



## A novel perspective for simulations of the MEW equation by trigonometric cubic B-spline collocation method based on Rubin-Graves type linearization

Nuri Murat Yağmurlu\* and Ali Sercan Karakaş

Inönü University, Department of Mathematics, Malatya, 44280, Turkey.

### Abstract

In the present study, the Modified Equal Width (MEW) wave equation is going to be solved numerically by presenting a new technique based on the collocation finite element method in which trigonometric cubic B-splines are used as approximate functions. In order to support the present study, three test problems; namely, the motion of a single solitary wave, the interaction of two solitary waves, and the birth of solitons are studied. The newly obtained results are compared with some of the other published numerical solutions available in the literature. The accuracy of the proposed method is discussed by computing the numerical conserved laws as well as the error norms  $L_2$  and  $L_\infty$ .

**Keywords.** Finite element method, Collocation method, Solitary waves, Modified equal width equation, Trigonometric B-splines.

**2010 Mathematics Subject Classification.** 65T40, 35Q51, 74J35, 65D07.

### 1. INTRODUCTION

In nature, phenomena that are nonlinear in their characteristics have a deterministic role in various fields of science such as waves, fluid mechanics, plasma physics, optics, solid state physics, kinetics and geology. Especially, in wave studies, all of the phenomena such as dispersion, dissipation, diffusion, reaction, and convection become important [42].

The widely used nonlinear modeling for wave phenomena is Korteweg de-Vries (KdV) [26] equation of the following form

$$u_t + 6uu_x + u_{xxx} = 0.$$

Then comes regularised long wave (RLW) equation for describing nonlinear dispersive wave phenomena of the form

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{1.1}$$

widely accepted as an alternative to the KdV equation. The third equation used for modeling those wave phenomena are known as the equal width (EW) equation and is presented in the following form [30]

$$u_t + uu_x - u_{xxt} = 0.$$

Finally comes to the modified equal width (MEW) equation closely in relation with the RLW (1.1) is given in the following form under the physical boundary conditions  $u \rightarrow 0$  if  $x \rightarrow \pm\infty$

Received: 08 August 2021 ; Accepted: 09 January 2022.

\* Corresponding author. Email:murat.yagmurlu@inonu.edu.tr .

$$u_t + \varepsilon u^2 u_x - \mu u_{xxt} = 0. \tag{1.2}$$

Here  $t$  and  $x$  denote time and space coordinates, respectively  $\varepsilon$  and  $\mu$  are positive parameters and  $u$  is related to the vertical displacement of the water surface. In order to obtain the numerical solutions of MEW (1.2) for  $x \in [a, b]$ , the following boundary conditions

$$\begin{aligned} u(a, t) &= u(b, t) = 0, \\ u_x(a, t) &= u_x(b, t) = 0, \\ u_{xxt}(a, t) &= u_{xxt}(b, t) = 0, \end{aligned}$$

and the following initial condition,

$$u(x, 0) = f(x), \quad a \leq x \leq b,$$

has been considered, where  $f(x)$  is a sufficiently smooth function. The MEW (1.2) equation has the following solitary wave solution

$$u(x, t) = A \operatorname{sech} [k(x - x_0 - ct)] \tag{1.3}$$

where  $k = 1/\sqrt{\mu}$  and  $c = A^2/2$ . In the literature, solitary waves are defined as traveling waves while retaining their shapes and speeds because of the delicate balance between nonlinearity and dispersive effects, whereas, a soliton is a very special type of solitary wave, retaining its shape and speed even after colliding with another wave [14]. Although, those solitary waves can have both positive and negative amplitudes, their speed is positive and proportional to the square of their amplitudes. Moreover, as with RLW equation, since all of them have the same number of waves  $k = 1/\sqrt{\mu}$ , they also have the same width [39]. The conservation constants of the MEW equation for the above boundary conditions are found by Olver [31] as follows

$$I_1 = \int_{-\infty}^{+\infty} u dx, \quad I_2 = \int_{-\infty}^{+\infty} (u^2 + \mu(u_x)^2) dx, \quad I_3 = \int_{-\infty}^{+\infty} u^4 dx.$$

In the literature, one can encounter several exact and approximate solutions of the MEW equation given with various initial and boundary conditions. Among others, Hamdi *et al.* [19] have obtained exact solitary wave solutions of the generalized equal width wave equation. Wazwaz [42] has studied the MEW equation and two of its variants with the help of tanh and sine-cosine methods. Esen and Kutluay [9] have utilized a linearized numerical scheme based on the finite difference method to find out solitary wave solutions of the one-dimensional MEW equation. Raslan [32] solved generalized EW equation numerically by collocation of cubic B-splines finite element method. Jin [21] has suggested an analytical approach based on the homotopy perturbation method for solving the MEW equation. Lu [29] has introduced the variational iteration method for finding the solutions of the MEW equation.

Esen [10] has obtained a numerical solution of the one-dimensional MEW equation with the help of a lumped Galerkin method using the quadratic B-spline finite element method. Çelikkaya [5] has used the operator splitting method for the numerical solution of a modified equal width equation. Essa [11] has applied the multigrid method for the numerical solution of the modified equal width wave equation. Zaki [39] has taken the solitary wave interactions for the MEW equation into consideration by collocation method based on quintic B-spline finite elements and he [40] also found the numerical solution of the EW equation by using the least-squares method. Karakoç and Zeybek [22] have obtained the numerical solutions of the generalized equal width (GEW) wave equation by using lumped Galerkin approach with the cubic B-spline functions and they [41] have also used quintic B-spline collocation algorithm with two different linearization techniques. Yağmurlu and Karakaş [37] have solved the equal width (EW) equation numerically. Roshan [33] has sought the solutions for the equation by using the Petrov-Galerkin method.

Geyikli and Karakoç [15] obtained numerical solutions of the MEW equation by using the collocation method with septic B-spline finite elements with three different linearization techniques and they [16] have also utilized the



subdomain finite element method with quartic B splines. Saka [35] has obtained numerical solutions for time split the MEW equation and space split the MEW equation using quintic B-spline collocation method. Karakoç and Geyikli [23] have obtained a numerical solution of the MEW equation using sextic B-splines. Geyikli and Karakoç [17] have applied Petrov–Galerkin method with the cubic B-splines for solving the MEW equation. Karakoç and Geyikli [24] have obtained a numerical solution of the modified equal width wave equation. Karakoç [25] has worked with applying the cubic B-spline function to develop a numerical method for approximating the analytic solution of the MEW equation. Evans and Raslan [12] have obtained solitary waves for the generalized equal width (GEW) equation. Kaplan and Dereli [18] have solved GEW equation by using the moving least squares collocation method.

Cheng [4] has applied the improved element-free Galerkin method applied to the MEW equation. Başhan *et al* [2] have used the finite difference method combined with differential quadrature method with Rubin and Graves linearizing technique for the numerical solution of the MEW equation. In a recently published article, Başhan *et al* [3] have presented a new perspective for the numerical solution of the MEW equation. One can see some recently published articles about physical phenomena such as given in Refs [1]-[13]. The presented method has the advantage of using less computer storage capacity and less running computational time. This resulted in accurate results in a short simulation time. More recently, Shallal *et al* [36] have obtained exact solutions of the conformable fractional EW and MEW equations by a new generalized expansion method.

The reader can also refer to the following articles about the different numerical approaches. Zadhan and Rashidina [38] have developed a non polynomial cubic spline function which they called "TS" spline based on trigonometric functions. Dehghan and Lakestani [8] have presented a numerical technique for the solution of a nonclassical problem for the one-dimensional wave equation. Daba and Dureessa [6] have presented a robust computational method involving an exponential cubic spline for solving singularly perturbed parabolic convection-diffusion equations arising in the modeling of neuronal variability. Lakestani and Dehghan [27] have presented a numerical technique for the solution of Fokker-Planck equation. Lakestani [28] has presented a numerical technique based on the finite difference and collocation methods for the solution of the Korteweg-de Vries (KdV) equation.

In this article, the error norms  $L_2$  and  $L_\infty$  are going to be used to compare the differences between exact and approximate solutions. Those widely used error norms  $L_2$  and  $L_\infty$  are computed by the following formulae

$$L_2 = \|u^{exact} - U_N\|_2 \simeq \sqrt{h \sum_{j=0}^N |u_j^{exact} - (U_N)_j|^2},$$

$$L_\infty = \|u^{exact} - U_N\|_\infty \simeq \max_j |u_j^{exact} - (U_N)_j|.$$

In the present article, the MEW equation is going to be handled using the finite element trigonometric B-spline cubic collocation method. During the solution process, a Rubin-Graves type linearization technique is going to be utilized to overcome the nonlinear term appearing in the equation. Then the newly obtained results are going to be compared with some of those available in the literature.

## 2. Implementation of the method for space discretization

The MEW equation is generally given in the following form

$$u_t + \varepsilon u^2 u_x - \mu u_{xxt} = 0, \quad a \leq x \leq b$$

together with the physical boundary conditions  $U \rightarrow 0$  as  $x \rightarrow \pm\infty$ , in which  $t$  is time,  $x$  is the space coordinate and  $\mu$  is a positive parameter. For the considered problems, the appropriate boundary conditions will be chosen as

$$\begin{aligned} U(a, t) &= 0, & U(b, t) &= 0, \\ U_x(a, t) &= 0, & U_x(b, t) &= 0. \end{aligned}$$

Let us consider the solution interval  $[a, b]$  is divided into  $N$  finite elements having equal lengths using the nodal points  $x_i$ ,  $i = 0(1)N$  in such a way that  $a = x_0 < x_1 \cdots < x_N = b$  and  $h = (x_{i+1} - x_i)$ . The trigonometric cubic B-splines  $T_m^3(x)$ , ( $m = -1(1)N + 1$ ), at the knots  $x_m$  are defined over the interval  $[a, b]$  by [7],



$$T_m^3(x) = \frac{1}{\theta} \begin{cases} \rho^3(x_{m-2}) & , \quad x_{m-2} \leq x \leq x_{m-1} \\ -\rho^2(x_{m-2})\rho(x_m) & \\ -\rho(x_{m-2})\rho(x_{m+1})\rho(x_{m-1}) & , \quad x_{m-1} \leq x \leq x_m \\ -\rho(x_{m+2})\rho^2(x_{m-1}) & \\ \rho(x_{m-2})\rho^2(x_{m+1}) & \\ +\rho(x_{m+2})\rho(x_{m-1})\rho(x_{m+1}) & , \quad x_m \leq x \leq x_{m+1} \\ +\rho^2(x_{m+2})\rho(x_m) & \\ -\rho^3(x_{m+2}) & , \quad x_{m+1} \leq x \leq x_{m+2} \\ 0 & , \quad \text{otherwise} \end{cases}$$

in which

$$\rho(x_m) = \sin\left(\frac{x-x_m}{2}\right), \quad \theta = \sin\left(\frac{h}{2}\right) \sin(h) \sin\left(\frac{3h}{2}\right), \quad m = 0(1)N.$$

The set of trigonometric cubic B-splines  $\{T_{-1}^3(x), T_0^3(x), \dots, T_{N+1}^3(x)\}$  forms a basis for the smooth functions defined over  $[a,b]$ . Therefore, an approximation solution  $U_N(x, t)$  can be written in terms of the trigonometric cubic B-splines as trial functions:

$$U(x, t) \approx U_N(x, t) = \sum_{i=m-1}^{m+2} T_i^3(x)\delta_i(t), \tag{2.1}$$

where  $\delta_i(t)$ 's are unknown, time dependent quantities to be determined from the boundary and trigonometric cubic B-spline collocation conditions. Each trigonometric cubic B-spline covers four elements so that each element  $[x_i, x_{i+1}]$  is covered by four trigonometric cubic B-splines. For this problem, the finite elements are identified with the interval  $[x_i, x_{i+1}]$ . Using the nodal values  $U_i, U'_i$  and  $U''_i$  are given in terms of the parameter  $\delta_i$  by:

$$\begin{aligned} U_i &= \alpha_1\delta_{i-1} + \alpha_2\delta_i + \alpha_1\delta_{i+1}, \\ U'_i &= \beta_1\delta_{i-1} + \beta_1\delta_{i+1}, \\ U''_i &= \gamma_1\delta_{i-1} + \gamma_2\delta_i + \gamma_1\delta_{i+1}, \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= \sin^2\left(\frac{h}{2}\right) \csc(h) \csc\left(\frac{3h}{2}\right), & \alpha_2 &= \frac{2}{(1+2\cos(h))}, \\ \beta_1 &= -\frac{3\csc\left(\frac{3h}{2}\right)}{4}, & \beta_2 &= \frac{3\csc\left(\frac{3h}{2}\right)}{4}, \\ \gamma_1 &= \frac{3((1+3\cos(h))\csc^2\left(\frac{h}{2}\right))}{16(2\cos\left(\frac{h}{2}\right) + \cos\left(\frac{3h}{2}\right))}, & \gamma_2 &= -\frac{3\cot^2\left(\frac{h}{2}\right)}{(2+4\cos(h))}. \end{aligned}$$

During the solution process, firstly, for the time discretization forward finite difference scheme and then for the space discretization finite element collocation method based on trigonometric cubic B-spline basis functions are going to be implemented.

### 3. Implementation of the method for time discretization

Now, the MEW wave equation is discretized as follows

$$u_t + \varepsilon u^2 u_x - \mu u_{xxt} = 0.$$



For this purpose, the Crank-Nicolson type scheme is implemented. Firstly the equation is discretized as,

$$\frac{U^{n+1} - U^n}{\Delta t} + \varepsilon \frac{(U^2 U_x)^{n+1} + (U^2 U_x)^n}{2} - \mu \frac{U_{xx}^{n+1} - U_{xx}^n}{\Delta t} = 0, \quad (3.1)$$

where Rubin and Graves type linearization technique [34] is used at the left hand side of the Eq. (3.1) to linearize the nonlinear terms as given below

$$(U^2 U_x)^{n+1} = U^{n+1} U^n U_x^n + U^n U^{n+1} U_x^n + U^n U^n U_x^{n+1} - 2U^n U^n U_x^n.$$

Accordingly, the following iterative scheme is obtained

$$U^{n+1} + \varepsilon \frac{\Delta t}{2} (U^{n+1} U^n U_x^n + U^n U^{n+1} U_x^n + U^n U^n U_x^{n+1} - U^n U^n U_x^n) - \mu U_{xx}^{n+1} = U^n + \varepsilon \frac{\Delta t}{2} (U^2 U_x)^n - \mu U_{xx}^n$$

This scheme results in a system of equations consisting of  $(N + 1)$  equations and  $(N + 3)$  unknowns. Using the appropriate boundary conditions given with the problem, the unknowns lying outside the solution domain of the problem are eliminated. Thus a solvable system of equations is obtained. Now utilizing this system, the calculations are carried out until the desired time level. But for this, first of all, the initial values of the unknowns at time  $t = 0$  are needed. The following section will deal with this step of the solution process.

**3.1. Initial state.** The initial vector  $d^0$  is determined from the initial and boundary conditions. So the approximation Eq. (2.1) must be rewritten for the initial condition

$$U_N(x, 0) = \sum_{m=-1}^{N+1} \delta_m^0(t) T_m^3(x)$$

where the  $\delta_m^0$ 's are unknown parameters. The initial numerical approximation  $U_N(x, 0)$  is required to satisfy the following conditions:

$$\begin{aligned} U_N(x, 0) &= U(x_i, 0), & i &= 0, 1, \dots, N \\ (U_N)_x(a, 0) &= 0, & (U_N)_x(b, 0) &= 0. \end{aligned}$$

Thus, these conditions lead to the matrix equation

$$W d^0 = b$$

where

$$d^0 = (\delta_0, \delta_1, \delta_2, \dots, \delta_{N-2}, \delta_{N-1}, \delta_N)^T$$

and

$$b = (U(x_0, 0), U(x_1, 0), U(x_2, 0), \dots, U(x_{N-2}, 0), U(x_{N-1}, 0), U(x_N, 0))^T.$$

#### 4. NUMERICAL EXAMPLES

In this section, three common test problems about the MEW equation are going to be solved and the results will be compared with some of those available in the literature. If the exact solution of the test problem is available, then the accuracy of the numerical method is going to be controlled by using the error norms  $L_2$  and  $L_\infty$ .



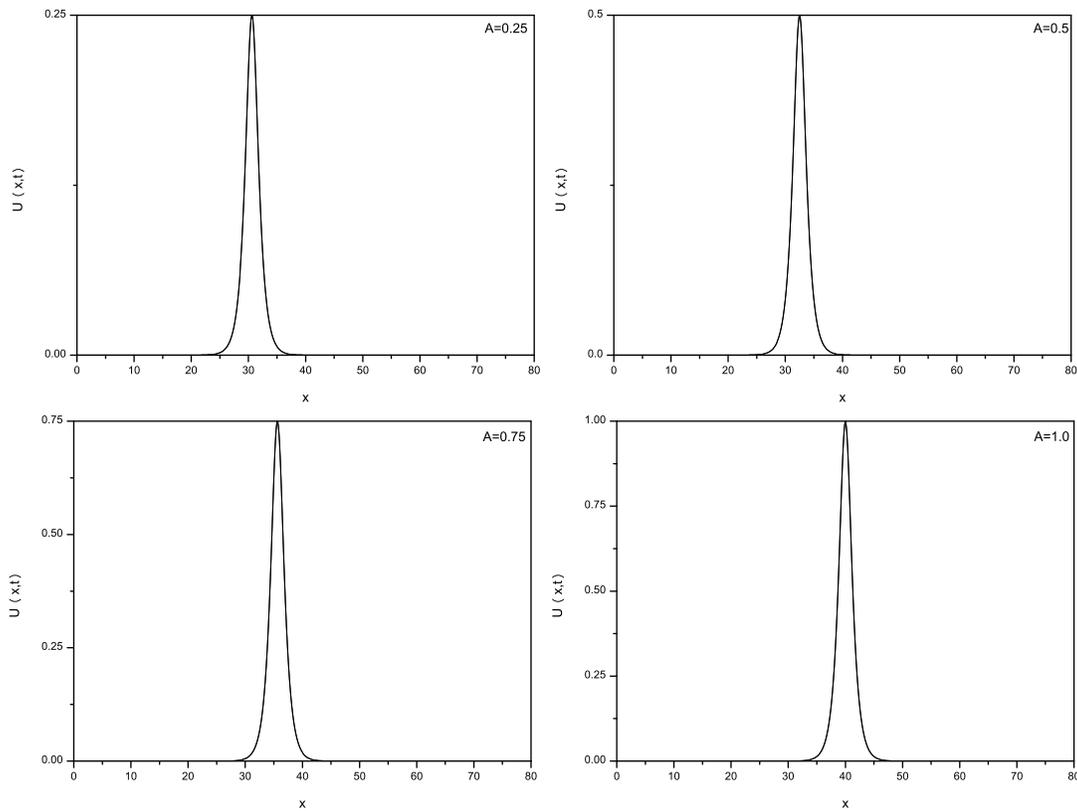


FIGURE 1. The single solitary wave solutions for values of  $A = 0.25, 0.50, 0.75$  and  $1.0$  at time  $t_{final} = 20$ .

4.1. **Problem I: Motion of a single solitary wave.** The solitary wave solution of the Eq.(1.2) is given by

$$u(x, t) = A \operatorname{sech}[k(x - x_0 - vt)]$$

where  $k = \sqrt{1/\mu}$ ,  $A = \sqrt{6v/\varepsilon}$ . This solution corresponds to the motion of a single solitary wave of magnitude  $A$ , initially centered at the position  $x_0$  and propagating to the right side with a constant velocity  $v$ . The solitary wave type solution (1.3) of Eq. (1.2) is not only on an unbounded region, but also at the same time it has a solitary wave solution on the closed interval  $a \leq x \leq b$ . The three invariants  $I_1, I_2$  and  $I_3$  satisfied by the MEW (1.2) equation are computed as follows by taking  $U_j$  and  $U'_j$  as the mesh values calculated from the numerical solution

$$I_1 = h \sum_{j=0}^N U_j, \quad I_2 = h \sum_{j=0}^N [U_j^2 + \mu(U'_j)^2], \quad I_3 = h \sum_{j=0}^N U_j^4.$$

For this problem the analytical values of the invariants are [39]

$$I_1 = \frac{A\pi}{k}, \quad I_2 = \frac{2A^2}{k} + \frac{2\mu k A^2}{3}, \quad I_3 = \frac{4A^4}{3k}.$$

In order to be able to make a comparison with some of the studies in the literature, the parameters as  $t = 20$ ,  $\mu = 1$ ,  $x_0 = 30$ ,  $A = 0.25$  and  $\Delta t = 0.05$  are used. In Figure 1, the movement of solitary wave has been given for various



TABLE 1. A comparison of the error norms  $L_2$  and  $L_\infty$  of Problem I for  $h = 0.1$ ,  $\Delta t = 0.2$ ,  $A = 0.25$ ,  $x_0 = 30$  on  $0 \leq x \leq 80$  at times  $t_{final} = 5, 10, 15, 20$ .

Method	$t_{final}$	$L_2 \times 10^4$	$L_\infty \times 10^4$	$I_1$	$I_2$	$I_3$
Present	5	1.276445	0.681926	0.7850300	0.1666259	0.0052058
	10	1.319189	0.748057	0.7850300	0.1666259	0.0052058
	15	1.385979	0.828043	0.7850300	0.1666259	0.0052058
	20	1.471099	0.897036	0.7850300	0.1666259	0.0052058
Ref. [9]	5	0.682986	0.610149	0.7853976	0.1664731	0.0052083
	10	1.362867	1.255591	0.7853984	0.1664732	0.0052083
	15	2.036756	1.916829	0.7853976	0.1664733	0.0052083
	20	2.701647	2.576377	0.7853977	0.1664736	0.0052083
Ref. [12]	5	0.473145	0.418872	0.7853712	0.1666095	0.0052078
	10	0.990390	0.840128	0.7853424	0.1665994	0.0052072
	15	1.499677	1.212955	0.7853751	0.1665922	0.0052067
	20	2.021476	1.569539	0.7852864	0.1665818	0.0052061
Ref. [2]	5	0.011570	0.006221	0.7854014	0.1666672	0.0052084
	10	0.010404	0.005784	0.7854008	0.1666668	0.0052084
	15	0.022265	0.014353	0.7854038	0.1666680	0.0052084
	20	0.011493	0.007664	0.7854013	0.1666670	0.0052084

TABLE 2. A comparison of the error norms  $L_2$ ,  $L_\infty$  and the invariants  $I_1, I_2, I_3$  of Problem I with some of the previous ones for  $h = 0.1$ ,  $\Delta t = 0.05$ ,  $A = 0.25$ ,  $x_0 = 30$  on  $0 \leq x \leq 80$  at time  $t_{final} = 20$ .

Method	$L_2 \times 10^3$	$L_\infty \times 10^3$	$I_1$	$I_2$	$I_3$
Present	0.146806	0.089667	0.7850300	0.1666259	0.0052058
Ref. [9]	0.269281	0.256997	0.7853977	0.1664735	0.0052083
Ref. [10]	0.079694	0.046552	0.7853898	0.1667614	0.0052082
Ref. [5]	0.175081	0.176288	0.7853982	0.1666666	0.0052083
Ref. [11]	0.005208	0.005456	0.7853965	0.1666638	0.0052081
Ref. [16]	0.051873	0.032113	0.7853967	0.1666664	0.0052083
Ref. [23]	0.051774	0.032114	0.7853967	0.1666663	0.0052083
Ref. [17]	0.080146	0.046121	0.7853967	0.1666663	0.0052083
Ref. [24]	0.080098	0.046061	0.7853967	0.1666663	0.0052083
Ref. [25]1	0.175277	0.176465	0.7853966	0.1666662	0.0052083
Ref. [25]2	0.175270	0.176459	0.7853966	0.1666662	0.0052083
Ref. [12]	0.290516	0.249892	0.7849545	0.1664765	0.0051995
Ref. [2]	0.001653	0.001194	0.7853979	0.1666671	0.0052084

values of amplitudes  $A = 0.25, 0.5, 0.75$  and  $1$ . From the figure, it is seen that the larger wave with a large amplitude has traveled a long way because of its faster velocity.

In Table 1,  $h = 0.1$ ,  $\Delta t = 0.2$ ,  $A = 0.25$ ,  $x_0 = 30$  are taken over the region  $0 \leq x \leq 80$  at times  $t_{final} = 5, 10, 15$ , and  $20$ . The newly obtained results are compared with some of those available in the literature. From the table it is clearly seen that the present results are better or in good agreement with those given in compared references.

In Table 2,  $h = 0.1$ ,  $\Delta t = 0.05$ ,  $A = 0.25$ ,  $x_0 = 30$  are taken over the region  $0 \leq x \leq 80$  at times  $t_{final} = 20$ . The newly obtained results are compared with some of those available in the literature. One can see that the newly obtained results are in good agreement with those given in references.

In Table 3, a comparison of the error norms  $L_2$ ,  $L_\infty$  and invariants  $I_1, I_2, I_3$  of Problem I with those in Ref. [9] for  $h = \Delta t = 0.01$ ,  $A = 0.25, 0.50, 0.75, 1.0$  and  $x_0 = 30$  on  $0 \leq x \leq 80$  at time  $t_{final} = 20$ . From the table one can see that the present error norms are better than those compared ones.



TABLE 3. A comparison of the error norms  $L_2$ ,  $L_\infty$  and the invariants  $I_1, I_2, I_3$  of Problem I with Ref. [9] for  $h = \Delta t = 0.01$ ,  $A = 0.25, 0.50, 0.75, 1.0$  and  $x_0 = 30$  on  $0 \leq x \leq 80$  at time  $t_{final} = 20$ .

$A$		$L_2 \times 10^3$	$L_\infty \times 10^3$	$I_1$	$I_2$	$I_3$
0.25	Present	0.0014686	0.0009014	0.7853945	0.1666663	0.0052083
	Ref. [9]	0.0026985	0.0026867	0.7853963	0.1666644	0.0052083
	Exact			0.7853982	0.1666667	0.0052083
0.50	Present	0.0057187	0.0038677	1.5707889	0.6666650	0.0833329
	Ref. [9]	0.0186465	0.0150972	1.5707920	0.6666588	0.0833333
	Exact			1.5707963	0.6666667	0.0833333
0.75	Present	0.0229900	0.0149503	2.3561834	1.4999963	0.4218729
	Ref. [9]	0.0519345	0.0366739	2.3561860	1.4999790	0.4218745
	Exact			2.3561945	1.5000000	0.4218750
0.10	Present	0.1010366	0.0626081	3.1415779	2.6666660	1.3333267
	Ref. [9]	0.1494558	0.0987068	3.1415790	2.6666350	1.3333310
	Exact			3.1415927	2.6666667	1.3333333

TABLE 4. A comparison of Problem II with those from Refs. [9], [12] with  $h = 0.1$ ,  $\Delta t = 0.2$  on  $0 \leq x \leq 80$ .

t	Present method						[9]			[12]		
	$I_1$	$I_2$	$I_3$	$I_1$	$I_2$	$I_3$	$I_1$	$I_2$	$I_3$	$I_1$	$I_2$	$I_3$
0	4.712388	3.333336	1.416669	4.712388	3.329462	1.416669	4.712388	3.332357	1.416670			
10	4.710180	3.331961	1.415419	4.712389	3.328927	1.416103	4.712022	3.324678	1.400768			
20	4.710180	3.331341	1.414833	4.712387	3.328361	1.415523	4.711697	3.324210	1.401182			
30	4.710181	3.329523	1.413184	4.712388	3.327818	1.413882	4.711242	3.346583	1.424847			
40	4.710181	3.329690	1.413358	4.712385	3.327112	1.414050	4.711017	3.321250	1.398239			
50	4.710180	3.330105	1.413629	4.712388	3.326632	1.414330	4.710804	3.320956	1.398729			
55	4.710180	3.329860	1.413359	4.712386	3.326393	1.414062	4.710630	3.323628	1.399068			
60	4.710180	3.329600	1.413079	4.712388	3.326228	1.413785						
70	4.710180	3.329056	1.412516	4.712388	3.325891	1.413228						
80	4.710180	3.328490	1.411954	4.712389	3.325434	1.412671						

4.2. **Problem II: Interaction of two solitary waves.** As a second test problem, Eq. (1.2) together with boundary conditions  $U \rightarrow 0$  as  $x \rightarrow \pm\infty$  and the initial condition for all linearization techniques is considered as

$$U(x, 0) = \sum_{j=1}^2 A_j \operatorname{sech}(k[x - x_j])$$

where  $k = \sqrt{1/\mu}$ . In order that the collision occurs, the solution domain is taken as  $0 \leq x \leq 80$  for values of  $h = 0.1$ ,  $\Delta t = 0.2$ ,  $\mu = 1$ ,  $A_1 = 1$ ,  $A_2 = 0.5$ ,  $x_1 = 15$ ,  $x_2 = 30$ . It is seen from Fig. 2 that the larger wave leaves the smaller one its behind. In addition, there was no elastic collision because the waves after the collision left small tail waves behind them. Because of this fact, these two solitary waves are not considered as solitons [39]. Moreover, for values of  $A_1 = 1$ ,  $A_2 = 0.5$ ,  $\Delta t = 0.2$ , a comparison has been made with those given in Refs. [9] and [12].

In Table 4, a comparison of the invariants  $I_1, I_2, I_3$  of Problem II with Refs. [9] and [12] is made for  $h = 0.1$ ,  $\Delta t = 0.2$  on  $0 \leq x \leq 80$  at various times. It is clearly seen that the invariants are well preserved after the initial time until the end of run-time.



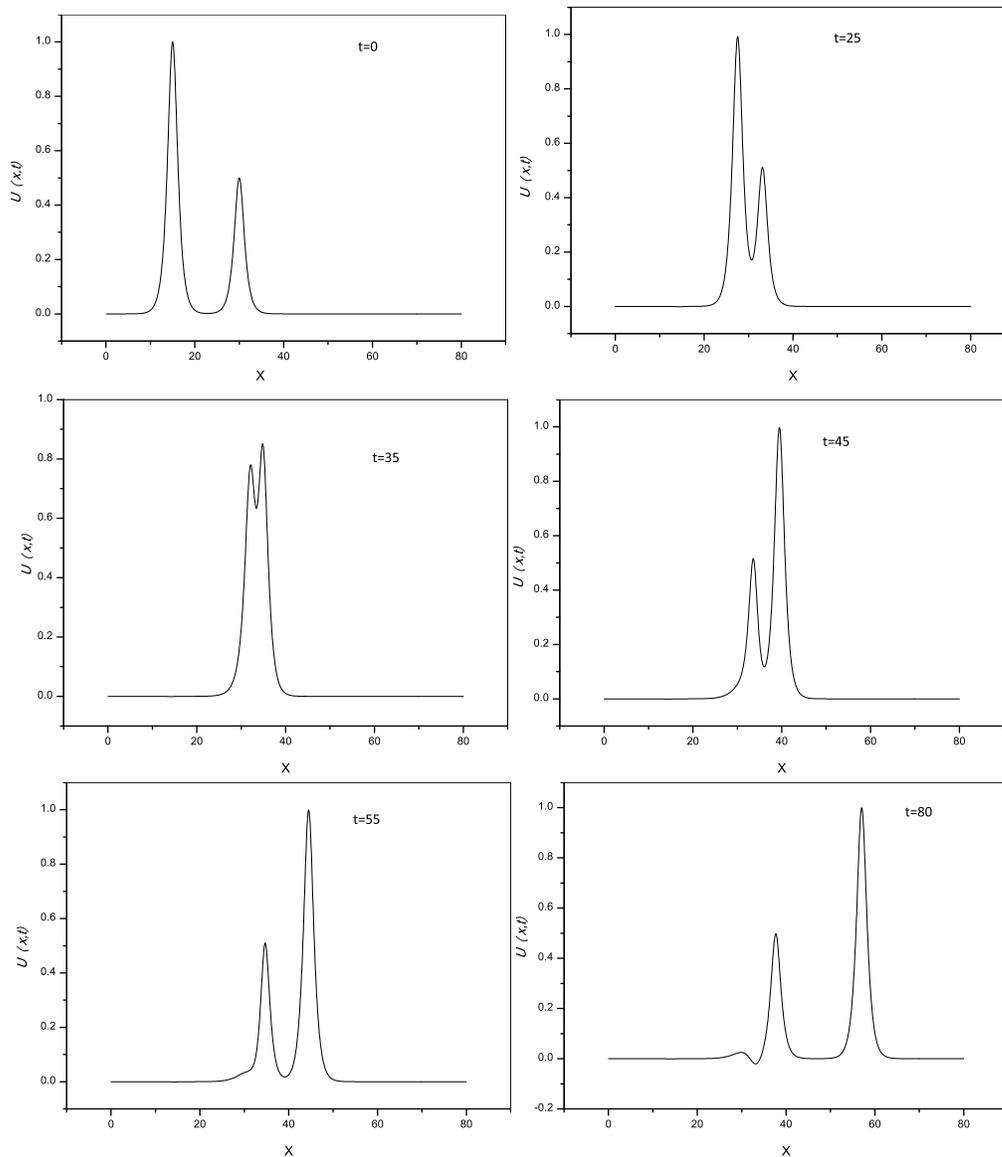


FIGURE 2. The interaction of two solitary waves at times  $t = 0, 25, 35, 45, 55$  and  $80$ .

**4.3. Problem III: The Maxwellian initial condition.** As the last test problem, the initial Maxwellian pulse is considered with the initial condition in solitary waves given by

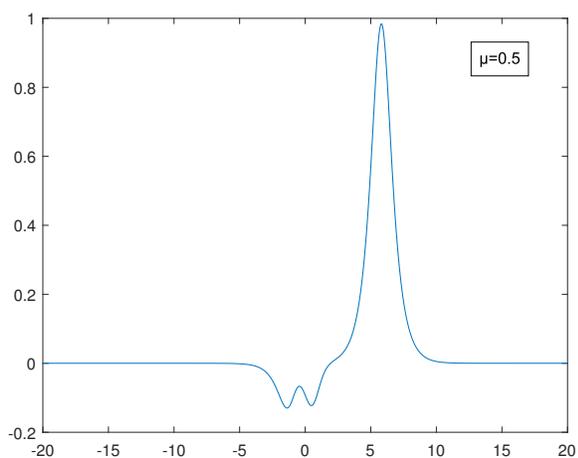
$$u(x, 0) = e^{-x^2} \quad (4.1)$$

with the boundary condition

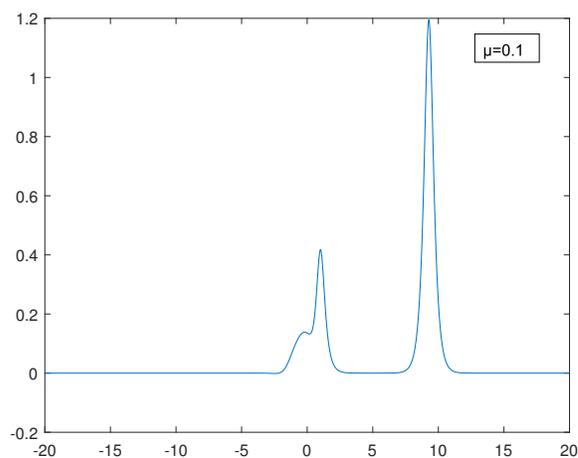
$$u(-20, 0) = u(20, 0) = 0, \quad t > 0.$$

Maxwellian initial condition Eq. (4.1) breaks up into a number of solitary waves depending on values of  $\mu$ . The calculations are carried out for values of  $\mu = 0.5, 0.1, 0.05, 0.02, 0.005, 0.0025$ ,  $h = 0.05$ ,  $\Delta t = 0.01$  and  $t = 12.5$ . Figure 3 shows Maxwellian initial condition for those parameters on  $-20 \leq x \leq 20$  at time  $t_{final} = 12.5$ .

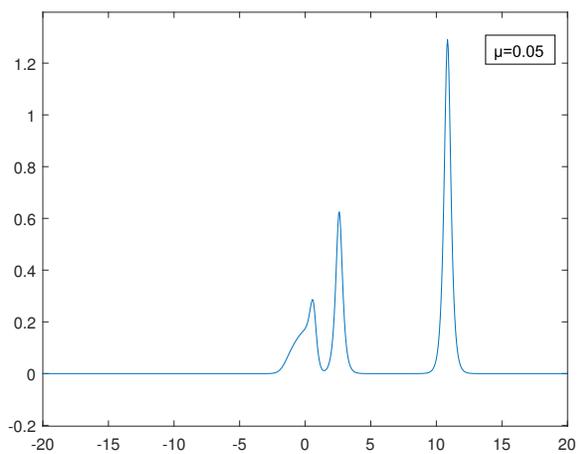




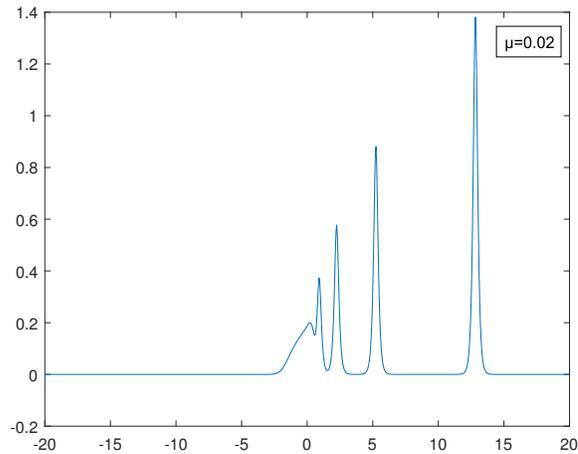
(A)



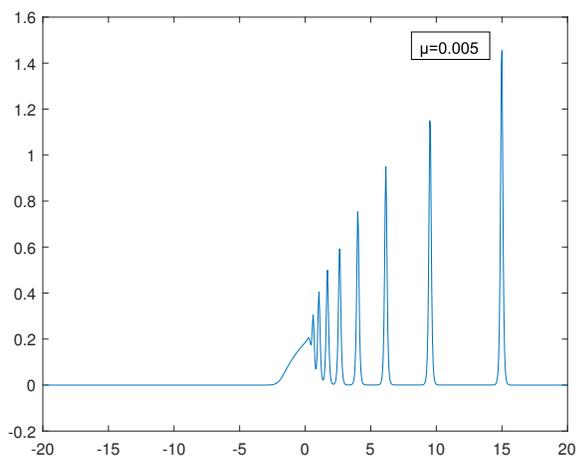
(B)



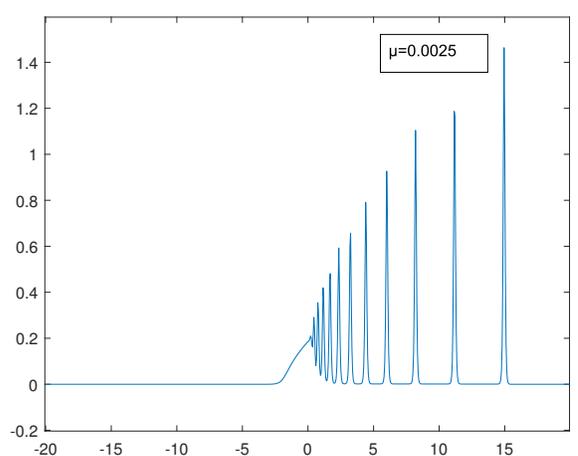
(C)



(D)



(E)



(F)



FIGURE 3. Maxwellian initial condition for  $h = 0.05$ ,  $\Delta t = 0.01$ ,  $\mu = 0.5, 0.1, 0.05, 0.02, 0.005, 0.0025$  on  $-20 \leq x \leq 20$  at time  $t_{final} = 12.5$ .

TABLE 5. The invariants  $I_1, I_2, I_3$  of Problem III for various values of  $\mu$  at time  $t = 12.5$ .

	$I_1$	$I_2$	$I_3$	$I_1$	$I_2$	$I_3$
	$\mu = 0.5$			$\mu = 0.1$		
Present	1.77235	1.87971	0.88597	1.77244	1.37783	0.88619
Ref. [5]	1.77245	1.88008	0.88623	1.77249	1.37774	0.88627
	$\mu = 0.05$			$\mu = 0.02$		
Present	1.77246	1.31444	0.88644	1.77256	1.27424	0.88660
Ref. [5]	1.77254	1.31431	0.88639	1.77275	1.27458	0.88717
	$\mu = 0.005$			$\mu = 0.0025$		
Present	1.77311	1.23603	0.86783	1.76963	1.19626	0.81240
Ref. [5]	1.77465	1.25032	0.89902	1.77868	1.24930	0.92893

In Table 5, a comparison of the invariants  $I_1, I_2, I_3$  of Problem III with Refs. [5] for various values of  $\mu = 0.5, 0.1, 0.05, 0.02, 0.005, 0.0025$  and  $h = 0.05, \Delta t = 0.01$  and  $t = 12.5$  is presented.

## 5. CONCLUSION

In this paper, numerical solutions of the MEW equation based on the trigonometric cubic B-spline finite element has been presented. Three test problems are worked out to examine the performance of the algorithms. The performance and accuracy of the method are shown by calculating the error norms  $L_2$  and  $L_\infty$ . For each linearization technique, the error norms are sufficiently small and the invariants are satisfactorily constant in all computer runs. The computed results show that the present method is a remarkably successful numerical technique for solving the MEW equation and is advisable for getting numerical solutions of other types of non-linear equations.

## ACKNOWLEDGMENT

The authors declare that there is no conflict of interests regarding the publication of this article.

## REFERENCES

- [1] K. K. Ali, R. Yilmazer, and H. Bulut, *Analytical Solutions to the Coupled Boussinesq–Burgers Equations via Sine-Gordon Expansion Method*, Adv. Intell. Syst. Comput., Springer, Cham, 1111 (2020), 233-240.
- [2] A. Başhan, N. M. Yağmurlu, Y. Uçar, and A. Esen, *Finite difference method combined with differential quadrature method for numerical computation of the modified equal width wave equation*, Numer. Methods Partial Differ. Equations, 37 (2009), 690-706.  
DOI: 10.1002/num.22547.
- [3] A. Başhan, N. M. Yağmurlu, Y. Uçar, and A. Esen, *A new perspective for the numerical solution of the Modified Equal Width wave equation*, Math. Methods Appl Sci.,  
DOI: 10.1002/mma.7322
- [4] R. J. Cheng and K. M. Liew, *Analyzing modified equal width (MEW) wave equation using the improved element-free Galerkin method*, Eng. Anal. Boundary Elem., 36 (2012), 1322–1330. DOI: 10.1016/j.enganabound.2012.03.013
- [5] I. Çelikkaya, *Operator splitting method for numerical solution of modified equal width equation*, Tbilisi Math. J., 12 (2019), 51–67.
- [6] I. T. Daba and G. F. Dureessa, *A Robust computational method for singularly perturbed delay parabolic convection-diffusion equations arising in the modeling of neuronal variability*, Comput. Methods Differ. Equations, 10(2) 475-488.  
DOI:10.22034/cmde.2021.44306.1873
- [7] İ. Dağ and B. Saka, *A cubic B-spline collocation method for the EW equation*, Math. Comput. Appl., 9(3) (2004), 381-392.  
DOI: 10.3390/mca9030381



- [8] M. Dehghan and M. Lakestani, *The Use of Cubic B-Spline Scaling Functions for Solving the One-dimensional Hyperbolic Equation with a Nonlocal Conservation Condition*, Wiley InterScience, 2006.  
DOI 10.1002/num.20209
- [9] A. Esen and S. Kutluay, *Solitary wave solutions of the modified equal width wave equation*, Commun. Non linear Sci. Numer. Simul. *13*(3) (2008), 1538–1546.  
DOI: 10.1016/j.cnsns.2006.09.018
- [10] A. Esen, *A lumped Galerkin method for the numerical solution of the modified equal-width wave equation using quadratic B-splines*, Int. J. Comput. Math., *83*(5-6) (2006), 449–459.  
DOI: 10.1080/00207160600909918
- [11] Y. M. A. Essa, *Multigrid method for the numerical solution of the modified equal width wave equation*, Appl. Math., *7* (2016), 1140–1147.  
DOI: 10.4236/am.2016.710102
- [12] D. J. Evans and K. R. Raslan, *Solitary waves for the generalized equal width (GEW) equation*, Int. J. Comput. Math., *82*(4) (2005), 445–455.  
DOI: 10.1080/0020716042000272539
- [13] W. Gao, H. F. Ismael, H. Bulut, and H. M. Baskonus, *Instability modulation for the (2+1)-dimension paraxial wave equation and its new optical soliton solutions in Kerr media*, Physica Scripta, *95*(3) (2020).
- [14] L. R. T. Gardner and G. A. Gardner, *Solitary waves of the EWE equation*, J. Comput. Phys. *101* (1992), 218–223.
- [15] T. Geyikli and S. B. G. Karakoç, *Septic B-Spline Collocation Method for the Numerical Solution of the Modified Equal Width Wave Equation*, Appl. Math., *2* (2011), 739–749.
- [16] T. Geyikli and S. B. G. Karakoç, *Subdomain Finite Element Method with Quartic B Splines for the Modified Equal Width Wave Equation*, Zh. Vychisl. Mat. Mat. Fiz., *55*(3) (2015), 410–421.
- [17] T. Geyikli and S. B. G. Karakoç, *Petrov–Galerkin method with cubic B-splines for solving the MEW equation*, Bull. Belg. Math. Soc. Simon Stevin, *19* (2012), 215–227.
- [18] A. G. Kaplan and Y. Dereli, *Numerical solutions of the GEW equation using MLS collocation method*, Int. J. Mod. Phys. C, *28*(1) (2017).  
DOI: 10.1142/S0129183117500115.
- [19] S. Hamdi, W. H. Enright, W. E. Schiesserand, and J. J. Gottlieb, *Exact solutions of the generalized equal width wave equation*, In Proceedings of the International Conference on Computational Science and its Applications, (Springer-Verlag), (2003), 725–734.
- [20] H. F. Ismael, H. Bulut, and H. M. Baskonus, *Optical soliton solutions to the Fokas–Lenells equation via sine-Gordon expansion method and  $(m + (G'/G))$ -expansion method*, Pramana- J. Phys., (2020), 94:35.
- [21] L. Jin, *Analytical Approach to the Modified Equal Width Equation*, Int. J. Contemp. Math. Sciences, *4*(23) (2009), 1113–1119.
- [22] S. B. G. Karakoç and H. Zeybek, *A cubic B-spline Galerkin approach for the numerical simulation of the GEW equation*, Statistics, Optimization & Information Computing, *4*(1) (2016), 30–41.  
DOI: 10.19139/soic.v4i1.167
- [23] S. B. G. Karakoç and T. Geyikli, *A numerical solution of the MEW equation using sextic B-splines*, J. Adv. Res. Appl. Math., *5* (2013), 51–65.  
DOI: 10.5373/jaram.1542.091012
- [24] S. B. G. Karakoç and T. Geyikli, *Numerical solution of the modified equal width wave equation*, Int. J. Diff. Equations, (2012), 1–15.  
DOI: 10.1155/2012/587208
- [25] S. B. G. Karakoç, Y. Ucar and N. M. Yağmurlu, *Different Linearization Techniques for the Numerical Solution of the MEW Equation*, Selçuk J. Appl. Math., *13*(2) (2012), 43–62.
- [26] D. J. Korteweg and G. de Vries, *On the change form of long waves advancing in a rectangular canal, and on a new type of long stationary wave*, Philosophical Magazine and Journal of Science, *39* (1895), 422–443.
- [27] M. Lakestani and M. Dehghan, *Numerical Solution of Fokker–Planck Equation Using the Cubic B-Spline Scaling Functions*, Wiley InterScience, 2008.



- DOI 10.1002/num.20352
- [28] M. Lakestani, *Numerical Solutions of the KdV Equation Using B-Spline Functions*, Iran. J. Sci. Technol. Trans. Sci., *41* (2017), 409–417.  
DOI 10.1007/s40995-017-0260-7
- [29] J. Lu, *He's variational iteration method for the modified equal width equation*, Chaos, Solitons and Fractals, *39*(5) (2007), 2102–2109.  
DOI: 10.1016/j.chaos.2007.06.104
- [30] P. J. Morrison, J. D. Meiss, and J. R. Carey, *Scattering of Regularized-Long-Wave Solitary Waves*, Physica D: Nonlinear Phenomena *11* (1984), 324–336.  
DOI: 10.1016/0167-2789(84)90014-9
- [31] P. J. Olver, *Euler operators and conservation laws of the BBM equation*, Math. Proc. Cambridge Philos. Soc., *85* (1979), 143–159.
- [32] K. R. Raslan, *Collocation method using the cubic B-spline for the generalized equal width equation*, Int. J. Simul. Process Model., *2* (2006), 37–44.  
DOI: 10.1504/IJSPM.2006.009019
- [33] T. Roshan, *A Petrov-Galerkin method for solving the generalized equal width (GEW) equation*, J. Comput. Appl. Math., *235*(6) (2011), 1641–1652.  
DOI: 10.1016/j.cam.2010.09.006
- [34] S. G. Rubin and R. A. Graves, *A cubic spline approximation for problems in fluid mechanics*, National aeronautics and space administration, Technical Report, Washington, 1975.
- [35] B. Saka, *Algorithms for numerical solution of the modified equal width wave equation using collocation method*, Math. Comput. Modell., *45* (2007), 1096–1117.  
DOI: 10.1016/j.mcm.2006.09.012
- [36] M. A. Shallal, K. K. Ali, K. R. Raslan, H. Rezazadeh, and A. Bekir, *Exact solutions of the conformable fractional EW and MEW equations by a new generalized expansion method*, J. Ocean Eng. Sci., *5*(3) (2020), 223–229.
- [37] N. M. Yağmurlu and A. S. Karakaş, *Numerical solutions of the equal width equation by trigonometric cubic B-spline collocation method based on Rubin–Graves type linearization*, Numer Methods Partial Differential Eq., *3*(6) (2020), 1170–1183.
- [38] H. Zadvan and J. Rashisidina, *Development of non polynomial spline and New B-spline with application to solution of Klein-Gordon equation*, Comput. Meth. Differ. Equations, *8*(4) (2020), 794–814.  
DOI:10.22034/cmde.2020.27847.1377
- [39] S. I. Zaki, *Solitary wave interactions for the modified equal width equation*, Computer Physics Communations, *126* (2000), 219–231.  
DOI: 10.1016/S0010-4655(99)00471-3
- [40] S. I. Zaki, *A least-squares Finite element scheme for the EW equation*, Comput. Methods Appl. Mech. Engrg. *189* (2000), 587–594.  
DOI: 10.1016/S0045-7825(99)00312-6
- [41] H. Zeybek and S. B. G. Karakoç, *Application of The Collocation Method With B-Splines To The GEW Equation*, Electron. Trans. Numer. Anal., *46* (2017), 71–88.
- [42] A. M. Wazwaz, *The tanh and the sine-cosine methods for a reliable treatment of the modified equal width equation and its variants*, Commun. Nonlinear Sci. Numer. Simul., *11*(2) (2006), 148–160.  
DOI: 10.1016/j.cnsns.2004.07.001

