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A numerical scheme for solving time-fractional Bessel differential equations

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Abstract

The object of this paper devotes on offering an indirect scheme based on time-fractional Bernoulli functions in the sense of Rieman-Liouville fractional derivative which ends up to the high credit of the obtained approximate fractional Bessel solutions. In this paper, the operational matrices of fractional Rieman-Liouville integration for Bernoulli polynomials are introduced. Utilizing these operational matrices along with the properties of Bernoulli polynomials and the least squares method, the fractional Bessel differential equation converts into a nonlinear system of algebraic. To solve these nonlinear algebraic equations which are a prominent the problem, there is a need to employ Newton's iterative method. In order to elaborate the study, the synergy of the proposed method is investigated and then the accuracy and the efficiency of the method are clearly evaluated by presenting numerical results.

Keywords. Fractional-order differential equation, Caputo and Rieman-Liouville fractional derivative and integral, Convergence analysis, Bernoulli functions, Least square method.

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1. Introduction

It is not surprising that a growing number of very active researches have studied fractional calculus. Fractional differential equations (FDEs) have been established generalizations of ordinary differential equations to an arbitrary order. From this perspective, a history of the expansion of fractional differential operators is located in [24]. In general, there could be not any exact solution to some FDEs. The nature of these equations entails some problems which require the application of certain techniques. Thereupon, it is essential to develop some reliable and efficient schemes to solve such FDEs. Approximation solution of the FDEs has attracted considerable attention from many researchers. In recent decades, copious numerical approaches have been developed. These methods comprise Laplace transforms method[35], Chebyshev polynomials[25], fractional optimal control problems[3, 5, 29], delay FDEs of pantograph[12], FDEs via Geraghty type[2], Fourier transforms method[19], fractional differential transform method[4], finite difference method[23], eigenvector expansion[9], Adomian decomposition method[31], variational iteration method[32], homotopy perturbation method[1], piecewise constant orthogonal functions[20], homotopy analysis method[13], orthogonal polynomials method[7, 30], wavelets method[14, 39], a numerical approach for variable-order fractional functional integral equations[15], a robust algorithm for nonlinear variable-order fractional control systems[22], extended algorithms for variable-order fractional derivatives[27], a numerical approach for variable-order fractional unified chaotic systems with time-delay[43], cubic spline variable-order fractional differential equations with time delay[42], a computational approach for variable-order fractional integro-differential equations[26] and an integro quadratic spline approach for a class of variable-order fractional initial value problems[28]. As we know, Bessel functions (BFs) are widely used functions that often appear in mathematical physics.

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Certainly, by conducting deep historical research, one can notice BFs particularly emerged as Daniel Bernoulli was investigating the fluctuation of the hanging chain while Euler was studying the vibration of a circular membrane and Bessel was busy with the research on planetary motion. Although notable cases of what would later be determined as Bessel functions had been investigated by Euler, Lagrange, and Bernoulli. F. W. Bessel first employed Bessel functions to explain three body motions, along with the Bessel functions show up in the series expansion on planetary perturbation. Furthermore, copious applications of the BFs have been discovered in mathematics, physics, and engineering such as propagation of waves, elasticity, fluid motion, and in several problems of potential theory and diffusion relating to cylindrical symmetry in [17, 38, 41]. Fractional-order Bessel differential the equation is investigated also in [6]. An uncomplicated category of the fractional modified Bessel equation is utilized in modeling the geometry of the human eye's corned [9]. In [10, 33, 37], the authors investigated the problem of the FDEs with variable coefficient and the fractional Bessel homogeneous equations with regard to the power series.

The available sets of orthogonal functions can be divided into three classes. The first class contains sets of constant function pieces, for example, Block-pulse, Haar, Walsh, etc. The second class contains a set of orthogonal polynomials, for example, Chebyshev, Laguerre, Legendre, etc. The third class includes the set of sine and cosine functions in the Fourier series. Orthogonal functions are meant to be applied when it comes to solving the various problems in dynamic systems. The main advantage of using orthogonal functions is that it is to be used to break down the problems in the dynamic system into algebraic transaction systems (linear and nonlinear) using operational matrices, derivatives, and integrals. The Bernoulli polynomials and Taylor series are not based on orthogonal functions, but in some cases, the inner product in the multiplication property is zero, however, they have the operational matrix of integration.

In this article we will propose a numerical scheme to solve the inhomogeneous fractional-order Bessel differential equation of order p as

$$x^{2\nu} D^{\nu} D^{\nu} z(x) + \nu x^{\nu} D^{\nu} z(x) + \nu^2 (x^{2\nu} - p^2) z(x) = \sum_{m=0}^{\infty} a_m x^{m\nu}, \tag{1.1}$$

with boundary conditions(Dirichlet conditions)

$$I^{1-\nu} z(x)|_{x=0} = \lambda, \qquad I^{1-\nu} z(x)|_{x=1} = \omega, \tag{1.2}$$

where D^{ν} denoted the Rieman-Liouville derivative ,$\lambda, \omega,$ are constants,
$0 < \nu < 1, \quad 0 < x \leq 1, \quad p$ is a positive non integer number.
If $\nu = 1$, then the equation is the inhomogeneous Bessel differential equation.
If $\sum_{m=0}^{\infty} a_m x^{m\nu} \equiv 0$, then the homogeneous fractional-order Bessel differential equation of order p achieved as

$$x^{2\nu} D^{\nu} D^{\nu} z(x) + \nu x^{\nu} D^{\nu} z(x) + \nu^2 (x^{2\nu} - p^2) z(x) = 0, \tag{1.3}$$

where $0 < \nu < 1$ and p is a positive non integer number.
If $\nu = 1$, therefore the equation is the homogeneous Bessel differential equation. Proposing analytical methods to solve these types of problems is almost impossible and must be solved numerically. To do this, there are copious methods. The proposed method with the help of using operational matrices has been recognized to illustrate high efficiency, accuracy, and straightforwardness to be applied. We will restrict our attention to attain numerically a solution to a time-fractional Bessel differential equation in the Rieman-Liouville fractional derivatives sense. Let us just go on with the knowledge that we already know, the power series are not appropriate for computation dealing with large values of arguments. All things considered, an approximation solution has been allotted to the fractional-order Bessel equation based on Bernoulli functions. In the current paper, a numerical scheme to promise a solution to the fractional Bessel differential equation is necessitated to be proposed. For this scope, operational matrices of time-fractional derivative and integration in the Rieman- Liouville sense are constructed. In this respect, all the known and unknown functions were approximated via the Bernoulli functions. It then, became crucial that electing the procedure of utilizing operational matrices of Bernoulli functions and the least square approximation scheme, the researcher decided to allot the system of nonlinear algebraic equations which needed to be solved with Newton's algorithm.

2. Fractional calculus

Let us go through some basic concept of the fractional calculus theory which promised to be applied later in the present article.

Definition 2.1. A real function $f(t), t > 0$ is said to be in space $C_\mu, \mu \in \mathbf{R}$ if there would be a real number $p > \mu$, in respect of $f(t) = t^p f_1(t)$ where
$f_1(t) \in C[0, \infty)$ and it is said to be in the space $C^n_\mu, \quad n \in N \bigcup \{0\}$ if $f^{(n)} \in C_\mu$, [8].

Definition 2.2. The Rieman-Liouville fractional integral operator of time $\nu \geq 0, f \in {}_\mu, \mu \geq -1$ is determined as [8, 34]:

$$I^\nu_a f(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-s)^{\nu-1} f(s) ds, \qquad I^0_a f(t) = f(t),$$

where $n - 1 < a \leq n, n \in N$ and $a \in R$.
The features of the operator I^ν which are required in this article, for $f, g \in C_\mu$, $\nu, \nu_1, \nu_2 \geq 0$, $\beta \geq -1$, $\mu \geq -1$ and constants λ_1, λ_2 as:

i: $I^{\nu_1} I^{\nu_2} f(t) = I^{\nu_1+\nu_2} f(t)$,
ii: $I^\nu(\lambda_1 f(t) + \lambda_2 g(t)) = \lambda_1 I^\nu f(t) + \lambda_2 I^\nu g(t)$,
iii: $I^\nu(t^\beta) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\nu+1)} t^{\beta+\nu}$.

Definition 2.3. The Rieman-Liouville time-fractional derivative operator of order $m - 1 < \nu \leq m$ is defined as [16]:

$$D^\nu z(t) = \frac{1}{\Gamma(m-\nu)} \frac{d^m}{dt^m} \int_0^t \frac{z(s)}{(t-s)^{\nu-m+1}} ds, \qquad t > 0.$$

The properties of the Rieman-liouville derivative are as following:

i: $I^\nu D^\nu z(t) = z(t) - \frac{t^{\nu-1}}{\Gamma(\nu)} I^{1-\nu} z(t)|_{t=0}, \qquad 0 < \nu < 1,$
ii: $D^\nu I^\nu z(t) = z(t)$,
iii: $D^\nu(t^\beta) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\nu+1)} t^{\beta-\nu}$.

Definition 2.4. (Generalized Taylor's formula) [14]
Let $D^{j\nu} f(t) \in C(0, 1]$, for $0 < \nu \leq 1$, $j = 0, 1, ..., m$. Thereupon, we get

$$f(t) = \sum_{j=0}^{m-1} \frac{t^{j\nu}}{\Gamma(j\nu+1)} D^{j\nu} f(0^+) + \frac{t^{m\nu}}{\Gamma(m\nu+1)} D^{m\nu} f(\eta),$$

with, $0 < \eta \leq t$, $\forall t \in (0, 1]$. Obviously having the following

$$|f(t) - \sum_{j=0}^{m-1} \frac{t^{j\nu}}{\Gamma(j\nu+1)} D^{j\nu} f(0^+)| < M_\nu \frac{t^{m\nu}}{\Gamma(m\nu+1)},$$

where $M_\nu \geq sup_{\eta \in (0,1]} |D^{m\nu} f(\eta)|$.

In sense of $\nu = 1$, we have Taylor's formula.

3. main results

The usefulness of the derivative operational matrix and integral operational matrix of the fractional order has been allotted essential in the presented method. Due to such an important role, it is imperative that we first recall the time-fractional Bernoulli functions (FBFs) and their properties. Of course, then, we obtain their operational matrices of fractional integration and derivative in the Rieman-Liouville sense.

3.1. **The FBFs and their properties.** As it was discussed earlier, the BFs have been proved useful in copious fields of mathematics such as number theory and the theory of finite differences. The classical Bernoulli polynomials of time m are determined on interval [0,1] as [40]:

$$\beta_m(t) = \sum_{i=0}^{m} \binom{m}{i} \beta_{m-i} t^i, \qquad 0 \leq x \leq 1,$$

where $\beta_i := \beta_i(0)$, $i = 0, 1, ..., m$, are Bernoulli numbers. Thereupon, the first Bernoulli polynomials recognized as

$$\beta_0(t) = 1, \quad \beta_1(t) = t - \frac{1}{2}, \quad \beta_2(t) = t^2 - t + \frac{1}{6}, \quad \beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t.$$

Also, the BFs attain the subsequent form [11]:

$$\int_0^1 \beta_n(t)\beta_m(t)dt = (-1)^{n-1}\frac{m!n!}{(m+n)!}\beta_{m+n}, \qquad m, n \geq 1. \tag{3.1}$$

The BFs establish a complete basis over the interval[0,1], [18].
The FBFs are presented in this subsection. Via employing the classical Bernoulli functions and the alternation of variable $t = x^\alpha \quad for \quad \alpha > 0$ (see [36]).
The (FBFs) $\beta_m(x^\alpha)$, are shortly denoted by $\beta_m^\alpha(x)$, is determined by

$$\beta_m^\alpha(t) = \sum_{i=0}^{m} \binom{m}{i} \beta_{m-i}^\alpha x^{i\alpha}, \qquad 0 \leq x \leq 1, \tag{3.2}$$

where $\beta_i := \beta_i(0)$, $i = 0, 1, ..., m$, are Bernoulli numbers .
By using equations (3.1) and (3.2) for the FBFs with the weight function
$w_\alpha(x) = x^{\alpha-1}$, we obtain

$$\int_0^1 \beta_n^\alpha(x)\beta_m^\alpha(x)w_\alpha(x)dx = \frac{1}{\alpha}(-1)^{n-1}\frac{m!n!}{(m+n)!}\beta_{m+n}, \qquad m, n \geq 1.$$

Also, the FBFs form a full basis over the interval [0,1], (see [18]).
Each function $z(x)$, defined over [0,1], may be developed regarding FBFs as below

$$z(x) \simeq z_m(x) = \sum_{k=0}^{m} c_k \beta_k^\alpha(x) = C^T \phi_\alpha(x),$$

where C and $\phi_\alpha(x)$ are $(m+1) \times 1$ vectors illustrated as

$$C = [c_0, c_1, ..., c_m]^T \qquad , \qquad \beta_\alpha(x) = [\beta_0^\alpha(x), ..., \beta_m^\alpha(x)]^T, \tag{3.3}$$

and

$$c_k = \int_0^1 y(x)\beta_k^\alpha(x)x^{\alpha-1}dx, \qquad k = 0, 1, ..., m.$$

3.2. **The FBFs operational matrix of the Fractional-order derivative.** The major purpose of the present subsection is recognized to extract the FBFs operator matrix of fractional derivative.
The Rieman-liouville fractional-order derivative of the vector $\beta^\alpha(x)$ given in equations (3.2) and (3.3), can be expressed by:

$$D^\nu \beta^\alpha(x) = F^{(\nu,\alpha)}\beta^\alpha(x), \tag{3.4}$$

where $F^{(\nu,\alpha)}$ is the operational matrix of fractional derivative of order ν in the Rieman-liouville sense with $(m+1) \times (m+1)$ dimension.

By employing equation (3.2) and features of operator D^{ν} for $i=0,1,...,m$, it is obtained

$$D^{\nu}\beta_i^{\alpha}(x)=D^{\nu}(\sum_{r=0}^{i}\binom{i}{r}\beta_{i-r}^{\alpha}x^{\alpha r})=\sum_{r=0}^{i}\binom{i}{r}\beta_{i-r}^{\alpha}D^{\nu}(x^{\alpha r})$$
$$=\sum_{r=0}^{i}\binom{i}{r}\beta_{i-r}^{\alpha}(\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1-\nu)})x^{\alpha r-\nu}=\sum_{r=0}^{i}b_{(i,r)}^{(\nu,\alpha)}x^{\alpha r-\nu}, \tag{3.5}$$

where

$$b_{(i,r)}^{(\nu,\alpha)}=(\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1-\nu)})\binom{i}{r}\beta_{i-r}^{\alpha},$$

can be expanded $x^{\alpha r-\nu}$ via the fractional-order Bernoulli polynomials as

$$x^{\alpha r-\nu}\simeq\sum_{j=0}^{m}\varphi_{(r,j)}^{(\nu,\alpha)}\beta_j^{\alpha}(x), \tag{3.6}$$

hence, by substituting the above equation into equation (3.5) for $i=0,1,...,m$, the following form is achieved

$$D^{\nu}\beta_i^{\alpha}(x)=\sum_{r=0}^{i}b_{(i,r)}^{(\nu,\alpha)}\sum_{j=0}^{m}\varphi_{(r,j)}^{(\nu,\alpha)}\beta_j^{\alpha}(x)=\sum_{j=0}^{m}(\sum_{r=0}^{i}\Theta_{(i,j,r)}^{(\nu,\alpha)})\beta_j^{\alpha}(x), \tag{3.7}$$

where

$$\Theta_{(i,j,r)}^{(\nu,\alpha)}=b_{(i,r)}^{(\nu,\alpha)}\varphi_{(r,j)}^{(\nu,\alpha)}.$$

Equation (3.7) can be rewritten as

$$D^{\nu}\beta_i^{\alpha}(x)=[\sum_{r=0}^{i}\Theta_{(i,0,r)}^{(\nu,\alpha)},\sum_{r=0}^{i}\Theta_{(i,1,r)}^{(\nu,\alpha)},...,\sum_{r=0}^{i}\Theta_{(i,m,r)}^{(\nu,\alpha)}]\beta^{\alpha}(x), \qquad i=0,...,m,$$

hence, $F^{(\nu,\alpha)}$ obtained as

$$F^{(\nu,\alpha)}=\begin{bmatrix}\Theta_{(0,0,0)}^{(\nu,\alpha)} & \Theta_{(0,1,0)}^{(\nu,\alpha)} & \cdots & \Theta_{(0,m,0)}^{(\nu,\alpha)}\\ \sum_{r=0}^{1}\Theta_{(1,0,r)}^{(\nu,\alpha)} & \sum_{r=0}^{1}\Theta_{(1,1,r)}^{(\nu,\alpha)} & \cdots & \sum_{r=0}^{1}\Theta_{(1,m,r)}^{(\nu,\alpha)}\\ \vdots & \vdots & \ddots & \vdots\\ \sum_{r=0}^{m-1}\Theta_{(m-1,0,r)}^{(\nu,\alpha)} & \sum_{r=0}^{m-1}\Theta_{(m-1,1,r)}^{(\nu,\alpha)} & \cdots & \sum_{r=0}^{m-1}\Theta_{(m-1,m,r)}^{(\nu,\alpha)}\\ \sum_{r=0}^{m}\Theta_{(m,0,r)}^{(\nu,\alpha)} & \sum_{r=0}^{m}\Theta_{(m,1,r)}^{(\nu,\alpha)} & \cdots & \sum_{r=0}^{m}\Theta_{(m,m,r)}^{(\nu,\alpha)}\end{bmatrix}.$$

3.3. **The FBFs operational matrix of the time-fractional integration.** The Rieman-Liouville fractional integration of the vector $\beta^{\alpha}(x)$ presented in equation (3.3) commented by:

$$I^{\nu}\beta^{\alpha}(x)=G^{(\nu,\alpha)}\beta^{\alpha}(x), \tag{3.8}$$

where $G^{(\nu,\alpha)}$ is the $(m+1)\times(m+1)$ fractional operational matrix of integration.
Due to the analytic form of $\beta^{\alpha}(x)$ in equation (3.2) and the features of operator I^{ν}, for $i=0,1,...,m$, subsequent form is provided:

$$I^{\nu}\beta_i^{\alpha}(x)=I^{\nu}(\sum_{r=0}^{i}\binom{i}{r}\beta_{i-r}^{\alpha}x^{\alpha r})=\sum_{r=0}^{i}\binom{i}{r}\beta_{i-r}^{\alpha}I^{\nu}(x^{\alpha r})$$
$$=\sum_{r=0}^{i}\binom{i}{r}\beta_{i-r}^{\alpha}(\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1-\nu)})x^{\alpha r+\nu}=\sum_{r=0}^{i}\delta_{(i,r)}^{(\nu,\alpha)}x^{\alpha r+\nu}, \tag{3.9}$$

where

$$\delta^{(\nu,\alpha)}_{(i,r)} = (\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1-\nu)})\binom{i}{r}\beta^{\alpha}_{i-r},$$

on the other hand, $x^{\alpha r+\nu}$ can be developed by the FBFs as

$$x^{\alpha r+\nu} \simeq \sum_{j=0}^{m} a^{(\nu,\alpha)}_{(r,j)}\beta^{\alpha}_{j}(x), \tag{3.10}$$

by putting the above equation into equation (3.9) for $i = 0, 1, ..., m$, it can be written

$$I^{\nu}\beta^{\alpha}_{i}(x) = \sum_{r=0}^{i}\delta^{(\nu,\alpha)}_{(i,r)}\sum_{j=0}^{m} a^{(\nu,\alpha)}_{(r,j)}\beta^{\alpha}_{j}(x) = \sum_{j=0}^{m}(\sum_{r=0}^{i}\Delta^{(\nu,\alpha)}_{(i,j,r)})\beta^{\alpha}_{j}(x), \tag{3.11}$$

where

$$\Delta^{(\nu,\alpha)}_{(i,j,r)} = \delta^{(\nu,\alpha)}_{(i,r)}a^{(\nu,\alpha)}_{(r,j)}.$$

Equation (3.11) can be rewritten as

$$I^{\nu}\beta^{\alpha}_{i}(x) = [\sum_{r=0}^{i}\Delta^{(\nu,\alpha)}_{(i,0,r)}, \sum_{r=0}^{i}\Delta^{(\nu,\alpha)}_{(i,1,r)}, ..., \sum_{r=0}^{i}\Delta^{(\nu,\alpha)}_{(i,m,r)}]\beta^{\alpha}(x), \qquad i = 0, ..., m,$$

hence, we have

$$G^{(\nu,\alpha)} = \begin{bmatrix} \Delta^{(\nu,\alpha)}_{(0,0,0)} & \Delta^{(\nu,\alpha)}_{(0,1,0)} & \cdots & \Delta^{(\nu,\alpha)}_{(0,m,0)} \\ \sum_{r=0}^{1}\Delta^{(\nu,\alpha)}_{(1,0,r)} & \sum_{r=0}^{1}\Delta^{(\nu,\alpha)}_{(1,1,r)} & \cdots & \sum_{r=0}^{1}\Delta^{(\nu,\alpha)}_{(1,m,r)} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{r=0}^{m-1}\Delta^{(\nu,\alpha)}_{(m-1,0,r)} & \sum_{r=0}^{m-1}\Delta^{(\nu,\alpha)}_{(m-1,1,r)} & \cdots & \sum_{r=0}^{m-1}\Delta^{(\nu,\alpha)}_{(m-1,m,r)} \\ \sum_{r=0}^{m}\Delta^{(\nu,\alpha)}_{(m,0,r)} & \sum_{r=0}^{m}\Delta^{(\nu,\alpha)}_{(m,1,r)} & \cdots & \sum_{r=0}^{m}\Delta^{(\nu,\alpha)}_{(m,m,r)} \end{bmatrix}.$$

4. Existence and uniqueness of the solution

We aim to investigate the existence and uniqueness of the solution for the fractional Bessel equation (1.1). In this article, $I_{\rho} = (-\rho, 0)\bigcup(0, \rho)$ for any $\rho > 0$ will be defined.

Theorem 4.1. *Let p be a positive non integer number, and suppose that ρ be a positive constant. Suppose that the radius of convergence of $\sum_{m=0}^{\infty} a_m x^{m\nu}$ is at least ρ and there exists a constant $\delta > 0$ satisfies the condition*

$$\mid \frac{a_{m+2}}{c_m} \mid \leq \frac{\Gamma[(m+2)\nu+1]\Gamma[(m+1)\nu+1]}{\delta^2\Gamma[(m+1)\nu+1]\Gamma(m\nu+1)}, \tag{4.1}$$

where

$$c_m = \begin{cases} \sum_{i=0}^{[\frac{m}{2}]} a_{2i}(-\nu^2)^{m-2i}\prod_{j=i}^{[\frac{m}{2}]}\frac{1}{\frac{\Gamma((2j)\nu+1)}{\Gamma(((2j)-2)\nu+1)}+\frac{\Gamma((2j)\nu+1)}{\Gamma(((2j)-1)\nu+1)}-\nu^2p^2} \\ \qquad\qquad (for \quad even \quad m), \\ \sum_{i=0}^{[\frac{m}{2}]} a_{2i+1}(-\nu^2)^{m-(2i+1)}\prod_{j=i}^{[\frac{m}{2}]}\frac{1}{\frac{\Gamma((2j+1)\nu+1)}{\Gamma(((2j+1)-2)\nu+1)}+\frac{\Gamma((2j+1)\nu+1)}{\Gamma(((2j+1)-1)\nu+1)}-\nu^2p^2} \\ \qquad\qquad (for \quad odd \quad m), \end{cases} \tag{4.2}$$

for all $m \in N_0$. Let $\rho_0 = min\{\rho, \delta\}$. Therefore each solution $z : I_{\rho_0} \to \mathcal{C}$ of the fractional Bessel'differential equation (1.1) can be commented by

$$z(x) = z_h(x) + \sum_{m=0}^{\infty} c_m x^{m\nu}, \tag{4.3}$$

for all $x \in I_{\rho_0}$, where $Z_h(x)$ is a solution of the equation (1.3).

Proof. Let $z : I_{\rho_0} \to \mathcal{C}$ is a function considered in the form (4.3) and $z_p(x) = z(x) - z_h(x) = \sum_{m=0}^{\infty} c_m x^{m\nu}$. Next, we evince that the function $z_p(x)$ is convenient for the inhomogeneous equation (1.1). From (4.2), we have:

$$x^{2\nu}D^{\nu}D^{\nu}z_p(x) + \nu x^{\nu}D^{\nu}z_p(x) + \nu^2(x^{2\nu} - p^2)z_p(x) = \sum_{m=2}^{\infty} \frac{\Gamma(m\nu+1)}{\Gamma((m-2)\nu+1)} c_m x^{m\nu}$$
$$+ \sum_{m=1}^{\infty} \nu \frac{\Gamma(m\nu+1)}{\Gamma((m-2)\nu+1)} c_m x^{m\nu} + \sum_{m=0}^{\infty} c_m \nu^2 x^{(m+2)\nu} - \sum_{m=0}^{\infty} p^2\nu^2 c_m x^{m\nu}$$
$$= \sum_{m=2}^{\infty} \frac{\Gamma(m\nu+1)}{\Gamma((m-2)\nu+1)} c_m x^{m\nu} + (\nu c_1 \Gamma(\nu+1)x^{\nu} + \sum_{m=2}^{\infty} \nu \frac{\Gamma(m\nu+1)}{\Gamma((m-1)\nu+1)} c_m x^{m\nu})$$
$$+ \sum_{m=2}^{\infty} \nu^2 c_{m-2} x^{m\nu} + (-\nu^2p^2c_0 - \nu^2p^2c_1x^{\nu} - \sum_{m=2}^{\infty} \nu^2 c_m x^{m\nu}) = a_0 + a_1x^{\nu} + \sum_{m=2}^{\infty} a_m x^{m\nu}, \tag{4.4}$$

since we obtain

$$c_0 = \frac{-1}{p^2\nu^2} a_0, \qquad c_1 = \frac{1}{\nu\Gamma(\nu+1) - p^2\nu^2} a_1,$$

and

$$(\frac{\Gamma(m\nu+1)}{\Gamma((m-2)\nu+1)} + \frac{\Gamma(m\nu+1)}{\Gamma((m-1)\nu+1)} - \nu^2p^2)c_m + \nu^2 c_{m-2} = a_m, \qquad form \geq 2,$$

from (4.1) and (4.2), we have:

$$\begin{aligned}
\lim_{m\to\infty} |\frac{c_{m+2}}{c_m}| &= \lim_{m\to\infty} \frac{1}{\frac{\Gamma((m+2)\nu+1)}{\Gamma(m\nu+1)} + \frac{\Gamma((m+2)\nu+1)}{\Gamma((m+1)\nu+1)} - \nu^2p^2} |\frac{a_{m+2}}{c_m} - \nu^2| \\
&\leq \lim_{m\to\infty} \frac{1}{\frac{\Gamma[(m+2)\nu+1]\Gamma[(m+1)\nu+1] + \Gamma[(m+2)\nu+1]\Gamma[m\nu+1] - \nu^2p^2\Gamma[m\nu+1]\Gamma[(m+1)\nu+1]}{\Gamma[m\nu+1]\Gamma[(m+1)\nu+1]}} |\frac{a_{m+2}}{c_m} - \nu^2| \\
&\leq \lim_{m\to\infty} \frac{\Gamma[m\nu+1]\Gamma[(m+1)\nu+1]}{\begin{pmatrix} \Gamma[(m+2)\nu+1]\Gamma[(m+1)\nu+1] \\ +\Gamma[(m+2)\nu+1]\Gamma[m\nu+1] - \nu^2p^2\Gamma[m\nu+1]\Gamma[(m+1)\nu+1] \end{pmatrix}} \\
&\times \frac{\Gamma[(m+2)\nu+1]\Gamma[(m+1)\nu+1]}{\delta^2\Gamma[(m+1)\nu+1]\Gamma(m\nu+1)} = \frac{1}{\delta^2}.
\end{aligned} \tag{4.5}$$

We learnt from the above that the power series for $z_p(x)$ converges for all $x \in I_{\rho_0}$, which evinces that $z_p(x)$ is a private answer of the heterogeneous equation (1.1).

On the other part, since each solution to (1.1) will be explained as a sum of a solution $z_h(x)$ of the homogeneous equation and a private answer $z_p(x)$ of the heterogeneous equation, each answer of (1.1) is sure of the form (4.3). ☐

5. Numerical method

In the present section, the fractional Bernoulli functions and its operational matrices are employed to obtain the approximate solution of the following Rieman-Liouville the time-fractional Bessel equation of order p .
Consider the fractional Bessel equation of order p

$$x^{2\nu}D^{\nu}D^{\nu}z(x) + \nu x^{\nu}D^{\nu}z(x) + \nu^2(x^{2\nu} - p^2)z(x) = \sum_{m=0}^{\infty} a_m x^{m\nu}, \tag{5.1}$$

with boundary conditions(Dirichlet conditions)

$$I^{1-\nu}z(x)|_{x=0} = \lambda, \qquad I^{1-\nu}z(x)|_{x=1} = \omega, \tag{5.2}$$

where D^{ν} denoted the Rieman-Liouville derivative and $0 < \nu < 1$, $0 < x \leq 1$, p is non integer number.
If $\nu = 1$, then the equation is classical Bessel equation.
Our method relies on the approximation of $D^{\nu} z(x)$ in manner of the fractional Bernoulli function as follows:

$$D^{\nu} z(x) \simeq D^{\nu} z_N(x) = \sum_{i=0}^{m} c_i \beta^{\alpha}(x) = C^T \beta^{\alpha}(x), \tag{5.3}$$

where, vector $C = [c_0, c_1, ..., c_m]^T$ is an unknown vector.
Applying the operational matrix of Rieman-Liouville integral (3.8), we have

$$\begin{aligned} z(x) \simeq z_N(x) &= \sum_{i=0}^{m} c_i I^{\nu} \beta_i^{\alpha}(x) + \sum_{j=1}^{n} \frac{x^{n-j}}{\Gamma(\nu - j + 1)} I^{(n-j)} z(x)|_{x=0} \\ &= C^T G^{(\nu,\alpha)} \beta^{\alpha}(x) + \frac{x^{\nu-1}}{\Gamma(\nu)} I^{1-\nu} z(x)|_{x=0} \qquad (5.4) \\ &= C^T G^{(\nu,\alpha)} \beta^{\alpha}(x) + \frac{\lambda}{\Gamma(\nu)} x^{1-\nu} \\ &\cong C^T G^{(\nu,\alpha)} \beta^{\alpha}(x) + d^T \beta^{\alpha}(x), \qquad (5.5) \end{aligned}$$

where $\frac{\lambda}{\Gamma(\nu)} x^{1-\nu}$ is estimated as $d^T \beta^{\alpha}(x)$. Using equations (3.4) and (5.3), and the feature of the Rieman-liouville derivative, we obtain

$$D^{\nu} D^{\nu} z(x) \simeq D^{\nu} D^{\nu} z_N(x) = \sum_{i=0}^{m} c_i D^{\nu} \beta_i^{\alpha}(x) \cong C^T F^{(\nu,\alpha)} \beta^{\alpha}(x). \tag{5.6}$$

Similarly, there can be expanded $x^{2\nu}$ and x^{ν} via the time-fractional Bernoulli functions as:

$$N_1(x) = x^{2\nu} \simeq \sum_{i=0}^{m} e_i \beta_i^{\alpha}(x) \simeq E_1^T \beta^{\alpha}(x), \tag{5.7}$$

and

$$N_2(x) = x^{\nu} \simeq \sum_{i=0}^{m} e_i^{'} \beta_i^{\alpha}(x) \simeq E_2^T \beta^{\alpha}(x), \tag{5.8}$$

where, $\qquad E_1 = [e_0, ..., e_m]^T, \qquad and \qquad E_2 = [e_0^{'}, ..., e_m^{'}]^T.$
Thereupon, substituting equations (5.3)-(5.8) in equation (5.1), we get

$$N_1(x) D^{\nu} D^{\nu} z_N(x) + \nu N_2(x) D^{\nu} z_N(x) + \nu^2 N_1(x) z_N(x) - \nu^2 p^2 z_N(x) - f(x) = 0. \tag{5.9}$$

By taking the least square approximate, we consider functional $W[c_0, c_1, ..., c_m]$ as follow

$$W[c_0, ..., c_m] = \int_0^1 N_1(x) D^{\nu} D^{\nu} z_N(x) + \nu N_2(x) D^{\nu} z_N(x) + \nu^2 N_1(x) z_N(x) - \nu^2 p^2 z_N(x) - f(x))^2 dx. \tag{5.10}$$

Now, to minimize $W[c_0, c_1, ..., c_m]$, we obtain $c_0, c_1, ..., c_m$ by following:

$$\frac{\partial W}{\partial c_k} = 0, \qquad k = 0, 1, ..., m. \tag{5.11}$$

The above equations yield $m + 1$ nonlinear equations with $m + 1$ unknown factors, Which requires to be solved by employing suitable algorithm. By obtaining C, one can obtain the estimation value of $z(x)$ in equation (5.4).

6. On the convergence of the method

In this section, we argue on the convergence analysis of the mentioned scheme in previous section. To this aim, we first will obtain an error upper bound for the operational matrices of the fractional derivative and integration. Then it can be shown that by increasing the number of FBFs, the errors vanish. Now we need to go through subsequent theorem.

Theorem 6.1. *Let H is a Hilbert space and Z is a closed subspace of H such that $dimZ < \infty$ and $z_1, z_2, ..., z_n$ is any basis for Z. Suppose that x be an arbitrary element in H and z_0 be the unique best approximation to x out of Z. Thereupon*

$$\|x - z_0\|_2 = (\frac{G(x, z_1, z_2, ..., z_n)}{G(z_1, z_2, ..., z_n)})^{\frac{1}{2}},$$

where

$$G(x, z_1, z_2, ..., z_n) = \begin{bmatrix} < x, x > & < x, z_1 > & \dots & < x, z_n > \\ < z_1, x > & < z_1, z_1 > & \dots & < z_1, z_n > \\ \vdots & \vdots & \ddots & \vdots \\ < z_n, x > & < z_n, z_1 > & \dots & < z_n, z_n > \end{bmatrix}.$$

Proof. see [18]. □

Theorem 6.2. *If $E_I^{(\nu,\alpha)}$ and $E_D^{(\nu,\alpha)}$ are the error vector of the operational matrix $G^{(\nu,\alpha)}$ and $F^{(\nu,\alpha)}$ respectively, then*

$$E_I^{(\nu,\alpha)} = [e_{I_0}^{(\nu,\alpha)}, ..., e_{I_m}^{(\nu,\alpha)}]^T = G^{(\nu,\alpha)}\beta^\alpha - I^\nu\beta^\alpha,$$

and

$$E_D^{(\nu,\alpha)} = [e_{D_0}^{(\nu,\alpha)}, ..., e_{D_m}^{(\nu,\alpha)}]^T = F^{(\nu,\alpha)}\beta^\alpha - D^\nu\beta^\alpha.$$

Proof. By approximation $x^{\alpha r+\nu}$, we had

$$x^{\alpha r+\nu} \simeq \sum_{j=0}^{m} a_{(r,j)}^{(\nu,\alpha)}\beta_j^\alpha(x),$$

where we can obtain $a_{(r,j)}^{(\nu,\alpha)}$ by taking the best approximation. Therefore, from the above theorem (6.1), we have

$$\|x^{\alpha r+\nu} - \sum_{j=0}^{m} a_{(r,j)}^{(\nu,\alpha)}\beta_j^\alpha(x)\|_2^2 = \frac{G(x^{\alpha r+\nu}, \beta_0^\alpha, ..., \beta_m^\alpha)}{G(\beta_0^\alpha, ..., \beta_m^\alpha)}.$$

Hence, according to equations (3.8)-(3.11), we obtain

$$\begin{aligned}
\|e_{Ii}^{(\nu,\alpha)}\|_2 &= \|I^{(\nu,\alpha)}\beta_i^\alpha(x) - \sum_{j=0}^{m}(\sum_{r=0}^{i}\Delta_{(i,j,r)}^{(\nu,\alpha)})\beta_j^\alpha(x)\|_2 \\
&\leq \sum_{r=0}^{i}\binom{i}{r}(\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1+\nu)})\beta_{i-r}^\alpha\|x^{\alpha r+\nu} - \sum_{j=0}^{m} a_{(r,j)}^{(\nu,\alpha)}\beta_j^\alpha(x)\|_2 \\
&\leq \sum_{r=0}^{i}\binom{i}{r}(\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1+\nu)})\beta_{i-r}^\alpha(\frac{G(x^{\alpha r+\nu}, \beta_0^\alpha, ..., \beta_m^\alpha)}{G(\beta_0^\alpha, ..., \beta_m^\alpha)})^{\frac{1}{2}}, \quad 0 \leq i \leq m.
\end{aligned}$$

In the same way for $E_D^{(\nu,\alpha)}$, by using equation (3.6), we had

$$x^{\alpha r-\nu} \simeq \sum_{j=0}^{m} \varphi_{(r,j)}^{(\nu,\alpha)} \beta_j^{\alpha}(x),$$

where we can obtain $\varphi_{(r,j)}^{(\nu,\alpha)}$ by taking the best approximation. Hence according to the theorem (6.1), we have

$$\|x^{\alpha r-\nu} - \sum_{j=0}^{m} \varphi_{(r,j)}^{(\nu,\alpha)} \beta_j^{\alpha}(x)\|_2^2 = \frac{G(x^{\alpha r-\nu}, \beta_0^{\alpha}, ..., \beta_m^{\alpha})}{G(\beta_0^{\alpha}, ..., \beta_m^{\alpha})}.$$

By using equations (3.4)-(3.7), we have

$$\begin{aligned}
\|e_{Di}^{(\nu,\alpha)}\|_2 &= \|D^{(\nu,\alpha)}\beta_i^{\alpha}(x) - \sum_{j=0}^{m}(\sum_{r=0}^{i} \Theta_{(i,j,r)}^{(\nu,\alpha)})\beta_j^{\alpha}(x)\|_2 \\
&\leq \sum_{r=0}^{i} \binom{i}{r} (\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1-\nu)})\beta_{i-r}^{\alpha} \|x^{\alpha r-\nu} - \sum_{j=0}^{m} \varphi_{(r,j)}^{(\nu,\alpha)} \beta_j^{\alpha}(x)\|_2 \\
&\leq \sum_{r=0}^{i} \binom{i}{r} (\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1-\nu)})\beta_{i-r}^{\alpha} (\frac{G(x^{\alpha r-\nu}, \beta_0^{\alpha}, ..., \beta_m^{\alpha})}{G(\beta_0^{\alpha}, ..., \beta_m^{\alpha})})^{\frac{1}{2}}, \quad 0 \leq i \leq m.
\end{aligned}$$

□

Theorem 6.3. *Assume* $D^{k\nu}z(x) \in C(0,1], \quad k = 0, 1, ..., m$ *and* $Z_m = span\{\beta_0(x), ..., \beta_m(x)\}$. *If* $z(x)$ *is approximated by* $q_m(x)$ *as*

$$z(x) \simeq q_m(x) = \sum_{i=0}^{m} c_i \beta i^{\alpha}(x) = C^T \beta^{\alpha}(x),$$

where $q_m(x)$ *is the best approximation out of* Z_m. *Consider*

$$S_m(z) = \int_0^1 [z(x) - q_m(x)]^2 dx.$$

Therefore, we have

$$\lim_{m\to\infty} S_m(z) = 0.$$

Proof. We define

$$z^*(x) = \sum_{j=0}^{m} \frac{x^{j\nu}}{\Gamma(j\nu+1)} D^{j\nu}(0^+).$$

From Definition (2.4), we get

$$|z(x) - z^*(x)| \leq M_\nu \frac{x^{(m+1)\nu}}{\Gamma((m+1)\nu+1)}$$

where $M_\nu \geq sup_{\eta\in(0,1]}|D^{(m+1)\nu}z(\eta)|$. Since $q_m(x)$ is the best approximation of $z(x)$ from Z_m, $z^*\in Z_m$ and from the above relation we obtain

$$\|z(x)-q_m(x)\|_2^2 \leq \|z(x)-z^*(x)\|_2^2 = \int_0^1 |z(x)-z^*(x)|^2 dx$$
$$\leq \int_0^1 M_\nu^2 \frac{x^{(2m+2)\nu}}{\Gamma((m+1)\nu+1)^2}dx = \frac{M_\nu^2}{\Gamma((m+1)\nu+1)^2}\int_0^1 x^{(2m+2)\nu}dx$$
$$= \frac{M_\nu^2}{\Gamma((m+1)\nu+1)^2((2m+2)\nu+1)}.$$

So, we obtain

$$\lim_{m\to\infty}\|z(x)-q_m(x)\|_2^2 = 0.$$

□

By considering the above theorems, we consider that as the number of FBFs increases, the errors tend to vanish. Now, the equation (5.10) can be written in the following form

$$W[z_N] = \int_0^1 (N_1(x)D^\nu D^\nu z_N(x) + \nu N_2(x)D^\nu z_N(x) + \nu^2 N_1(x)z_N(x) - \nu^2 p^2 z_N(x) - f(x))^2 dx. \tag{6.1}$$

Evidently the set of the FBFs provide a basis for Banach space of $C^1[0,1]$ with the uniform norm $\|f\| = \|f\|\infty + \|f'\|\infty$. Take it to the consideration that K_n be an n-dimensional subspace of $A = (C^1[0,1], \|.\|)$ established by $\{\beta_0, ..., \beta_n\}$. Therefore each element of K_n is in the form $\mu_0\beta_o + \mu_1\beta_1 + ... + \mu_n\beta_n$, where $\mu_0, ..., \mu_n$ are real constants and on each set K_n the functional W ends up to a function $W[\mu_0\beta_o + \mu_1\beta_1 + ... + \mu_n\beta_n]$ of variables $\mu_0, ..., \mu_n$. We choose $\mu_0, ..., \mu_n$ to lessen W , we indicate the minimum of W by η_n , and the element of K_n which prepare the least amount by y_n. Obviously

$$K_n \subset K_{n+1},$$

therefore, we obtain

$$\eta_m \geq \eta_{m+1}.$$

From the above argue, we can deduce as following .

Theorem 6.4. *Observe the functional W, therefore [21]*

$$\lim_{m\to\infty}\eta_n = \eta,$$

where

$$\eta = inf_{z\in A}W[z].$$

It is considered that η_m is the optimal value of the W on the set K_m. Also for each of $z(x)$ in K_m, we get

$$z(x) = z_0\beta_0^\alpha + ... + z_m\beta_m^\alpha = Z^T\beta^\alpha(x),$$

therefore we obtain

$$D^\nu z(x) = D^\nu(Z^T\beta^\alpha(x)) = Z^T D^\nu\beta^\alpha(x) = Z^T.F^{(\nu,\alpha)}.\beta^\alpha(x) + Z^T.E_D^{(\nu,\alpha)}. \tag{6.2}$$

By taking $Z^T F^{(\nu,\alpha)}$ as C^T, we obtain

$$z(x) = C^T.G^{(\nu,\alpha)}.\beta^\alpha(x) + C^T.E_I^{(\nu,\alpha)} + Z_0 I^\nu E_D^{(\nu,\alpha)} + d^T\beta^\alpha(x), \tag{6.3}$$

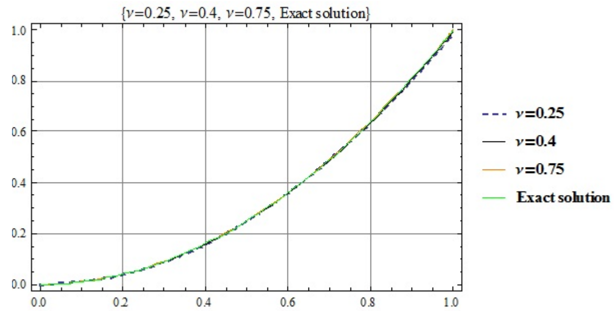

FIGURE 1. The comparison of $z(x)$ for $m = 5$, $\alpha = \nu = 0.25, 0.40, 0.75$, and the exact solution for Example7.1.

also, we have

$$D^{\nu}D^{\nu}z(x) = D^{\nu}(Z^T.F^{(\nu,\alpha)}.\beta^{\alpha} + Z^T.E_D^{(\nu,\alpha)}).$$

Assume $Z^T F^{(\nu,\alpha)} = C^T$, then we obtain

$$D^{\nu}D^{\nu}z(x) = C^T.F^{(\nu,\alpha)}.\beta^{\alpha}(x) + Z^T D^{\nu} E_D^{(\nu,\alpha)} + C^T.E_D^{(\nu,\alpha)}. \tag{6.4}$$

We obtained the approximated form of problem (5.1)on λ_m by taking the operational matrices as follow

$$\begin{aligned} W = \int_0^1 & [(d_1^T \beta^{\alpha}(x))(C^T.F^{(\nu,\alpha)}.\beta^{\alpha}(x) + Z^T D^{\nu} E_D^{(\nu,\alpha)} + C^T.E_D^{(\nu,\alpha)}) \\ & + \nu(d_2^T \beta^{\alpha}(x))(Z^T.F^{(\nu,\alpha)}.\beta^{\alpha}(x) + Z^T.E_D^{(\nu,\alpha)}) + \nu^2((d_1^T \beta^{\alpha}(x)) - p^2) \\ & \times (C^T.G^{(\nu,\alpha)}.\beta^{\alpha}(x) + C^T.E_I^{(\nu,\alpha)} + Z_0.I^{\nu}.E_D^{(\nu,\alpha)} + d^T \beta^{\alpha}(x))]^2 dx. \end{aligned}$$

Theorem(6.2) ensure that as $m \to \infty$, thereupon $E_D^{(\nu,\alpha)}$, $E_I^{(\nu,\alpha)}$, tend to vanish. So, if $z_m(x)$ is a solution to (5.1), thereupon we have $z_m(x) \to z(x)$ as $m \to \infty$. From theorem (6.4), we had

$$\lim_{m \to \infty} \eta_m = \eta,$$

where

$$\eta = inf_{z \in A} W[z].$$

TABLE 1. comparison of the approximate solutions for $\nu = \alpha = 0.25, 0.75$, and different of m, for problem 7.1.

	$\alpha = \nu = 0.25$		$\alpha = \nu = 0.75$	
x	$m = 5$	$m = 7$	$m = 5$	$m = 7$
0.1	0.0120820	0.0124998	0.0103574	0.00958479
0.3	0.0861657	0.0896002	0.0905577	0.09246760
0.5	0.247876	0.2453460	0.2510070	0.25232600
0.7	0.4896930	0.4866100	0.4916030	0.49334200
0.9	0.8027460	0.8070520	0.8122810	0.81368400

TABLE 2. comparison of the approximate solutions for $\nu = \alpha = 1/2$, and different of m, for problem 7.1.

x	exact solution	m=5	m=7
0.1	0.01	0.0100129	0.0101009
0.3	0.09	0.0900084	0.0905181
0.5	0.25	0.2500070	0.2501570
0.7	0.49	0.4900080	0.4901320
0.9	0.81	0.8100060	0.8105220

TABLE 3. The absolute errors with some m and different values of ν for problem 7.1.

	$m=5$		$m=7$	
ν	$\alpha=1$	$\alpha=\nu$	$\alpha=1$	$\alpha=\nu$
0.25	$1.28e^{-5}$	$2.01e^{-5}$	$5.53e^{-4}$	$1.13e^{-5}$
0.40	$6.05e^{-6}$	$7.08e^{-6}$	$1.91e^{-4}$	$3.58e^{-5}$
0.50	$1.52e^{-7}$	$7.37e^{-11}$	$3.81e^{-7}$	$2.72e^{-5}$
0.75	$5.15e^{-6}$	$1.90e^{-6}$	$1.13e^{-2}$	$7.58e^{-6}$

TABLE 4. The absolute errors for $\nu = 1/2$, with $m = 5$, and various of α for problem 7.1.

x	$\alpha=.25$	$\alpha=.50$	$\alpha=.75$	$\alpha=1$	$\alpha=1.5$
0.1	$2.04e^{-3}$	$1.28e^{-5}$	$2.31e^{-5}$	$5.86e^{-4}$	$6.04e^{-3}$
0.3	$2.96e^{-3}$	$8.38e^{-6}$	$7.12e^{-6}$	$3.54e^{-4}$	$4.21e^{-3}$
0.5	$1.20e^{-4}$	$7.40e^{-6}$	$1.74e^{-6}$	$3.64e^{-4}$	$3.47e^{-3}$
0.7	$3.73e^{-3}$	$8.08e^{-6}$	$1.84e^{-5}$	$3.68e^{-4}$	$3.52e^{-3}$
0.9	$1.01e^{-3}$	$6.27e^{-6}$	$1.35e^{-5}$	$1.92e^{-4}$	$2.92e^{-3}$

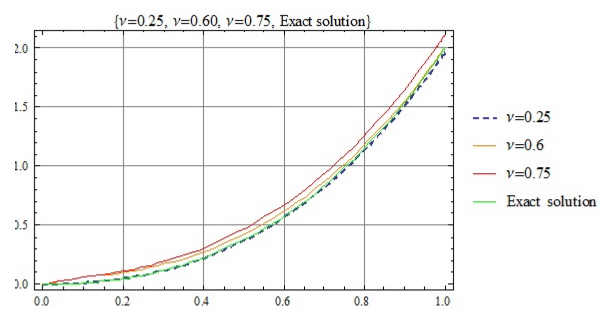

FIGURE 2. The comparison of $z(x)$ for $m = 7$,$\alpha = \nu = 0.25, 0.40, 0.60, 0.75$, and the exact solution for Example7.2.

7. Numerical examples

Evidently it is crucial to gather data in research, such data meant to provide an easier understanding of a proposed method. Three examples of the FBFs are chosen in terms of the study purpose with the assumptions that every one of them would prepare rich information of value to the study as well as the efficiency of the proposed FBFs approach in section 5 .

TABLE 5. comparison of the approximate solutions for $\nu = \alpha = 0.25, 0.40$, and various of m, for problem 7.2.

	$\alpha = \nu = 0.25$		$\alpha = \nu = 0.40$	
x	$m = 5$	$m = 7$	$m = 5$	$m = 7$
0.1	0.0265149	0.0121692	0.0109748	0.0261518
0.3	0.1009140	0.1159200	0.1169260	0.1296830
0.5	0.3620220	0.3680210	0.3725030	0.3838700
0.7	0.8349210	0.8197870	0.8350140	0.8440880
0.9	1.5092600	1.5118200	1.5443400	1.5525900

TABLE 6. comparison of the approximate solutions for $\nu = \alpha = 1/2$, and various of m, for problem 7.2.

x	exact solution	m=5	m=7
0.1	0.011	0.0106818	0.0181927
0.3	0.117	0.1171430	0.1249840
0.5	0.375	0.3743390	0.3815600
0.7	0.833	0.8323520	0.8387270
0.9	1.539	1.5380400	1.5453500

TABLE 7. The absolute errors with some m and different values of ν for problem 7.2.

	$m = 5$		$m = 7$	
ν	$\alpha = 1$	$\alpha = \nu$	$\alpha = 1$	$\alpha = \nu$
0.25	$2.22e^{-4}$	$4.50e^{-4}$	$2.93e^{-4}$	$2.17e^{-4}$
0.40	$8.60e^{-6}$	$7.51e^{-6}$	$7.96e^{-2}$	$1.73e^{-4}$
0.50	$1.25e^{-6}$	$5.32e^{-7}$	$2.99e^{-7}$	$4.75e^{-5}$
0.60	$5.93e^{-4}$	$6.60e^{-4}$	$1.95e^{-2}$	$1.68e^{-3}$
0.75	$3.19e^{-6}$	$6.80e^{-6}$	$1.13e^{-2}$	$7.62e^{-3}$

TABLE 8. The absolute for $\nu = 1/2$, with $m = 7$, and different of α for problem 7.2.

x	$\alpha = .25$	$\alpha = .50$	$\alpha = 1$	$\alpha = 2$
0.1	$8.45e^{-3}$	$7.19e^{-3}$	$8.21e^{-4}$	$5.51e^{-4}$
0.3	$6.17e^{-3}$	$7.98e^{-3}$	$6.20e^{-4}$	$4.91e^{-4}$
0.5	$6.24e^{-3}$	$6.56e^{-3}$	$3.84e^{-4}$	$8.00e^{-4}$
0.7	$5.41e^{-3}$	$5.72e^{-3}$	$5.43e^{-6}$	$2.43e^{-4}$
0.9	$8.19e^{-3}$	$6.35e^{-3}$	$5.99e^{-4}$	$9.64e^{-4}$

Example 7.1. Firstly, consider the fractional Bessel equation

$$x^{2\nu} D^{\nu} D^{\nu} z(x) + \nu x^{\nu} D^{\nu} z(x) + \nu^2 x^{2\nu} z(x) = (2 + \frac{4}{3\sqrt{\pi}})x^2 + \frac{1}{4}x^3, \tag{7.1}$$

with boundary condition

$$I^{1-\nu} z(x)|_{x=0,1} = 0.$$

For problem (7.1), the real answer is $z(x) = x^2$ for $\nu = 0.5$.
Figure 1 shows the numerical results of problem (7.1) for $m = 5$, $\alpha = \nu = 0.25, 0.40, 0.50, 0.75$ and the exact solution.

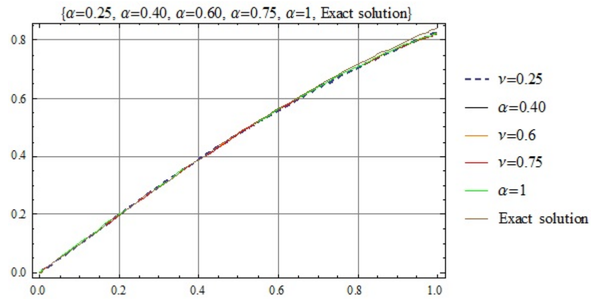

FIGURE 3. The comparison of $z(x)$ for $m = 5$, $\alpha = 0.25, 0.40, 0.60, 0.75, 1$, and the exact solution for Example 7.3.

TABLE 9. Comparison of The absolute errors for different values of α for problem 7.3.

	$m=5$			$m=7$		
x	$\alpha=0.25$	$\alpha=0.5$	$\alpha=1$	$\alpha=0.25$	$\alpha=0.5$	$\alpha=1$
0.1	$1.91e^{-3}$	$2.85e^{-5}$	$1.78e^{-3}$	$4.54e^{-3}$	$3.63e^{-4}$	$1.02e^{-3}$
0.3	$2.12e^{-3}$	$3.56e^{-4}$	$1.92e^{-3}$	$2.83e^{-3}$	$6.51e^{-4}$	$3.32e^{-4}$
0.5	$1.64e^{-3}$	$2.08e^{-3}$	$7.08e{-4}$	$2.78e^{-3}$	$1.69e^{-3}$	$1.30e^{-3}$
0.7	$9.21e^{-3}$	$5.94e^{-3}$	$4.86e^{-3}$	$7.93e^{-3}$	$5.61e^{-3}$	$5.27e^{-3}$
0.9	$1.31e^{-2}$	$1.24e^{-2}$	$1.04e^{-2}$	$1.54e^{-2}$	$1.14e^{-2}$	$1.15e^{-2}$

We see that the approximate solutions are in high agreement with the accurate solution, when $\nu = 1/2$. Therefore, we state the solution for $\nu = 0.25$ and $\nu = 0.75$ is also credible. Table 1 illustrate the values of the solutions for $\alpha = \nu = 0.25$ and shows them for $\alpha = \nu = 0.75$ and table2 gives the values of the solutions for $\alpha = \nu = 0.5$. Also, the approximate solutions for $\alpha = \nu = 0.5$ are compared with the exact solution in Table 2. We know that the exact solution for the values of $\nu \neq 0.5$ are unknown. Therefore to illustrate the present scheme for this problem, we use estimated error in section 6. Table 3, displays graph of error for some m and different values of ν. These tables and figures demonstrate the advantage and the accuracy of the FBFs for solving the time-fractional Bessel differential equation. Also, Table 4 demonstrates the effect of parameter α for this problem. From above tables and figures, it could be claimed that the best cases of α for problem (7.1) is $\alpha = \nu$.

Example7.2. Consider the fractional Bessel equation as follow

$$x^{2\nu} D^\nu D^\nu z(x) + \nu x^\nu D^\nu z(x) + \nu^2 (x^{2\nu} - 1) z(x) = 0.25x^4 + 3.9x^3 + 2.5x^2, \tag{7.2}$$

with boundary condition

$$I^{1-\nu} z(x)|_{x=0,1} = 0.$$

For problem (7.2), the exact solution is $z(x) = x^3 + x^2$ for $\nu = 0.5$.
Figure 2 depicts the graph of the real solution and the approximate results of problem (7.2) for $m = 7$, $\alpha = \nu = 0.25, 0.40, 0.50, 0.60, 0.75$. It could be claimed that the approximate solutions are in high agreement with the exact solution, when $\nu = 1/2$. Therefore, we state the solution for $\nu = 0.25$ and $\nu = 0.75$ is also credible. Table 5 demonstrates the values of the solutions for $\alpha = \nu = 0.25$ and shows them for $\alpha = \nu = 0.75$ and table 6 gives the values of the solutions for $\alpha = \nu = 0.5$. Also, the approximate solutions for $\alpha = \nu = 0.5$ are compared with the exact solution in table6. We know that the exact solution for the values of $\nu \neq 0.5$ are unknown. Therefore to show efficient of the present scheme for problem (7.2), we use estimated error in section 6. Table 7, displays graph of error for some m and copious values of ν. These tables and figures demonstrate the advantage and the accuracy of the time-fractional Bernoulli functions for solving the fractional Bessel differential equation. Also, Table 8 display the effect of parameter α for

this problem. From above tables and figures, it is convenient to say that the best cases of α for problem (7.2) is $\alpha = \nu$.

Example 7.3. Consider the fractional Bessel equation

$$xD^{0.5}D^{0.5}z(x) + 0.5x^{0.5}D^{0.5}z(x) + 0.25xz(x) = 1.565x + .25x^2 - .725x^3 \\ -.042x^4 + 0.057x^5 + 0.002x^6 - 0.0018x^7 - 0.00004x^8, \quad (7.3)$$

with boundary condition

$$I^{1-\nu}z(x)|_{x=0,1} = 0.$$

The real solution of problem (7.3) is $z(x) = Sin(x)$ for $p = 0$.
In Table 9, the absolute error between the real and numerical solutions for different of α and various m is exposed. The comparison shows that the best sense of α for this problem is $\alpha = 0.5$ and it is observed that the errors decline when m increase.
Figure3 demonstrates the exact solution and the numerical results of problem (7.3) for $m = 5$, $\alpha = 0.25, 0.40, 0.60, 0.75, 1$. The comparison shows that the numerical solutions tend to $z(x) = Sin(x)$, when $\alpha = \nu \to 1/2$.

8. Conclusion

This paper attempted to obtain numerical solutions to the FBFs. The FBFs are taken to be an eminent factor in mathematics, physics, and engineering, which are usually not straightforward to be solved analytically. They require the application of techniques. Simply put, in this article, the researcher has decided to regard convenient mechanization of the time-fractional Bernoulli function and least square method as being an appropriate and applicable proposed method to solve Bessel equations. That is to say, this method transforms the under study problem into a nonlinear algebraic system. Through the study, one can pinpoint that the solution of the resulting system is used to compute unknown Bernoulli coefficients of the solution functions. This can able the researcher to draw inferences about the suggested approach. Last but not least, it has been described more clearly that the proposed method must be consistent with the Bessel equations. One should bear in mind that, using the suggested scheme, it does seem quite justified to expect achieved solutions to illustrate the best case of α for this problem, that is $\alpha = \nu$. The results obtained from the given examples also happen to determine the ability, reliability, and trustworthiness of the present method.

References

[1] O. Abdulaziz, I. Hashim, and S. Momani, *solving sestem of fractional differential equations by homotopy perturbation method*, Phys. Let. A, *372*(4) (2008), 451-459.

[2] R.S. Adguzel, U. Aksoy, E. Karapinar, and I.M. Erhan, *On the solutions of fractional differential equations via Geraghty type hybrid contractions*, Appl. Comput. Math, *20*(2) (2021), 313-333.

[3] S.A. Alavi, A. Haghighi, A. Yari, and F. Soltanian, *A numerical method for solving fractional optimal control problems using the operational matrix of Mott polynomials*, Computational Methods for Differential Equations, *10*(3) (2022), 755-773, DOI:10.22034/cmde.2021.39419.1728.

[4] A. Arikoglu and I. Ozkol, *Solution of fractional integro-differential equations by using fractional differential transform method*, Chaos Solitons Fractals, *40*(2) (2009), 521-529.

[5] E. Ashpazzadeh, M. Lakestani, and A. Fatholahzadeh, *Spectral Methods Combined with Operational Matrices for Fractional Optimal Control Problems: A Review*, Appl. Comput. Math, *20*(2) (2021), 209-235.

[6] A. H. Bhrawy , M. M. Tharwat, and A. Yildirim, *A new formula for fractional integrals of Chebyshev polynomials: application for solving multi-term fractional differential equations*, Appl. Math. Model, *37*(6) (2013), 4245-4252.

[7] A. H. Bhrawy and M.A. Zaky, *A shifted fractional-order jacobi orthogonal functions: An application for system of fractional differential equations*, Appl. Math. Model, *40*(2) (2016), 832-845.

[8] V. Daftardar-geiji and H. Jafari, *Adomian decomposition:a tool for solving a system of fractional differential equations*, J.Math. Anal. Appl, *301*(2) (2005), 508-518.

[9] E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, *A new Jacobi operational matrix: an application for solving fractional differential equations*, Appl. Math. Model, *36*(10) (2012), 4931-4943.

[10] A. Gokdogan, E. Unal, and E. Celik, *Conformable Fractional Bessel Equation and Bessel Functions*, classical Analysis and ODES, (2015), arXiv:1506.07382(math).
[11] N. Haddadi, Y. Ordokhani, and M. Razzaghi, *Optimal control of delay systems by using a hybrid functions approximation*, J. Optim. Theory. Appl, *153*(2) (2012),338-356.
[12] M.S. Hashemi, E. Ashpazzadeh, M. Moharrami, and M. Lakestani, *Fractional order Alpert multiwavelets for discretizing delay fractional differential equation of pantograph type*, Appl. Numer. Math, *170* (2021), 1-13.
[13] I. Hashim, O. Abdulaziz, and S. Momani, *Homotopy analysis method for fractional IVPs*, Commun. Nonlinear Sci. Numer. Simul, *14*(3) (2009), 674-684.
[14] S. Kazem, S. Abbasbandy, and S. Kumar, *Fractional-order legendre functions for solving fractional-order differential equation*, Appl Math Model, *37*(7) (2013), 5498-5510.
[15] F. Kh. Keshi, B. P. Moghaddam, and A. Aghili, *A numerical approach for solving a class of variable-order fractional functional integral equations*, Comput. Appl. Math, *37*(1) (2018),4821-4834.
[16] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics studies, *204*(2006), 1-523.
[17] B. G. Korenev, *Bessel Functions and their Applications*, London; New York: Taylor and Francis, 2002.
[18] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley and Sons Press, New York, 1987.
[19] S. Kumar, D. Kumar, S. Abbasbandy, and M. M. Rashidi, *Analytical solution of fractional Navier-stokes equation by using modified Laplace decomposition method*, Ain Shams Eng. J, *5*(2) (2014), 569-574.
[20] Y. Li and N. Sun, *Numerical solution of fractional differential equations using the generalized bloc pulse operational matrix*, Comput. Math. Apple, *62*(3) (2011), 1046-1054.
[21] A. Lotfi, M. Dehghan, and S. A. Yousefi, *A nemerical technique for solving fractional optimal control problem*, Comput. Math. Appl, *62*(3)(2011), 1055-1067.
[22] J. A. T. Machado and B. P. Moghaddam, *A Robust Algorithm for Nonlinear Variable-Order Fractional Control Systems with Delay Publication*, Int. J. Nonlinear. Sci. Numer. Simul, *19*(3-4) (2018), 231-238.
[23] M. Meerschaert and C. Tadjeran, *Finite difference approximations for two-sided space -fractional partial differential equations*, Appl. Numer. Math, *56*(2006), 80-90.
[24] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John-Wily and Sons Inc.New York, 1993.
[25] S. Mockary, A. Vahidi, and E. Babolian, *An efcient approximate solution of Riesz fractional advection-difusion Equation*, Computational Methods for Differential Equations,*10*(2) (2022), 307-319, DOI:10.22034/cmde.2021.41690.1815.
[26] B. P. Moghaddam and J. A. T. Machado, *A computational approach for the solution of a class of variable-order fractional integro-differential equations with weakly singular kernels De Gruyter*, *20*(4) (2017), 1021-1042.
[27] B. P. Moghaddam and J. A. T. Machado, *Extended Algorithms for Approximating Variable Order Fractional Derivatives with Applications*, Journal of Scientific Computing, *71*(3) (2017), 1351-1374.
[28] B. P. Moghaddam, J. A. T. Machado and H. Behforooz, *An integro quadratic spline approach for a class of variable-order fractional initial value problems*, Chaos, Solitons and Fractals, *102*(c) (2017), 354-360.
[29] M. A. Moghaddam, Y. E. Tabriz, and M. Lakestani, *Solving fractional optimal control problems using Genocchi polynomials*, Computational Methods for Differential Equations, *9*(1) (2021), 79-93.
[30] P. Mokhtary, F. GHoreishi, and H. M. Srivastava, *The Muntz-Legendre Tau method for fractional differential equations*, Appl. Math. Model, *40*(2)(2016), 671-684.
[31] S. Momani and K. Al-Khaled, *Numerical solutions for systems of fractional differential equations by thr decomposition method*, Appl. Math. Comput, *162*(3) (2005), 1351-1365.
[32] Z. Odibat and S. Momani, *Application of variational iteration method to nonlinear differential equations of fractional order*, Int.J.Nonlinear Sci. Numer.Simul, *7*(1) (2006), 27-34.
[33] W. Okrasinski and L. Plociniczak, *A not on fractional equation and its asymptotics*, fractional calculus and applied analysis, *16*(3) (2013), 559-572.
[34] I. Podlubny, *Fractional Differential Equations*, Academic Prees, San Diego, CA, 1999.

[35] I. Podlubny, *Fractional differential equations: An introduction to fractionl derivatives, Fractional Differential equations, to Methods of Their Solution and Some of Their Applications*, Elsevier, 1998.

[36] P. Rahimkhani, Y. Ordokhani, and E. Babolian, *Fractional-order Bernoulli wavelets and their applications*, Appl. Num. Math, *40*(17-18) (2016), 8087-8107.

[37] M. Rivero, L. Rodriguez-Germa, and J. J. Trujillo, *Linear fractional differential equations with variable coefficients*, A Mathematics Letters, *21*(9) (2008), 892-897 .

[38] G. Simmons, *Differential Equations: With Applications and Historical Notes*, New York: McGraw-Hill Companies, 1972.

[39] S.C. Shiralashetti and A. B. Deshi, *An efficient Haar wavelet collocation method for the numerical solution of multi-term fractional differential equations*, Nonlinear Dynam, *83*(1) (2016), 293-303.

[40] E. Tohidi, A. H. Bhrawy, and Kh. Erfani, *A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation*, Appl. Math. Model, *37*(6) (2013), 4283-4294.

[41] G. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., London; New York: Cambridge University Press, 1995.

[42] SH. Yaghoobi, B. P. Moghaddam and K. Ivaz, *An efficient cubic spline approximation for variable-order fractional differential equations with time delay*, Nonlinear Dynamics, *87*(2) (2017), 815-826.

[43] SH. Yaghoobi, B. P. Moghaddam and K. Ivaz, *A numerical approach for variable-order fractional unified chaotic systems with time-delay*, Computational Methods for Differential Equations, *6*(4) (2018), 396-410.