. Comput. Math., *2* (2016) 575–591.

[27] B. H. Lichae, J. Biazar and Z. Ayati, *A class of RungeKutta methods for nonlinear Volterra integral equations of the second kind with singular kernels*, Adv. Differ. Equ., *349* (2018).

[28] K. Maleknejad, N. Aghazadeh and M. Rabbani, *Numerical solution of second kind fredholm integral equations system by using a taylor-series expansion method*, Appl. Math. Comput., *175* (2006) 1229–1234.

[29] K. Maleknejad, S. Sohrabi and B. Baranji, *Application of 2D-BPFs to nonlinear integral equations*, Commun. Nonlinea.r Sci. Numer. Simul., *15* (2010) 527–535.

[30] K. Maleknejad and P. Torabi, *Application of fixed point method for solving nonlinear Volterra-Hammerstein integral equation*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., *74* (2012) 45–56.

[31] S. McKee, T. Tang and T. Diogo, *An Euler-type method for two-dimensional volterra integral equations of the first kind*, J. Inst. Math. Its Appl., *20* (2000) 423–440.

[32] F. Mirzaee and S. Alipour, *Solving two-dimensional non-linear quadratic integral equations of fractional order via Operational matrix method*, Multidiscipl. Model. Mater. Struct., *15* (2019) 1136–1151.

[33] S. Najafalizadeh and R. Ezzati, *A block pulse operational matrix method for solving two-dimensional nonlinear integro-differential equations of fractional order*, J. Comput. Appl. Math., *326* (2017) 159–170.

[34] B. G. Pachpatte, *Multidimensional integral equations and inequalities*, Springer Science & Business Media, 2011.

[35] A. G. Ramm, *Dynamical systems method for solving operator equations*, Commun. Nonlinear Sci. Numer. Simul., *9* (2004) 383–402.

[36] Qi. Tang and D. Waxman, *An integral equation describing an asexual population in a changing environment*, Nonlinear Anal. Theory Methods Appl., *53* (2003) 683–699.

[37] A. Tari Marzabad and S. M. Torabi, *Numerical solution of two-dimensional integral equations of the first kind by multi-step methods*, Comput. Methods Diff. Equations., *4* (2016)128–138.

[38] F. G. Tricomi, *Integral equations*, volume 5, Dover Publications, 1982.

[39] A.-M. Wazwaz, *Linear and nonlinear integral equations: methods and applications*, Springer Science & Business Media, 2011.