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Obtaining soliton solutions of equations combined with the Burgers and equal width wave equations using a novel method

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Abstract

In the present paper, a modified simple equation method is used to obtain exact solutions of the equal width wave Burgers and modified equal width wave Burgers equations. By giving specific values to the parameters, particular solutions are obtained and the plots of solutions are drawn. It shows that the proposed method can be easily generalized to solve a variety of non-linear equations by implementing a robust and straightforward algorithm without the need for any tools.

Keywords. Simple equation method, Burgers equation, Modified equal width wave equation, Soliton solution.

2010 Mathematics Subject Classification. 35D99, 34A05.

1. Introduction

Many of the equations resulting from the mathematical modeling of various problems are through a nonlinear differential equation. Therefore, obtaining an analytical or approximate solution to these equations is very important. In the last decade, various methods have been introduced and used for this purpose, such as the exponential function method [5, 7, 9, 10, 12, 36, 37], Adomian decomposition method [8], Homotopy perturbation method [11], variational iteration method [13, 14], tanh method [24, 29], G'/G expansion method [30], and many others [1–4, 6, 15–17, 19, 23, 25, 28, 33]. One of the most important methods is the simple equation method (SE method) [18, 20–22, 26, 27, 31, 32, 34, 35]. The primary purpose is to extend this method for obtaining the exact solution of equal width wave Burgers (EW-Burgers) and modified equal width wave Burgers (MEW-Burgers) equations. The structure and steps of the proposed method with the help of mathematical software are quite simple.

The continuous process of the article is as follows. Section 2 describes the modified simple equation (MSE) method. In section 3, the MSE method is used to obtain the exact solution of the MEW-burgers equation. The EW-Burgers equation is discussed in section 4. Finally, a conclusion is presented in section 5.

2. The MSE method

To describe the MSE method, let us consider a following nonlinear PDE

$$p(u, u_t, u_x, u_{xx}, u_{tt}, \ldots) = 0. \tag{2.1}$$

The implementation of this method can be divided into several stages.

Step 1. By changing the variable

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$$\xi = kx + wt, \tag{2.2}$$

where w and k are constant , Eq. (2.1) can be written as follows:

$$Q(u, u', u'', u''', \dots) = 0. \tag{2.3}$$

Step 2. In this step, we consider the following solution

$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{F'(\xi)}{F(\xi)}\right)^i, \tag{2.4}$$

where $a_i' s$ are anonymous constants, and$F(\xi)$ is an unknown function.

Step 3. From (2.4), we get the following results

$$u'(\xi) = m a_m \left(\frac{F'(\xi)}{F(\xi)}\right)^{m-1} \left(\frac{F''}{F} - \frac{F'^2}{F^2}\right) + (m-1)a_{m-1} \left(\frac{F'(\xi)}{F(\xi)}\right)^{m-2} \left(\frac{F''}{F} - \frac{F'^2}{F^2}\right) + \dots$$
$$= b_m \frac{F'^{(m+1)}(\xi)}{F^{(m+1)}(\xi)} + \frac{A(\xi)}{F^m(\xi)} + \dots$$
$$u''(\xi) = c_m \frac{F'^{(m+1)}(\xi)}{F^{(m+1)}(\xi)} + \frac{B(\xi)}{F^m(\xi)} + \dots,$$
$$\vdots$$

and

$$u^n(\xi) = {a_m}^n \frac{F'^{(nm)}(\xi)}{F^{(nm)}(\xi)} + \frac{B(\xi)}{F^{nm-1}(\xi)} + \cdots.$$

Therefore, the most power of $F(\xi)$ in denominator of u', u'', $\dots$is $m+1$, $m+2$, respectively, and this power in u^n is equal to nm. To obtain the number m, we balance the highest derivative order and the highest nonlinear order in Eq. (2.3).

Step 4. In this step, we put Eq. (2.4) into the ordinary differential equation obtained. Therefore, a polynomial of $\frac{F'(\xi)}{F(\xi)}$ and its derivatives will be obtained. By considering the coefficients of $F^{-i}(\xi)$, $i = 0, 1, 2, 3, \dots$, as zero, a system of equations that can be obtained to determine unknown constants, $F(\xi)$and $F'(\xi)$. Finally, solutions of Eq. (2.1) will be obtained by putting obtained results into Eq. (2.4).

3. Application MSE method to MEW- Bergers

In this part we use the above method for the MEW-Burgers equation as follows

$$u_t + \alpha u^2 u_x + \gamma u_{xx} - \beta u_{txx} = 0. \tag{3.1}$$

By considering (2.2), Eq. (3.1) turns to the following ODE,

$$wu' + k\alpha u^2 u' + \gamma k^2 u'' - \beta k^2 w\, u''' = 0. \tag{3.2}$$

We take the integrals of the both sides of equation (3.2) and set the integral to zero for constant simplicity.

$$wu + k\alpha \frac{u^3}{3} + \gamma k^2 u' - \beta k^2 w\, u'' = 0, \tag{3.3}$$

which is a special case of the second order nonlinear ordinary differential equation (2.3) obtained from the second order nonlinear partial differential equation (2.1) in [19]. In [19], if $D = -\beta k^2 w$, $c_1 - c = \gamma k^2$, $-\gamma = \frac{k\alpha}{3}$, $\alpha = w$, $\beta = 0$, $A_0 = a_0$, $A_1 = a_1$, and $\Psi = F$, then the results of this manuscript about the ordinary differential equation (3.3) can be obtained from [19]. Also Eq. (3.3) is the same as the second order nonlinear ordinary differential equation (2.4) in

[17]. In [17], if $c^2-1=-\beta k^2 w,\ -c=\gamma k^2,\ \beta=\frac{k\alpha}{3},\ \alpha=w,\ A_0=a_0,\ A_1=a_1,$ and $\Psi=F$, then the results of this manuscript about the ordinary differential equation (3.3) can be obtained from [17].

By balancing the order and the degree of the equation (3.3), we have $m+2=3m$. So, we derive $m=1$. Let us assume that

$$u(\xi)=a_1\left(\frac{F'}{F}\right)+a_0,\quad a_1\neq 0. \tag{3.4}$$

Putting (3.4) in (3.3) and equalizing coefficient $F^{-i}(\xi),\ i=0,\ 1,\ 2,\ 3$ to zero, leads to

$$wa_0+\frac{k\alpha}{3}a_0^3=0, \tag{3.5}$$

$$-\beta k^2 wa_1F'''+\gamma k^2 a_1F''+\left(wa_1+k\alpha a_0^2a_1\right)F'=0, \tag{3.6}$$

$$3\beta k^2 wa_1F'F''+(-\gamma k^2a_1+k\alpha a_0{a_1}^2)F'^2=0, \tag{3.7}$$

$$\left(\frac{k\alpha}{3}a_1^3-2\beta k^2 wa_1\right)(F')^3=0. \tag{3.8}$$

From Eqs. (3.5) and (3.8) and by considering $a_1\neq 0$ from (3.4), we have

$$\begin{aligned} a_0 &= 0,\ \pm\sqrt{\tfrac{-3w}{k\alpha}},\\ a_1 &= \pm\sqrt{\tfrac{6kw\beta}{\alpha}}. \end{aligned}$$

Case 1: If $a_0=0$, (3.6) and (3.7) change to

$$(-\beta k^2wF'''+\gamma k^2F''+wF')a_1=0, \tag{3.9}$$

$$(3\beta wF'F''-\gamma F'^2)k^2a_1=0. \tag{3.10}$$

If $k=0$, then $\xi=wt$ and the solution will be obtained in t which is useless. By solving Eq. (3.10), we obtain

$$F'=Ae^{\frac{\gamma}{3\beta w}\xi}.$$

Therefore we derive

$$F=A\frac{3\beta w}{\gamma}e^{\frac{1}{\beta}((\pm\sqrt{\frac{-\beta}{2}})x+\frac{\gamma}{3}t)}+B.$$

By placing above solution into (3.9), we derive

$$k=\pm\frac{3w}{\gamma}\sqrt{\frac{-\beta}{2}}.$$

By replacing $F$and F'into Eq. (3.4), the general solution of Eq. (3.1) has been obtained as

$$u(x,t)=\pm\sqrt{\frac{6kw\beta}{\alpha}}\left(\frac{A\gamma e^{\frac{1}{\beta}((\pm\sqrt{\frac{-\beta}{2}})x+\frac{\gamma}{3}t)}}{3A\beta we^{\frac{1}{\beta}((\pm\sqrt{\frac{-\beta}{2}})x+\frac{\gamma}{3}t)}+\gamma B}\right). \tag{3.11}$$

The plots of above solution for $\beta=-2,\alpha=-1,w=1,\gamma=1,\ A=1,$and $B=1$illustrated in Figure 1.

Case 2: If $a_0=+\sqrt{\frac{-3w}{k\alpha}}$ Eqs. (3.6) and (3.7) turns to

$$-\beta k^2wF'''+\gamma k^2F''-2wF'=0, \tag{3.12}$$

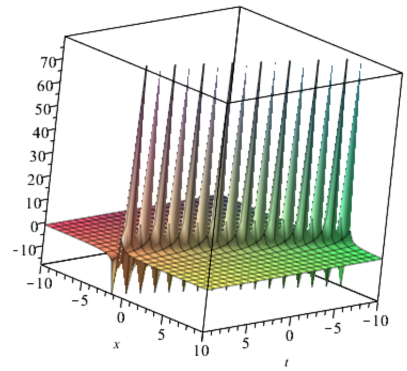

FIGURE 1. Plot of solution (3.11) for$\beta=-2,\alpha=-1,w=1,\gamma=1$, A=1 and B=1.

$$3\beta kwF'F''+(-\gamma k\pm 3w\sqrt{-2\beta})F'^2=0. \tag{3.13}$$

By considering Eq. (3.13), we obtain

$$F'=Ae^{(\frac{\gamma}{3\beta w}\mp\frac{\sqrt{-2\beta}}{\beta k})\xi}, \tag{3.14}$$

where $\beta<0$ and $A$is an arbitrary constant. Therefore, from (3.14) and (3.12), we have

$$k=\pm\frac{3w\sqrt{-2\beta}}{2\gamma}.$$

and

$$F'=Ae^{\frac{-\gamma}{3\beta w}\xi}.$$

So,

$$F=\frac{3\beta wA}{(-\gamma)}e^{e^{\frac{-\gamma}{3\beta w}\xi}}+B.$$

By substituting F and F' into Eq. (3.4), the general solution of Eq. (3.1) has been obtained as

$$u(x,t)=\pm\sqrt{\frac{6kw\beta}{\alpha}}\left(\frac{-\gamma Ae^{(-\frac{1}{2\beta})(\pm\sqrt{-2\beta}x+6\gamma t)}}{3\beta wAe^{(-\frac{1}{2\beta})(\pm\sqrt{-2\beta}x+6\gamma t)}+B}\right)+\sqrt{\frac{-3w}{\alpha k}}. \tag{3.15}$$

The plots of above solution for $\beta=-2,\alpha=-1,w=1,\gamma=1,\ A=1,$and $B=1$illustrated in Figure 2.

Case 3: When $a_0=-\sqrt{\frac{-3w}{k\alpha}}$, the following solutions have been obtained by performing a process similar to case 2.

$$u(x,t)=\pm\sqrt{\frac{6kw\beta}{\alpha}}\left(\frac{-\gamma Ae^{(-\frac{1}{2\beta})(\mp\sqrt{-2\beta}x+6\gamma t)}}{3\beta wAe^{(-\frac{1}{2\beta})(\mp\sqrt{-2\beta}x+6\gamma t)}+B}\right)-\sqrt{\frac{-3w}{\alpha k}}. \tag{3.16}$$

The plots of above solution for $\beta=-2,\alpha=-1,w=1,\gamma=1,\ A=1,$and $B=1$ illustrated in Figure 3.

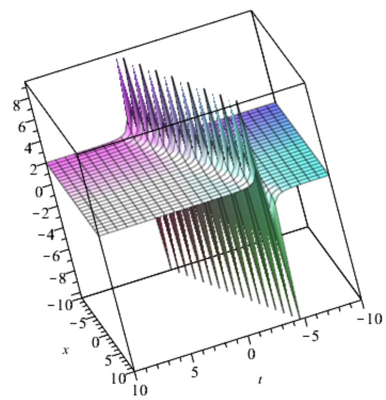

FIGURE 2. Plot of solution (3.15) for $\beta=-2, \alpha=-1, w=1, \gamma=1$, A=1 and B=1.

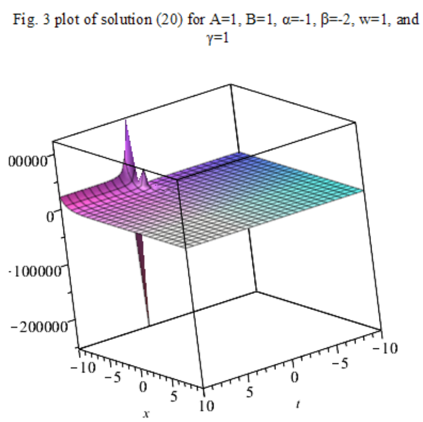

FIGURE 3. Plot of solution (3.16) for $\beta=-2, \alpha=-1, w=1, \gamma=1$, A=1 and B=1.

4. Application MSE method to EW- Bergers

Let's consider the following EW-Berger equation,

$$u_t+\alpha\, u\, u_x-\delta\, u_{xx}-\mu\, u_{xxt}=0. \tag{4.1}$$

Using (2.2), it has been result

$$wU'+\alpha kUU'-k^2\delta U''-k^2w\mu U'''=0. \tag{4.2}$$

Integrating (4.2) with respect ξ, leads to

$$wU+\alpha k\frac{U^2}{2}-k^2\delta U'-k^2w\mu U''=0. \tag{4.3}$$

Presumably $u(\xi)$ can be illustrated in the form (2.4). Balancing U'' and U^2 leads to $m+2=2m$. So $m=2$. Let us assume that

$$u(\xi)=a_2\left(\frac{F'}{F}\right)^2+a_1\left(\frac{F'}{F}\right)+a_0, \quad a_2\neq 0. \tag{4.4}$$

Putting (4.4) in (4.3) and equalizing coefficient $F^{-i}(\xi)$, $i = 0, 1, 2, 3,$ 4to zero, leads to

$$\frac{\alpha k}{2} a_0^2 + \mathrm{w} a_0 = 0, \tag{4.5}$$

$$-\mu k^2 w a_1 F''' - \delta k^2 a_1 F'' + (\alpha k a_0 a_1 + w a_1) F' = 0, \tag{4.6}$$

$$-\mu k^2 w a_2 F''^2 + (-2\delta k^2 a_2 - \mu k^2 w(2a_2 - 3a_1))F'F'' + (w a_2 + \frac{\alpha k}{2} {a_1}^2 + \alpha k a_0 a_2 + \delta k^2 a_1) F'^2 = 0, \tag{4.7}$$

$$10\mu k^2 w a_2 F'^2 F'' + (-2a_1 \mu k^2 w + \alpha k a_1 a_2 + 2\delta k^2 a_2) F'^3 = 0, \tag{4.8}$$

$$\left(\frac{\alpha k}{2} {a_2}^2 - 6\mu k^2 w a_2\right) (F')^4 = 0. \tag{4.9}$$

By solving Eqs. (4.5) and (4.9) and by considering $a_2 \neq 0$ from (4.4), we drive

$$a_0 = 0, \frac{-2w}{\alpha k}, \; a_2 = \frac{12\mu k w}{\alpha}.$$

Case 1:

If $a_0 = 0$,and $a_2 = \frac{12\mu k w}{\alpha}$Eqs. (4.6), (4.7), and (4.8) yield to

$$-\mu k^2 w a_1 F''' - \delta k^2 a_1 F'' + w a_1 F' = 0, \tag{4.10}$$

$$-\frac{12\mu^2 k^3 w^2}{\alpha} F''^2 + (\frac{-24\delta k^3 \mu w}{\alpha} - \mu k^2 w(\frac{24k\mu w}{\alpha} - 3a_1))F'F'' + (\frac{12\mu k w^2}{\alpha} + \frac{\alpha k}{2} {a_1}^2 + \delta k^2 a_1) F'^2 = 0, \tag{4.11}$$

$$\frac{120\mu^2 k^3 w^2}{\alpha} a_2 F'^2 F'' + (-2a_1 \mu k^2 w + 12\mu k^2 w a_1 + 24\frac{\mu \delta k^3 w}{\alpha}) F'^3 = 0. \tag{4.12}$$

By solving Eq. (4.12), we obtain

$$F' = A e^{-\frac{5\alpha a_1 + 12\delta k}{60 w k \mu}\xi}, \tag{4.13}$$

where $A$is an arbitrary constant. By substituting (4.13) into Eqs. (4.10) and (4.11), we get

$$60\mu w^2 - 7\delta k a_1 \alpha - \frac{5}{12} {a_1}^2 \alpha^2 - \frac{72}{5} \delta^2 k^2 = 0,$$

$$12\mu k^2 w - \frac{25}{12} k {a_1}^2 \alpha^2 + \frac{14}{5} a_1 \alpha \delta k^2 + \frac{108}{25} \delta^2 k^3 + 2\mu k^2 w a_1 \alpha + \frac{24}{5} \mu k^3 w \delta = 0.$$

By solving the above equation, we derive

$$a_1 = \frac{1}{\alpha}(-\frac{42}{5}\delta k \pm 6\sqrt{\delta^2 k^2 + 4\mu w^2}).$$

Therefore, we have

$$F = -A \frac{60 w k \mu}{5\alpha a_1 + 12\delta k} e^{-\frac{5\alpha a_1 + 12\delta k}{60 w k \mu}\xi} + B.$$

Replacing F and F' into Eq. (4.4), the general solution of Eq. (4.1) has been obtained

$$u(\xi) = \frac{12\mu k w}{\alpha} \left(\frac{A e^{-\frac{5\alpha a_1 + 12\delta k}{60 w k \mu}\xi}}{-A \frac{60 w k \mu}{5\alpha a_1 + 12\delta k} e^{-\frac{5\alpha a_1 + 12\delta k}{60 w k \mu}\xi} + B} \right)^2 + a_1 \left(\frac{A e^{-\frac{5\alpha a_1 + 12\delta k}{60 w k \mu}\xi}}{-A \frac{60 w k \mu}{5\alpha a_1 + 12\delta k} e^{-\frac{5\alpha a_1 + 12\delta k}{60 w k \mu}\xi} + B} \right). \tag{4.14}$$

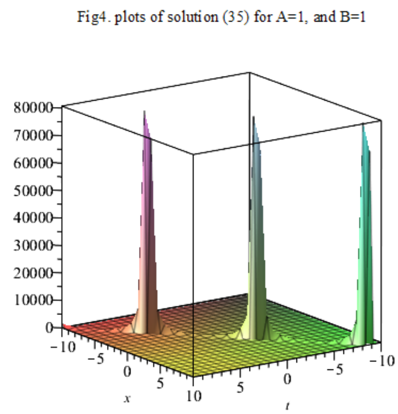

FIGURE 4. Plot of solution (4.15) for A=1 and B=1.

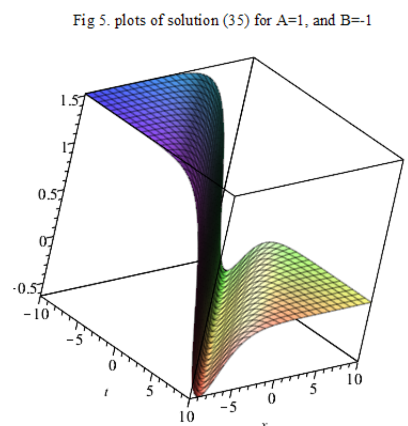

FIGURE 5. Plot of solution (4.15) for A=1 and B=-1.

By considering$\alpha = \delta = \mu = k = 1$, we get $w = 1.0625$, $a_1 = 5.6912$, $a_2 = 12.75$, and $w = -0.7498, a_1 = 2.4143$, $a_2 = -8.9972$. So we derive the following exact solutions

$$u(\xi) = 12.75\left(\frac{Ae^{-0.6346(x+1.0625t)}}{-1.5758Ae^{-0.6346(x+1.0625t)}+B}\right)^2 + 5.6912\left(\frac{Ae^{-0.6346(x+1.0625t)}}{-1.5758Ae^{-0.6346(x+1.0625t)}+B}\right), \tag{4.15}$$

and

$$u(\xi) = -8.9972\left(\frac{Ae^{0.5351(x-0.7498t)}}{1.8688Ae^{0.5351(x-0.7498t)}+B}\right)^2 + 2.4143\left(\frac{Ae^{0.5351(x-0.7498t)}}{1.8688Ae^{0.5351(x-0.7498t)}+B}\right). \tag{4.16}$$

The plots of solutions (4.15) and (4.16) for $A = 1$ and $B = 1$, -1illustrated in Figures 4-7.

Case 2: When $a_0 = \frac{-2w}{\alpha k}$, and $a_2 = \frac{12\mu k w}{\alpha}$ Eqs. (4.6), (4.7), and (4.8) yield to

$$-\mu k^2 w a_1 F''' - \delta k^2 a_1 F'' - w a_1 F' = 0,$$

$$-\frac{12\mu^2 k^3 w^2}{\alpha}F''^2 + \left(\frac{-24\delta k^3 \mu w}{\alpha} - \mu k^2 w\left(\frac{24k\mu w}{\alpha} - 3a_1\right)\right)F'F'' + \left(\frac{12\mu k w^2}{\alpha} + \frac{\alpha k}{2}{a_1}^2 + \delta k^2 a_1 - \frac{24k\mu w^2}{\alpha}\right)F'^2 = 0,$$

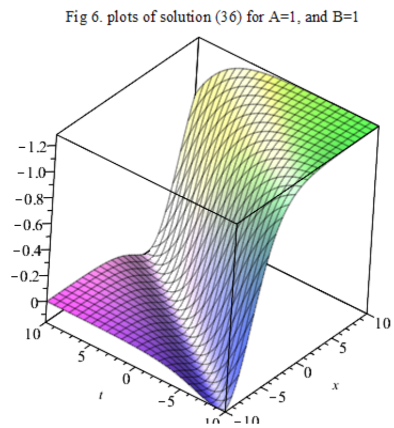

FIGURE 6. Plot of solution (4.16) for A=1 and B=1.

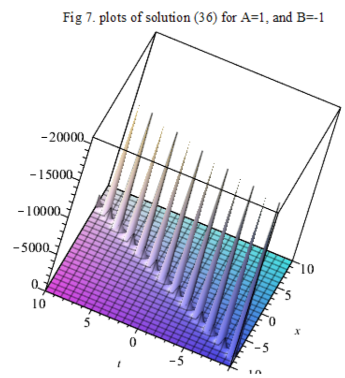

FIGURE 7. Plot of solution (4.16) for A=1 and B=-1.

$$\frac{120\mu^2k^3w^2}{\alpha}a_2F'^2F'' + (-2a_1\mu k^2w + 12\mu k^2wa_1 + 24\frac{\mu\delta k^3w}{\alpha})F'^3 = 0.$$

The following results are obtained by performing a process similar to case 1.

$$u(\xi) = \frac{12\mu kw}{\alpha}\left(\frac{Ae^{-\frac{5\alpha a_1+12\delta k}{60wk\mu}\xi}}{-A\frac{60wk\mu}{5\alpha a_1+12\delta k}e^{-\frac{5\alpha a_1+12\delta k}{60wk\mu}\xi}+B}\right)^2 + a_1\left(\frac{Ae^{-\frac{5\alpha a_1+12\delta k}{60wk\mu}\xi}}{-A\frac{60wk\mu}{5\alpha a_1+12\delta k}e^{-\frac{5\alpha a_1+12\delta k}{60wk\mu}\xi}+B}\right) - \frac{2w}{\alpha k}, \tag{4.17}$$

where $a_1 = \frac{1}{\alpha}(-\frac{42}{5}\delta k \pm 6\sqrt{\delta^2k^2 - 4\mu w^2})$.

By considering$\alpha = \delta = \mu = k = 1$, we derive $w = 0.5702,\ a_1 = -2.7035,\ a_2 = 6.8424$, and $w = -0.0975,\ a_1 = -14.096,\ a_2 = -1.17$.

So we derive the following exact solutions

$$u(\xi) = 6.8424\left(\frac{Ae^{0.0444(x+0.5702t)}}{22.5225Ae^{0.0444(x+0.5702t)}+B}\right)^2 - 2.7035\left(\frac{Ae^{0.0444(x+0.5702t)}}{22.522Ae^{0.0444(x+0.5702t)}+B}\right) - 1.1404, \tag{4.18}$$

and

$$u(\xi) = -1.17\left(\frac{Ae^{-9.9966(x-0.0975t)}}{-0.1Ae^{-9.9966(x-0.0975t)}+B}\right)^2 - 14.096\left(\frac{Ae^{-9.9966(x-0.0975t)}}{-0.1Ae^{-9.9966(x-0.0975t)}+B}\right) + 0.195. \tag{4.19}$$

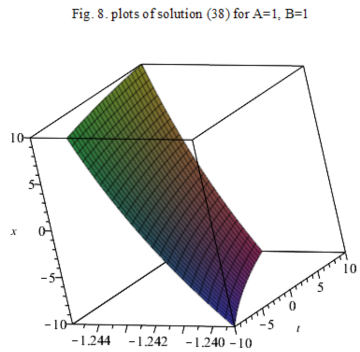

FIGURE 8. Plot of solution (4.18) for A=1 and B=1.

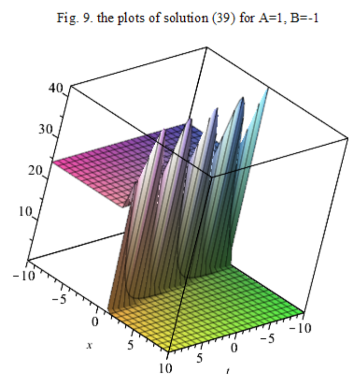

FIGURE 9. Plot of solution (4.19) for A=1 and B=-1.

The plots of solutions (4.18) and (4.19) for $A = 1$ and $B = 1, -1$illustrated in Figures 8 and 9.

5. CONCLUSION

In the current research, a well-known and useful method was applied to search the exact solutions of two NPDE and the new solutions of these equations were obtained. The outcomes indicated that the modified simple equation method was a powerful method for solving nonlinear evolution equations. It should be mentioned that this prevalent method can be algorithmically generalized to solve all kinds of nonlinear equations. The primary merits of procedure used in this research over other analytical methods was that for $F(\xi)$ there was no condition to be fulfilled. So, the probability to get new and varied solutions using this method increased outstandingly.

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