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Asymptotic method for solution of oscillatory fractional derivative

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Abstract

In the paper, an oscillatory system with liquid dampers is considered, when the mass of the head is large enough. By means of expedient transformations, the equation of motion with fractional derivatives is reduced to an equation of fractional order containing a small parameter. The corresponding nonlocal boundary value problem is solved and the zero and first approximations of solutions of the relative small parameter are constructed. The results are illustrated on the concrete example, where the solution differs from the analytical solution by 10^{-2} order.

Keywords. Oscillatory systems of fractional derivative, Small parameter, Asymptotic method, Zero and first approximations.

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1. Introduction

In oil production [13, 21], one of the main methods is the procedure with a sucker rod pumping unit [15], where the plunger is inside the Newtonian fluid [4]. This allows, in contrast to the well-known motion [15], to describe linear ordinary differential equations of the second order with fractional derivatives in subordinate numbers [1, 4, 6, 14, 16, 19]. Since the construction of solutions for this case of the corresponding problem with nonseparated boundary conditions is rather difficult [14, 17, 20], it makes sense to introduce a small parameter (by means of a rather heavy mass head) and construct an asymptotic solution in the first approximation.

2. Problem statement

Let consider the following boundary value problem [2, 3, 12]

$$y''(x) + \sum_{k=0}^{2q-1} \varepsilon a_k D^{\frac{k}{q}} y(x) = \varepsilon f(x), \quad 0 < x_0 < x < l, \tag{2.1}$$

$$\sum_{k=0}^{2q-1} \left[\alpha_{jk} D^{\frac{k}{q}} \; y(x)|_{x=x_0} + \beta_{jk} D^{\frac{k}{q}} y(x)|_{x=l} \right] = \gamma_j, \quad j = \overline{1, 2q}, \tag{2.2}$$

where by means of the corresponding transformations [9, 10] is reduced to the following boundary value problem of the matrix form [7, 8, 11, 18]

$$D^{\frac{1}{q}} Z(x, \varepsilon) = A(\varepsilon) Z(x, \varepsilon) + B(x, \varepsilon), \tag{2.3}$$

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$$\alpha z(x_0,\varepsilon)+\beta z(l,\varepsilon)=\gamma, \tag{2.4}$$

$$z(x,\varepsilon)=\left(z_0(x,\varepsilon),\ z_1(x,\varepsilon),\ z_2(x,\varepsilon),\ \dots, z_{2q-1}(x,\varepsilon)\,\right)^T, \tag{2.5}$$

$$A(\varepsilon)=A_0+\varepsilon A_1, \tag{2.6}$$

$$B(x,\varepsilon)=\varepsilon l_{2q}f(x), \tag{2.7}$$

$$\alpha=(\alpha_{jk})_{j=\overline{1,2q};k=\overline{0,2q-1}},\ \ \beta=(\beta_{jk})_{j=\overline{1,2q};k=\overline{0,2q-1}}, \tag{2.8}$$

$$\gamma=(\gamma_1,\,\gamma_2,\,...,\gamma_{2q})^T, \tag{2.9}$$

$$A_0=\begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 & 0\\ 0 & 0 & 1 & 0 & ... & 0 & 0\\ . & . & . & . & . & . & .\\ 0 & 0 & 0 & 0 & ... & 0 & 1\\ 0 & 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix},$$

$$A_1=-\begin{pmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0\\ 0 & 0 & 0 & 0 & ... & 0 & 0\\ . & . & . & . & . & . & .\\ 0 & 0 & 0 & 0 & ... & 0 & 0\\ a_0 & a_1 & a_2 & a_3 & ... & a_{2q-2} & a_{2q-1} \end{pmatrix}, \tag{2.10}$$

$$l_{2q}=(0\ 0\ 0...0\ 1)^T. \tag{2.11}$$

It is easy to see that from (2.10)

$$A_0^{2q}=0,\ A_1^k=(-1)^{k-1}a_{2q-1}^{k-1}A_1=(-1)^k a_{2q-1}^{k-1}\begin{pmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0\\ 0 & 0 & 0 & 0 & ... & 0 & 0\\ . & . & . & . & . & . & .\\ 0 & 0 & 0 & 0 & ... & 0 & 0\\ a_0 & a_1 & a_2 & a_3 & ... & a_{2q-2} & a_{2q-1} \end{pmatrix}. \tag{2.12}$$

Then the vector solution of system (2.3) can be represented in the form:

$$z(x,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k z^{(k)}(x). \tag{2.13}$$

Substituting (2.13) into (2.3) and (2.4), we arrive at the following problems for a system

$$D^{\frac{1}{q}}z^{(0)}(x)=A_0z^{(0)}(x), \tag{2.14}$$

$$D^{\frac{1}{q}}z^{(1)}(x)=A_0z^{(1)}(x)+(A_1z^{(0)}(x)+l_{2q}f(x)),\quad \dots \tag{2.15}$$

$$D^{\frac{1}{q}}z^{(s)}(x)=A_0z^{(s)}(x)+A_1z^{(s-1)}(x),\quad s\geq 2 \tag{2.16}$$

with the corresponding boundary conditions:

$$\alpha z^{(0)}(x_0) + \beta z^{(0)}(l) = \gamma, \tag{2.17}$$

$$\alpha z^{(1)}(x_0) + \beta z^{(1)}(l) = 0, \tag{2.18}$$

$$\alpha z^{(s)}(x_0) + \beta z^{(s)}(l) = 0, \;\; s \geq 2. \tag{2.19}$$

Taking into account that for the equation

$$D^{\frac{1}{q}} y(t) = a y(t), \;\; t > 0,$$

the solutions are obtained from the shifted Mittag-Leffler series [5, 20] in the form

$$y(t) = \sum_{k=0}^{\infty} a^k \frac{t^{-1+\frac{k+1}{q}}}{(-1+\frac{k+1}{q})!},$$

i.e.

$$D^{\frac{1}{q}} y(t) = \sum_{k=0}^{\infty} a^k \frac{t^{-1+\frac{k}{q}}}{(-1+\frac{k}{q})!} = \frac{t^{-1}}{(-1)!} + a \frac{t^{-1+\frac{1}{q}}}{(-1+\frac{1}{q})!} + a^2 \frac{t^{-1+\frac{2}{q}}}{(-1+\frac{2}{q})!} + ... = \delta(t) + a y(t).$$

3. Construction of the zero approximation

Then the matrix solution to system (2.14) from the nilpotency condition of the matrices A_0 from (2.10) will have the form

$$Z^0(x) = \sum_{k=0}^{\infty} A_0^k \frac{t^{-1+\frac{k+1}{q}}}{(-1+\frac{k+1}{q})!} = \sum_{k=0}^{2q-1} A_0^k \frac{t^{-1+\frac{k+1}{q}}}{(-1+\frac{k+1}{q})!}, \tag{3.1}$$

and the general solution of the vector system (2.14) in the form:

$$z^{(0)}(x) = Z^{(0)}(x) C, \tag{3.2}$$

where the vector column C will be

$$C = (C_0, C_1, ..., C_{2q-1})^T, \tag{3.3}$$

C_j are arbitrary constant real numbers.

Substituting (3.2) into the boundary condition (2.17), we obtain:

$$[\alpha Z^{(0)}(x_0) + \beta Z^{(0)}(l)] C = \gamma. \tag{3.4}$$

If

$$\det[\alpha Z^{(0)}(x_0) + \beta Z^{(0)}(l)] \neq 0, \tag{3.5}$$

then from (3.4) we get

$$C = [\alpha Z^{(0)}(x_0) + \beta Z^{(0)}(l)]^{-1} \gamma. \tag{3.6}$$

Thus, we obtain solutions to the boundary value problem (2.14), (2.17) in the following form:

$$z^{(0)}(x) = Z^{(0)}(x) [\alpha Z^{(0)}(x_0) + \beta Z^{(0)}(l)]^{-1} \gamma. \tag{3.7}$$

4. First approximation

Now let's return to the vector system (2.15). It is easy to see that the homogeneous vector system corresponding to (2.15) has the form:

$$D^{\frac{1}{q}} z^{(1)}(x) = A_0 z^{(1)}(x), \tag{4.1}$$

the general solution (4.1) is represented in the form

$$z^{(1)}(x) = Z^{(0)}(x)C, \tag{4.2}$$

where the column vector C is given in the form (3.3).

To calculate one of the particular solutions of the inhomogeneous vector system (2.15), firstly we calculate the right-hand side of this system:

$$\begin{aligned} A_1 z^{(0)}(x) + l_{2q} f(x) = (0\ 0 \ldots 0\ f(x)+ \\ A_{1,2q} Z^{(0)}(x)[\alpha Z^{(0)}(x_0) + \beta Z^{(0)}(l)]^{-1}\gamma)^T, \end{aligned} \tag{4.3}$$

where

$$A_{1,2q} = -(a_0\ a_1\ a_2 \ldots a_{2q-2}\ a_{2q-1}) \tag{4.4}$$

vector of rows (the last row of the matrix $A1$).

Thus, all equations of the vector system (2.15), except the last one, are homogeneous, and the last equation has the form:

$$D^{\frac{1}{q}} \tilde{z}^{(1)}_{2q-1}(x) = f(x) + A_{1,2q} Z^{(0)}(x)[\alpha Z^{(0)}(x_0) + \beta Z^{(0)}(l)]^{-1}\gamma, \tag{4.5}$$

the solution which is obtained easily by integrating order $\frac{1}{q}$, i.e.

$$\tilde{z}^{(1)}_{2q-1}(x) = I^{\frac{1}{q}} \left\{ f(x) + A_{1,2q} Z^{(0)}(x)[\alpha Z^{(0)}(x_0) + \beta Z^{(0)}(l)]^{-1}\gamma \right\}, \tag{4.6}$$

where $\tilde{z}$- is one of the particular solutions of the inhomogeneous equation (4.5). Then the general solution (2.15) is presented in the form

$$z^{(1)}(x) = Z^{(0)}(x)C + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I^{\frac{1}{q}} \left\{ f(x) + A_{1,2q} Z^{(0)}(x)[\alpha Z^{(0)}(x_0) + \beta Z^{(0)}(l)]^{-1}\gamma \right\} \end{pmatrix}. \tag{4.7}$$

Substituting $z^1(\cdot)$ from (4.7) to the boundary condition (2.18), we have

$$[\alpha Z^{(0)}(x_0) + \beta Z^{(0)}(l)]C = -\beta \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tilde{A}(l) \end{pmatrix} - \alpha \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tilde{A}(x_0) \end{pmatrix}, \tag{4.8}$$

where

$$\tilde{A}(x) = I^{\frac{1}{q}} \left\{ f(x) + A_{1,2q} Z^{(0)}(x)[\alpha Z^{(0)}(x_0) + \beta Z^{(0)}(l)]^{-1}\gamma \right\}. \tag{4.9}$$

Let the following restrictions hold

$$\det[\alpha Z^{(0)}(x_0) + \beta Z^{(0)}(l)] \neq 0, \tag{4.10}$$

then from (2.3) we find

$$C = -[\alpha Z^{(0)}(x_0) + \beta Z^{(0)}(l)]^{-1} \left\{ \beta \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tilde{A}(l) \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tilde{A}(x_0) \end{pmatrix} \right\}. \tag{4.11}$$

The solution to the boundary value problem (2.15), (2.18) has the form:

$$z^{(1)}(x) = -Z^{(0)}(x)[\alpha Z^{(0)}(x_0) + \beta Z^{(0)}(l)]^{-1} \times \left\{ \beta \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tilde{A}(l) \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tilde{A}(x_0) \end{pmatrix} \right\} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tilde{A}(x) \end{pmatrix}. \tag{4.12}$$

The solution to the boundary value problems (2.16), (2.19) for $s \geq 2$ is investigated. Thus the proposed solution to the boundary value problem (2.1), (2.2) (or (2.3), (2.4)) will be

$$z(x) = z^0(x) + \varepsilon z(x),$$

which differs from the exact solution 10^{-2} order.

Example 1. Let consider the case when in (2.1) ($a_k = 0$), i.e.

$$y''(x) = \varepsilon f(x), \tag{4.13}$$

with periodic boundary condition

$$D^{\frac{k}{q}}\, y(x)|_{x=x_0} = D^{\frac{k}{q}} y(x)|_{x=l}\,, \quad k = \overline{0, 2q-1}. \tag{4.14}$$

By making an expedient transformation: $y(x) = z_0(x)$, $D^{\frac{k}{q}} y(x) = D^{\frac{k}{q}} z_0(x) = D^{\frac{k-1}{q}} z_1(x) = D^{\frac{k-2}{q}} z_2(x) = ... = D^{\frac{1}{q}} z_{k-1}(x) = z_k(x)$,

$$D^{\frac{2k}{q}} y(x) = D^{\frac{2k}{q}} z_0(x) = D^{\frac{1}{q}} z_{2q-1}(x) = \varepsilon f(x), \;\; k = \overline{1, 2q-1}.$$

Consider the following boundary value problem of matrix form

$$D^{\frac{1}{q}} z(x) = Az(x) + \varepsilon f(x), \quad z(x_0) = z(l), \tag{4.15}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & 0 & ... & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & ... & 0 & 1 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix} - \text{nilpotent matrix of order } 2q,$$

$F(x) = (0 \;\, 0 \;...\; 0 \;\, f(x))^T$ with the length $2q$,

$$z(x) = (z_0(x) \;\; z_1(x) \;...\; z_{2q-1}(x))^T$$

$$z(x) = \sum_{k=0}^{2q-1} A^k \frac{x^{-1+\frac{k+1}{q}}}{(-1+\frac{k+1}{q})!} C + \varepsilon I_{x_0}^{\frac{1}{q}} F(x), \tag{4.16}$$

where

$$I_{x_0}^{\frac{1}{q}}F(x)=\int\limits_{x_0}^{x}\frac{(x-t)^{\frac{1}{q}-1}}{(\frac{1}{q}-1)!}F(t)dt, \tag{4.17}$$

$$\left[\left(\sum_{k=0}^{2q-1}A^k\frac{x_0^{-1+\frac{k+1}{q}}}{(-1+\frac{k+1}{q})!}\right)-\left(\sum_{k=0}^{2q-1}A^k\frac{l^{-1+\frac{k+1}{q}}}{(-1+\frac{k+1}{q})!}\right)\right]C=\varepsilon\int\limits_{x_0}^{l}\frac{(l-t)^{\frac{1}{q}-1}}{(\frac{1}{q}-1)!}F(t)dt. \tag{4.18}$$

If

$$\Delta=\det\sum_{k=0}^{2q-1}A^k\frac{x_0^{-1+\frac{k+1}{q}}-l^{-1+\frac{k+1}{q}}}{(-1+\frac{k+1}{q})!}\neq 0, \tag{4.19}$$

then from the system (4.18) we obtain:

$$C=\varepsilon\left(\sum_{k=0}^{2q-1}A^k\frac{x_0^{-1+\frac{k+1}{q}}-l^{-1+\frac{k+1}{q}}}{(-1+\frac{k+1}{q})!}\right)^{-1}\int\limits_{x_0}^{l}\frac{(l-t)^{\frac{1}{q}-1}}{(\frac{1}{q}-1)!}F(t)dt, \tag{4.20}$$

and from (4.16) we have

$$z(x)=\varepsilon\left(\sum_{k=0}^{2q-1}A^k\frac{x^{-1+\frac{k+1}{q}}}{(-1+\frac{k+1}{q})!}\right)\times\left(\sum_{k=0}^{2q-1}A^k\frac{x_0^{-1+\frac{k+1}{q}}-l^{-1+\frac{k+1}{q}}}{(-1+\frac{k+1}{q})!}\right)^{-1}\int\limits_{x_0}^{l}\frac{(l-t)^{\frac{1}{q}-1}}{(\frac{1}{q}-1)!}F(t)dt, \tag{4.21}$$

these are solutions to the boundary value problem (4.15).

Example 2. Let consider the classical boundary value problem:

$$y''(x)=\varepsilon f(x), \tag{4.22}$$

$$y(x_0)=x(l),\quad y'(x_0)=y'(l). \tag{4.23}$$

The solution to the problem (4.22), (4.23) has the form:

$$y(x)=c_0+\frac{\varepsilon x}{l-x_0}\int\limits_{x_0}^{l}tf(t)dt+\varepsilon\int\limits_{x_0}^{x}(x-t)f(t)dt, \tag{4.24}$$

$$D^{\frac{1}{q}}\ y(x)|_{x=x_0}=D^{\frac{1}{q}}y(x)|_{x=l}\,,$$

where $1/q$ - order derivatives, $f(x)$ can also be periodic.

But in fact from

$$c_0\frac{x_0^{-\frac{1}{q}}}{(-\frac{1}{q})!}+\frac{\varepsilon}{l-x_0}\int\limits_{x_0}^{l}tf(t)dt\frac{x_0^{1-\frac{1}{q}}}{(1-\frac{1}{q})!}+\varepsilon D^{\frac{1}{q}}\int\limits_{x_0}^{x}(x-t)f(t)dt|_{x=x_0}$$

we get

$$c_0\frac{l^{-\frac{1}{q}}}{(-\frac{1}{q})!}+\frac{\varepsilon}{l-x_0}\int\limits_{x_0}^{l}tf(t)dt\frac{l^{1-\frac{1}{q}}}{(1-\frac{1}{q})!}+\varepsilon D^{\frac{1}{q}}\int\limits_{x_0}^{x}(x-t)f(t)dt|_{x=l}\,,$$

i.e. solution to the boundary value problem (4.22)-(4.23) does not satisfy the boundary condition (4.24).

5. Conclusion

The asymptotic solution of the oscillatory system with liquid dampers for a sufficiently large head mass is given. It is shown that the obtained solution does not satisfy the boundary condition of the similar classical problem.

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