Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 10, No. 3, 2022, pp. 639-655
DOI:10.22034/cmde.2021.44502.1879

Hybrid shrinking projection extragradient-like algorithms for equilibrium and fixed point problems

Dejene Shewakena Bedane and Anteneh Getachew Gebrie*

Department of Mathematics, College of Computational and Natural Science, Debre Berhan University, Debre Berhan, P.O. Box 445, Ethiopia.

Abstract

Based on the extragradient-like method combined with shrinking projection, we propose two algorithms, the first algorithm is obtained using sequential computation of extragradient-like method and the second algorithm is obtained using parallel computation of extragradient-like method, to find a common point of the set of fixed points of a nonexpansive mapping and the solution set of the equilibrium problem of a bifunction given as a sum of the finite number of Hölder continuous bifunctions. The convergence theorems for iterative sequences generated by the algorithms are established under widely used assumptions for the bifunction and its summands.

Keywords. Common fixed point problem, Equilibrium problem, Hölder continuity, Extragradient method, Shrinking projection.
2010 Mathematics Subject Classification. 90C25, 90C33, 65K10.

1. Introduction

Let H be real Hilbert space and C be a nonempty closed convex subset of H. For a bifunction $f : C \times C \to \mathbb{R}$, the problem

$$\text{find } z^* \in C \text{ such that } f(z^*, z) \geq 0,\ \forall z \in C \tag{1.1}$$

is called *equilibrium problem* (Fan inequality [9]) of f on C, denoted by $\mathrm{EP}(f, C)$. The set of all solutions of $\mathrm{EP}(f, C)$ is denoted by $\mathrm{SEP}(f, C)$ and is given by $\mathrm{SEP}(f, C) = \{z^* \in C : f(z^*, z) \geq 0,\ \forall z \in C\}$. Various algorithms have been proposed to solve (1.1), see for example [1, 6, 10, 22, 26]. Following the introduction of the equilibrium problem, many iterative algorithms are proposed to find $\bar{x} \in FixT \bigcap \mathrm{SEP}(f, C)$ where $T : C \to C$ is a nonexpansive mapping and $FixT = \{x \in C : Tx = x\}$ is the set of fixed points of T; see [2, 3, 20, 23, 24]. Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|,\ \forall x, y \in C$.

For many years, equilibrium problems and fixed point problems become an attractive fields for many researchers both in theory and applications, see in [7, 28], and due to the importance of the solutions to such problems, many researchers are working in this area and studying on existence and approximation of the solutions to such problems. The problem under consideration in this paper is

$$\text{find } x^* \in FixT \text{ such that } \sum_{i=1}^{N} f_i(x^*, y) \geq 0\ \ \forall y \in C, \tag{1.2}$$

where $f_i : C \times C \to \mathbb{R}$ is bifunction for $i \in I = \{1, \dots, N\}$ and $T : C \to C$ is nonexpansive mapping. Let Ω denotes the solution set of (1.2), i.e., $\Omega = \mathrm{SEP}\big(\sum_{i=1}^{N} f_i, C\big) \bigcap FixT$ where $SEP\big(\sum_{i=1}^{N} f_i, C\big)$ is the solution of the equilibrium

Received: 11 February 2021 ; Accepted: 13 August 2021.
∗ Corresponding author. Email: antgetm@gmail.com .

problem $\mathrm{EP}\big(\sum_{i=1}^{N} f_i, C\big)$:

$$\text{find } x^* \in C \text{ such that } \sum_{i=1}^{N} f_i(x^*, y) \geq 0, \ \ \forall y \in C. \tag{1.3}$$

Recently, the problem of finding a common point of the solution set of the equilibrium problem $\mathrm{EP}\big(\sum_{i=1}^{N} f_i, C\big)$ becomes an attractive field for many researchers (see [12, 15–17, 19]). In [17], the weakly convergent algorithm is proposed for $\mathrm{EP}(f_1+f_2, C)$ using parallel or sequential computation of resolvent operator defined in [6]. Unlike results in [12, 15, 17, 19], Moudafi in [16] used Barycentric projected-subgradient method to generate strongly convergent splitting algorithm for solving equilibrium problems $\mathrm{EP}\big(\sum_{i=1}^{N} f_i, C\big)$ under suitable assumptions.
Here we recall some useful notions.
Let C be a subset of a real Hilbert space H and $f : C \times C \to \mathbb{R}$ is a bifunction. Then, f is said to be

- **(i):** *τ-Hölder continuous* on C with constant $L > 0$ if there exists $\tau \in (0, 1]$ such that at least one of the following is satisfied:
  (a) $|f(x, y) - f(z, y)| \leq L\|x - z\|^{\tau}, \ \forall x, y, z \in C$;
  (b) $|f(x, y) - f(x, z)| \leq L\|y - z\|^{\tau}, \ \forall x, y, z \in C$.
  We call f is τ-Hölder continuous in the first argument (resp. in the second argument) if f satisfies (a) (resp. f satisfies (b)).
- **(ii):** *Lipschitz-type continuous* on C if there exist two positive constants c_1, c_2 such that
$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\| - c_2\|y - z\|, \ \forall x, y, z \in C.$$
- **(iii):** *new type of Lipschitz continuous (defined in [19])* on C with constant $L > 0$ if
$$|f(x, y) + f(y, z) - f(x, z)| \leq L\|x - y\|\|y - z\|, \ \forall x, y, z \in C.$$

It is well shown in [19] that the new type of Lipschitz type continuous bifunction is Lipschitz type continuous bifunction. When $I = \{1, 2\}$, Hai and Vinh in [12] and Pham and Trinh in [19] used proximal operator for f_1 and f_2 which is similar to extragradient algorithm:

$$\begin{cases} y^k = \arg\min\{\lambda_k f_1(x^k, y) + \frac{1}{2}\|x^k - y\|^2 : y \in C\}, \\ z^k = \arg\min\{\lambda_k f_2(y^k, y) + \frac{1}{2}\|y^k - y\|^2 : y \in C\}, \end{cases} \tag{1.4}$$

and

$$\begin{cases} y^k = \arg\min\{\lambda_k f_1(x^k, y) + \frac{1}{2}\|x^k - y\|^2 : y \in C\}, \\ z^k = \arg\min\{\lambda_k f_2(x^k, y) + \frac{1}{2}\|x^k - y\|^2 : y \in C\}, \end{cases} \tag{1.5}$$

in solving (1.2). Pham and Trinh in [19], considered the problem (1.2) when f_1 is τ_1-Hölder continuous in the first or second argument and f_2 is the new type of Lipschitz continuous. On the other hand, Hai and Vinh in [12] obtained a weakly convergent algorithm to some point $p \in SEP(f_1 + f_2, C)$ under certain assumptions where f_1 and f_2 are Hölder continuous (in the first or second argument).

One of the purposes of this paper is to show that if some appropriate additional step of iteration is performed in algorithms proposed in [12], then one has algorithms, possibly under weaker assumptions, converging strongly to some point solving (1.2). Inspired by the practical application of equilibrium problems and motivated by results in [12, 17], we propose two strongly convergent algorithms for solving the problem (1.2) where the first one starts with the sequential extragradient method and the second one starts with the parallel extragradient method. To obtain a strong convergence result, we took some additional steps of the iteration involving resolvent operator and shrinking projection method in the algorithms proposed in [12]. Our algorithms may be computationally expensive than algorithms in [17] and [12] as there is the additional step of an iteration involving resolvent operator and shrinking projection, see more about the resolvent operator and shrinking projection-type results in [5, 13, 17, 23, 27]. Despite this, our algorithms generate a sequence strongly converging to the solution set Ω of the problem (1.2). It is also clear to see that the problem (1.2) is general for the problem considered in [12].

This paper is organized as follows. Section 2 briefly explains the necessary mathematical background. Section 3 presents the proposed algorithms and proves that it converges to $SEP\big(\sum_{i=1}^{N} f_i, C\big) \bigcap FixT$ under certain assumptions. Some applications are provided in section 4.

2. Preliminary

In this section, we recall some definitions and results for further use. Let H be a real Hilbert space with the inner product $\langle ., . \rangle$ and the induced norm $\|.\|$. Let C be a nonempty closed convex subset of H. We write $x^k \to x$ means that the sequence $\{x^k\}$ strongly converges to x as $k \to \infty$. The metric projection on C is a mapping $P_C : H \to C$ defined by

$$P_C(x) = \arg\min\{\|y - x\| : y \in C\}, \quad x \in H.$$

Lemma 2.1. *[25] Let C be a closed convex subset of H. Given $x \in H$ and a point $z \in C$, then $z = P_C(x)$ if and only if $\langle x - z, y - z \rangle \leq 0 \;\; \forall y \in C$.*

Lemma 2.2. *[18] Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C is a metric projection on C. Then,*

$$\|x - P_C(y)\|^2 + \|P_C(y) - y\|^2 \leq \|x - y\|^2 \;\; \forall x \in C, y \in H.$$

Let C be a subset of a real Hilbert space H and $f : C \times C \to \mathbb{R}$ be a bifunction. Then, f is said to be

- **(i):** *strongly monotone* on C, if there is $M{>}0$ (shortly M-strongly monotone on C) iff

$$f(x, y) + f(y, x) \leq -M\|y - x\|^2, \; \forall x, y \in C,$$

- **(ii):** *monotone* on C iff $f(x, y) + f(y, x) \leq 0, \; \forall x, y \in C$,
- **(iii):** *pseudomonotone* on C with respect to $x \in C$ iff

$$f(x, y) \geq 0 \;\text{ implies }\; f(y, x) \leq 0, \; \forall y \in C.$$

Clearly, (i)$\Rightarrow$ (ii)$\Rightarrow$(iii) for every $x \in C$.

For a subset C of a real Hilbert space H, Id_C is mapping from C onto C given by $Id_C(x) = x$ for all $x \in C$.

Lemma 2.3. *[11] Suppose C is closed convex subset of a Hilbert space H and $U : C \to C$ be nonexpansive mapping. Then,*

- **(i):** *If U has a fixed point, then $FixU$ is a closed convex subset of H.*
- **(ii):** *$Id_C - U$ is demiclosed, i.e., whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(Id_C - U)x_n\}$ strongly converges to some y, it follows that $(Id_C - U)x = y$.*

Lemma 2.4. *[4] Let $\{x_1, \ldots, x_d\} \subset H$, $\{\lambda_1, \ldots, \lambda_d\} \subset \mathbb{R}$ with $\sum\limits_{i=1}^{d} \lambda_i = 1$. Then,*

$$\Big\| \sum_{i=1}^{d} \lambda_i x_i \Big\|^2 = \sum_{i=1}^{d} \lambda_i \|x_i\|^2 - \sum_{i=1}^{d} \sum_{j=1}^{d} \lambda_i \lambda_j \frac{\|x_i - x_j\|^2}{2}.$$

Given $\lambda \in [0, 1]$, $x, y \in H$ where H is Hilbert space. Then using Lemma 2.4, we have

$$\|\lambda x + (1 - \lambda) y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Let function $\Psi : H \to \mathbb{R}$ be a function and $x \in H$. Then, the subdifferential of Ψ at x is defined by

$$\partial\Psi(x) = \{w \in H : \langle w, y - x \rangle \leq \Psi(y) - \Psi(x), \;\; \forall y \in H\}.$$

We recall that the normal cone of C at $x \in C$ is defined as follows:

$$N_C(x) = \{w \in H : \langle w, y - x \rangle \leq 0, \;\; \forall y \in C\}.$$

Lemma 2.5. *[8] Let C be a convex subset of a real Hilbert space H and $g : C \to \mathbb{R}$ be a convex and subdifferentiable function on C. Then, x^* is a solution to the following convex problem*

$$\min\{g(x) : x \in C\},$$

if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(x^*)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^*.

For a closed convex subset C of H and for bifunctions $f_i : C \times C \to \mathbb{R}$, $i \in I$ and $f = \sum_{i=1}^{N} f_i$ consider the assumptions given bellow.

Assumption 1

- **(A1):** f is monotone on C; for each $y \in C$, $f(.y)$ is upper semicontinuous;
- **(A2):** for each $x, y, z \in C$, $\limsup\limits_{\lambda \to 0^+} f(\lambda x + (1-\lambda)y, z) \le f(y, z)$.

Assumption 2

- **(B1):** for all $x \in C$, $f_i(x, x) = 0$, $i \in I$,
- **(B2):** for each $x \in C$, the bifunction $f_i(x, .)$ is lower semicontinuous, convex and subdifferentiable on C, $i \in I$.

From ***Assumption 1*** and ***Assumption 2*** above we have

- **(C1):** for all $x \in C$, $f(x, x) = 0$,
- **(C2):** f is monotone on C,
- **(C3):** for each $x, y, z \in C$,

$$\limsup_{\lambda \to 0^+} f(\lambda x + (1-\lambda)y, z) \le f(y, z),$$

- **(C4):** for each $x \in C$, the bifunction $f(x, .)$ is convex and lower semicontinuous.

The following two results are from Equilibrium Programming in Hilbert Spaces.

Lemma 2.6. *[6, Lemma 2.12] Let f satisfies (C1)-(C4). Then, for each $r > 0$ and $x \in H$, there exists $v \in C$ such that*

$$f(v, y) + \frac{1}{r}\langle v - y, y - x\rangle \ge 0, \ \forall y \in C.$$

Lemma 2.7. *[6, Lemma 2.12] Let f satisfies (C1)-(C4). Then, for each $r > 0$ and $x \in H$, define a mapping (called resolvant of f), given by*

$$T_r^f(x) = \{v \in C : f(v, y) + \frac{1}{r}\langle y - v, v - x\rangle \ge 0, \ \forall y \in C\}.$$

Then the followings holds:

- **(i):** *T_r^f is single-valued;*
- **(ii):** *T_r^f is a firmly nonexpansive, i.e., for all $x, y \in H$,*

$$\|T_r^f(x) - T_r^f(y)\|^2 \le \langle T_r^f(x) - T_r^f(y), x - y\rangle;$$

- **(iii):** *$Fix(T_r^f) = SEP(f, D)$, where $Fix(T_r^f)$ is the fixed point set of T_r^f;*
- **(iv):** *$SEP(f, D)$ is closed and convex.*

3. Main Result

In this section, using extragradient and shrinking projection method we propose two algorithms for solving (1.2) and analyse the strong convergence of the sequences generated by the algorithms by assuming that the solution set Ω is nonempty.

3.1. **Shrinking projection and sequential extragradient method.** Assume that

(a1): $f = \sum_{i=1}^{N} f_i$ satisfy Assumption 1,
(a2): each f_i satisfy Assumption 2 for all $i \in I$,
(a3): f_1 is τ_1-Hölder continuous in the first or in second argument with constant Q_1 and f_i is τ_i-Hölder continuous in the first argument with constant Q_i for each $i \in I - \{1\}$.

Lemma 3.1. *Ω is closed convex subset of H.*

Proof. By Lemma 2.3, $FixT$ is closed convex subset of H.
Let $\{u^k\}$ be a sequence in $\mathrm{SEP}(f,C) = \mathrm{SEP}\big(\sum_{i=1}^{N} f_i, C\big)$ such that $u^k \to x^*$. For each $y \in C$, from the upper semicontinuity of $f(.,y)$, we have

$$0 \le \limsup_{k\to\infty} f(u^k, y) \le f(x^*, y).$$

Hence, $x^* \in SEP(f,C)$ implying that $SEP(f,C)$ is closed. Let $x_1^*,\ x_2^* \in \Omega$ and $\gamma \in [0,1]$. Then for all $y \in C$ we have

$$f(y, \gamma x_1^* + (1-\gamma)x_2^*) \le \gamma f(y, x_1^*) + (1-\gamma) f(y, x_2^*) \le 0.$$

For each $\gamma \in [0,1]$ set

$$p_\gamma = \gamma x_1^* + (1-\gamma)x_2^*.$$

Take $\sigma \in (0,1]$. Then by the convexity of $f(\sigma y + (1-\sigma)p_\gamma, .)$ one has

$$\begin{aligned} 0 &= f(\sigma y + (1-\sigma)p_\gamma, \sigma y + (1-\sigma)p_\gamma) \\ &\le \sigma f(\sigma y + (1-\sigma)p_\gamma, y) + (1-\sigma) f(\sigma y + (1-\sigma)p_\gamma, p_\gamma) \\ &= \sigma f(\sigma y + (1-\sigma)p_\gamma, y) + (1-\sigma) f(\sigma y + (1-\sigma)p_\gamma, \gamma x_1^* + (1-\gamma)x_2^*) \\ &\le \sigma f(\sigma y + (1-\sigma)p_\gamma, y), \quad \forall y \in C. \end{aligned}$$

Letting $\sigma \to 0$ and by and using the Assumption 1 (A2) it follows that $f(p_\gamma, y) \ge 0,\ \forall y \in C$. It means that $p_\gamma \in SEP(f,C)$.
Thus, $\mathrm{SEP}\big(\sum_{i=1}^{N} f_i, C\big)$ is convex. Hence, $\mathrm{SEP}\big(\sum_{i=1}^{N} f_i, C\big)$ is closed and convex subset of H.
Therefore, Ω is closed convex subset of H. □

Algorithm 3.1
Initialization: Choose $x^0 \in C$. Let $C = C_0 = D_0$, $\{\lambda_k\}$, $\{r_k\}$, $\{\delta_k\}$, and $\{\alpha_k\}$ be real sequences such that $0<\lambda_k,\ \ r_k \ge r>0,\ \ 0<\delta_k<1,\ \ 0<\alpha_k<1$.

Step 1: Solve N strongly convex optimization programs

$$y_i^k = \arg\min\{\lambda_k f_i(y_{i-1}^k, y) + \frac{1}{2}\|y_{i-1}^k - y\|^2 : y \in C\}, \quad i \in I,$$

where $y_0^k = x^k$. If $x^k = y_1^k = \ldots = y_N^k$, then take $v^k = x^k$ and go to Step 3. Otherwise, go to Step 2.

Step 2: Find v^k such that

$$v^k \in T_{r_k}^f(y_N^k) = \{v \in C : f(v,y) + \frac{1}{r_k}\langle y - v, v - y_N^k\rangle \ge 0,\ \forall y \in C\}.$$

Step 3: Find $t^k = \delta_k v^k + (1-\delta_k)T(v^k)$.
If $x^k = y_1^k = \ldots = y_N^k$ and $t^k = T(x^k)$, stop. Otherwise, go to Step 4.

Step 4: Evaluate $s^k = \alpha_k x^k + (1-\alpha_k)t^k$.

Step 5: Evaluate

$$x^{k+1} = P_{C_{k+1}\cap D_{k+1}}(x^0)$$

where

$$C_{k+1} = \{y \in C_k : \|s^k - y\|^2 \le \|x^k - y\|^2 + L\eta_k\},$$

$$D_{k+1} = \{y \in D_k : \|t^k - y\| \le \|v^k - y\| \le \|y_N^k - y\|\},$$

for $\eta_k = \sum_{i=2}^{N}\sum_{j=1}^{i-1}\lambda_k^{\frac{\tau_i}{2-\tau_j}+1} + \sum_{i=1}^{N}\lambda_k^{\frac{2}{2-\tau_i}}$, $L = 2QM$ such that $Q = \max\{Q_i : i \in I\}$ and $M = \max\{Q^{\frac{\tau_i}{2-\tau_j}} : i, j \in I\}$.

Step 6: Set $k := k+1$ and go to Step 1.

Lemma 3.2. *For the sequences $\{x^k\}$, $\{s^k\}$ and $\{y_i^k\}$ generated by Algorithm 3.1, we have*

(i): $\|y_N^k - y\|^2 \le \|x^k - y\|^2 - \sum_{i=2}^{N}\|y_{i-1}^k - y_i^k\|^2 + 2f(x^k, y) + L\eta_k, \quad \forall y \in C$

(ii):

$$\|s^k - x^*\|^2 \le \|x^k - x^*\|^2 - (1-\alpha_k)\sum_{i=2}^{N}\|y_{i-1}^k - y_i^k\|^2 + L\eta_k - \delta_k(1-\alpha_k)(1-\delta_k)\|T(v^k) - v^k\|^2, \quad \forall x^* \in \Omega,$$

(iii): $\|s^k - x^*\|^2 \le \|x^k - x^*\|^2 - (1-\alpha_k)\|v^k - y_N^k\|^2 + L\eta_k, \quad \forall x^* \in \Omega,$

where $\eta_k = \sum_{i=2}^{N}\sum_{j=1}^{i-1}\lambda_k^{\frac{\tau_i}{2-\tau_j}+1} + \sum_{i=1}^{N}\lambda_k^{\frac{2}{2-\tau_i}}$, $L = 2QM$ *such that* $Q = \max\{Q_i : i \in I\}$ *and* $M = \max\{Q^{\frac{\tau_i}{2-\tau_j}} : i, j \in I\}$.

Proof. *(i)* Using Lemma 2.5 and

$$y_i^k = \arg\min\{\lambda_k f_i(y_{i-1}^k, y) + \frac{1}{2}\|y_{i-1}^k - y\|^2 : y \in C\},$$

one has

$$0 \in \partial\left\{\lambda_k f_i(y_{i-1}^k, y) + \frac{1}{2}\|y_{i-1}^k - y\|^2\right\}(y_i^k) + N_C(y_i^k).$$

There exists $w_i \in \partial f_i(y_{i-1}^k, y_i^k)$ and $q_i \in N_C(y_i^k)$ such that

$$0 = \lambda_k w_i + y_i^k - y_{i-1}^k + q_i.$$

From the definition of the normal cone and $q_i \in N_C(y_i^k)$, we have

$$\langle y_{i-1}^k - \lambda_k w_i - y_i^k, y - y_i^k\rangle \le 0, \quad \forall y \in C. \tag{3.1}$$

Moreover, from $w_i \in \partial f_i(y_{i-1}^k, y_i^k)$, we have

$$\langle w_i, y - y_i^k\rangle \le f_i(y_{i-1}^k, y) - f_i(y_{i-1}^k, y_i^k) \quad \forall y \in H. \tag{3.2}$$

From (3.1) and (3.2), we have

$$\langle y_{i-1}^k - y_i^k, y - y_i^k\rangle \le \lambda_k(f_i(y_{i-1}^k, y) - f_i(y_{i-1}^k, y_i^k)), \quad \forall y \in C. \tag{3.3}$$

The result above together with

$$2\langle y_{i-1}^k - y_i^k, y - y_i^k\rangle = \|y_{i-1}^k - y_i^k\|^2 + \|y_i^k - y\|^2 - \|y_{i-1}^k - y\|^2,$$

yields

$$\|y_i^k - y\|^2 \le 2\lambda_k(f_i(y_{i-1}^k, y) - f_i(y_{i-1}^k, y_i^k)) - \|y_{i-1}^k - y_i^k\|^2 + \|y_{i-1}^k - y\|^2 \tag{3.4}$$

for all $y \in C$. Taking $y = y_{i-1}^k$ in (3.4) and using $f_i(x, x) = 0$ and τ_i-Hölder continuity of f_i in the first or second argument with constant Q_i, we get

$$\|y_{i-1}^k - y_i^k\|^2 \le -\lambda_k f_i(y_{i-1}^k, y_i^k) = |\lambda_k f_i(y_{i-1}^k, y_i^k)| \le \lambda_k Q_i\|y_{i-1}^k - y_i^k\|^{\tau_i}. \tag{3.5}$$

Hence, for each $i \in I$, we have

$$\|y_{i-1}^k - y_i^k\| \le (\lambda_k Q_i)^{\frac{1}{2-\tau_i}}. \tag{3.6}$$

Noting $x^k = y_0^k$ and using condition (a3) together with (3.5) and (3.6) gives

$$
\begin{aligned}
2\lambda_k\Big(\sum_{i=1}^N \big(f_i(y_{i-1}^k, y) - f_i(y_{i-1}^k, y_i^k)\big)\Big) &= 2\lambda_k\Big(\sum_{i=1}^N \big(f_i(y_{i-1}^k, y) - f_i(y_{i-1}^k, y_i^k)\big) + \sum_{i=2}^N f_i(x^k, y) - \sum_{i=2}^N f_i(x^k, y)\Big) \\
&\le 2\lambda_k\Big(\sum_{i=1}^N f_i(x^k, y) + \sum_{i=2}^N \big|f_i(y_{i-1}^k, y) - f_i(y_0^k, y)\big| + \sum_{i=1}^N \big|f_i(y_{i-1}^k, y_i^k)\big|\Big) \\
&\le 2\lambda_k\Big(\sum_{i=1}^N f_i(x^k, y) + \sum_{i=2}^N\sum_{j=1}^{i-1} \big|f_i(y_j^k, y) - f_i(y_{j-1}^k, y)\big| + \sum_{i=1}^N \big|f_i(y_{i-1}^k, y_i^k)\big|\Big) \\
&\le 2\lambda_k\Big(\sum_{i=1}^N f_i(x^k, y) + \sum_{i=2}^N\sum_{j=1}^{i-1} Q_i\|y_j^k - y_{j-1}^k\|^{\tau_i} + \sum_{i=1}^N Q_i\|y_{i-1}^k - y_i^k\|^{\tau_i}\Big) \\
&\le 2\lambda_k\Big(\sum_{i=1}^N f_i(x^k, y) + \sum_{i=2}^N\sum_{j=1}^{i-1} Q_i(\lambda_k Q_j)^{\frac{\tau_i}{2-\tau_j}} + \sum_{i=1}^N Q_i(\lambda_k Q_i)^{\frac{\tau_i}{2-\tau_i}}\Big) \\
&\le 2\lambda_k f(x^k, y) + 2\lambda_k\Big(\sum_{i=2}^N\sum_{j=1}^{i-1} Q_i(\lambda_k Q_j)^{\frac{\tau_i}{2-\tau_j}} + \sum_{i=1}^N Q_i(\lambda_k Q_i)^{\frac{\tau_i}{2-\tau_i}}\Big). \qquad (3.7)
\end{aligned}
$$

Let $Q = \max\{Q_i : i \in I\}$ and $M = \max\{Q^{\frac{\tau_i}{2-\tau_j}} : i, j \in I\}$. From (3.7), we have

$$
2\lambda_k\Big(\textstyle\sum_{i=1}^N \big(f_i(y_{i-1}^k, y) - f_i(y_{i-1}^k, y_i^k)\big)\Big) \le 2\lambda_k f(x^k, y) + L\Big(\sum_{i=2}^N\sum_{j=1}^{i-1} \lambda_k^{\frac{\tau_i}{2-\tau_j}+1} + \sum_{i=1}^N \lambda_k^{\frac{2}{2-\tau_i}}\Big) \qquad (3.8)
$$

where $L = 2QM$. Thus, combining (3.4), (3.7), and (3.8) it follows

$$
\begin{aligned}
\|y_N^k - y\|^2 &\le \|x^k - y\|^2 - \sum_{i=1}^N \|y_{i-1}^k - y_i^k\|^2 + 2\lambda_k\Big(\sum_{i=1}^N \big(f_i(y_{i-1}^k, y) - f_i(y_{i-1}^k, y_i^k)\big)\Big) \\
&\le \|x^k - y\|^2 - \sum_{i=1}^N \|y_{i-1}^k - y_i^k\|^2 + 2\lambda_k f(x^k, y) + L\eta_k \qquad (3.9)
\end{aligned}
$$

where $\eta_k = \sum_{i=2}^N\sum_{j=1}^{i-1} \lambda_k^{\frac{\tau_i}{2-\tau_j}+1} + \sum_{i=1}^N \lambda_k^{\frac{2}{2-\tau_i}}$. Since $\Omega \subset C$, take $y = x^* \in \Omega$ in (3.9). From the pseudomonotonicity of f, we obtain

$$
\|y_N^k - x^*\|^2 \le \|x^k - x^*\|^2 - \sum_{i=1}^N \|y_{i-1}^k - y_i^k\|^2 + L\eta_k, \quad \forall x^* \in \Omega. \qquad (3.10)
$$

(ii) Let $x^* \in \Omega$. Then,

$$
f(x^*, y) + \frac{1}{r_k}\langle y - x^*, x^* - x^*\rangle = f(x^*, y) = \sum_{i=1}^N f_i(x^*, y) \ge 0, \ \forall y \in C.
$$

Thus by definition of $T_{r_k}^f$, $x^* = T_{r_k}^f(x^*)$ and hence

$$
\begin{aligned}
\|v^k - x^*\|^2 = \|T_{r_k}^f(y_N^k) - T_{r_k}^f(x^*)\|^2 &\le \langle T_{r_k}^f(y_N^k) - T_{r_k}^f(x^*), y_N^k - x^*\rangle \\
&\le \|T_{r_k}^f(y_N^k) - T_{r_k}^f(x^*)\|\|y_N^k - x^*\|
\end{aligned}
$$

implying that

$$
\|v^k - x^*\| \le \|y_N^k - x^*\|. \qquad (3.11)
$$

By definition of t^k and Lemma 2.4, we have

$$\begin{aligned}\|t^k - x^*\|^2 &= \|\delta_k v^k + (1-\delta_k)T(v^k) - x^*\|^2 \\ &= \|\delta_k(v^k - x^*) + (1-\delta_k)(T(v^k) - x^*)\|^2 \\ &= \delta_k\|v^k - x^*\|^2 + (1-\delta_k)\|T(v^k) - x^*\|^2 - \delta_k(1-\delta_k)\|T(v^k) - v^k\|^2 \\ &= \delta_k\|v^k - x^*\|^2 + (1-\delta_k)\|T(v^k) - T(x^*)\|^2 - \delta_k(1-\delta_k)\|T(v^k) - v^k\|^2 \\ &\leq \delta_k\|v^k - x^*\|^2 + (1-\delta_k)\|v^k - x^*\|^2 - \delta_k(1-\delta_k)\|T(v^k) - v^k\|^2 \\ &= \|v^k - x^*\|^2 - \delta_k(1-\delta_k)\|T(v^k) - v^k\|^2. \end{aligned} \tag{3.12}$$

Again by definition of s^k, and using (3.10), (3.11), and (3.12), we have

$$\begin{aligned}\|s^k - x^*\|^2 &\leq \alpha_k\|x^k - x^*\|^2 + (1-\alpha_k)\|\|t^k - x^*\|^2 \\ &\leq \alpha_k\|x^k - x^*\|^2 + (1-\alpha_k)(\|v^k - x^*\|^2 - \delta_k(1-\delta_k)\|T(v^k) - v^k\|^2) \\ &\leq \|x^k - x^*\|^2 - (1-\alpha_k)\sum_{i=1}^{N}\|y_{i-1}^k - y_i^k\|^2 + (1-\alpha_k)L\lambda_k^{\frac{2}{2-\tau}} - \delta_k(1-\alpha_k)(1-\delta_k)\|T(v^k) - v^k\|^2.\end{aligned}$$

(iii) By Lemma 2.7,

$$\begin{aligned}\|v^k - x^*\|^2 &= \|T_{r_k}^f(y_N^k) - T_{r_k}^f(x^*)\|^2 \\ &\leq \langle T_{r_k}^f(y_N^k) - T_{r_k}^f(x^*), y_N^k - x^*\rangle \\ &\leq \langle v^k - x^*, y_N^k - x^*\rangle \\ &= \tfrac{1}{2}(\|v^k - x^*\|^2 + \|y_N^k - x^*\|^2 - \|v^k - y_N^k\|^2)\end{aligned}$$

implying that

$$\|v^k - x^*\| \leq \|y_N^k - x^*\|^2 - \|v^k - y_N^k\|^2. \tag{3.13}$$

Then, by definition of s^k and Lemma 2.4 combined with results in (3.12), (3.13), and (3.10) results

$$\begin{aligned}\|s^k - x^*\|^2 &\leq \alpha_k\|x^k - x^*\|^2 + (1-\alpha_k)\|t^k - x^*\|^2 \\ &\leq \alpha_k\|x^k - x^*\|^2 + (1-\alpha_k)\|v^k - x^*\|^2 \\ &\leq \|x^k - x^*\|^2 - (1-\alpha_k)\|v^k - y_N^k\|^2 + L\eta_k.\end{aligned} \tag{3.14}$$

This ends the proof of the Lemma. □

Remark 3.3. *For $k = 0$ we have $x^* \in C = C_0 = D_0$. From (3.11) and (3.12), we see that for all $x^* \in \Omega$*

$$\|t^k - x^*\| \leq \|v^k - x^*\| \leq \|y_N^k - x^*\|, \quad k \geq 0$$

and from (3.14), we also see that for all $x^* \in \Omega$

$$\|s^k - x^*\|^2 \leq \|x^k - x^*\|^2 + L\eta_k, \quad k \geq 0.$$

Therefore, $\Omega \subset C_k \cap D_k$, $k \geq 0$.
Let

$$E_k = \{y \in D_{k-1} : \|t^k - y\| \leq \|v^k - y\|\} \quad \forall k \geq 1,$$
$$F_k = \{y \in D_{k-1} : \|v^k - y\| \leq \|y_N^k - y\|\} \quad \forall k \geq 1.$$

Thus, $D_k = D_{k-1} \cap E_k \cap F_k$, for $k \geq 1$. Note that E_k and F_k are either the halfspaces or the whole space H for all $k \geq 1$. Hence, they are closed and convex. Since $D_0 = C$, E_k and F_k are closed and convex for all $k \geq 1$, D_k is also closed and convex $k \geq 0$. Moreover, for each k, C_k is closed and convex, (see Lemma 1.3 in [14]).
Therefore, $C_k \cap D_k$ is nonempty closed and convex subset of H. It is easy to see that

$$\ldots \subset C_{k+1} \cap D_{k+1} \subset C_k \cap D_k \subset \ldots \subset C_1 \cap D_1 \subset C_0 \cap D_0 = C.$$

Remark 3.4. **(a):** *Since the problems in step 1 are strongly convex and C is nonempty, they are uniquely solvable.*

(b): *If condition 1 and 2 is satisfied, f satisfies (C1)-(C4) and hence by Combettes and Hirstoaga in [6], for each r_k and for each $x \in C$ the mapping $T^f_{r_k}$ is single-valued.*

(c): *From remark 3.3, $C_k \cap D_k$ is nonempty closed and convex subset of H, for each k. Moreover, $\emptyset \neq \Omega \subset C_k \cap D_k$ for all $k \geq 0$ and hence $P_{C_{k+1}\cap D_{k+1}}$ is well defined.*

Hence, Algorithm 3.1 is well defined.

Remark 3.5. **(a):** *If $x^k = y^k_1 = \ldots = y^k_N$, then from (3.4) we have $f_i(y^k_{i-1}, y) \geq 0, \ \ \forall y \in C$, hence $\sum_{i=1}^N f_i(x^k, y) \geq 0, \ \ \forall y \in C$, implying that $x^k \in SEP\big(\sum_{i=1}^N f_i, C\big)$. Moreover,*

$$f(x^k, y) + \frac{1}{r_k}\langle y - x^k, x^k - y^k_N\rangle = f(x^k, y) \geq 0, \ \ \forall y \in C.$$

Hence, $v^k = T^f_{r_k}(y^k_N) = x^k$.

(b): *If $x^k = y^k_1 = \ldots = y^k_N$ and $t^k = T(x^k)$ then $x^k \in SEP\big(\sum_{i=1}^N f_i, C\big)$ and $x^k \in FixT$, i.e., $x^k \in \Omega$. Therefore, Algorithm 3.1 terminates at step 3 when $x^k = y^k_1 = \ldots = y^k_N$ and $t^k = T(x^k)$.*

By the argument of Remark 3.5, we can conclude that if Algorithm 3.1 terminates at some iterate k, then x^k is the solution of (1.2). Otherwise, if Algorithm 3.1 does not stop, then we have the following strong convergence Theorem.

Theorem 3.6. *If the real sequences $\{\alpha_k\}$, $\{\delta_k\}$, $\{\lambda_k\}$ satisfy the following restrictions:*

(i): $0 < \liminf_{k\to\infty} \delta_k \leq \limsup_{k\to\infty} \delta_k < 1$,

(ii): $\limsup_{k\to\infty} \alpha_k < 1$,

(iii): *for each $i \in \{2, 3, \ldots, N\}$ and $j \in \{1, 2, \ldots, N-1\}$*

$$\lim_{k\to+\infty} \lambda_k^{\frac{\tau_i}{2-\tau_j}+1} = 0,$$

,

(iv): *for each $i \in I$*

$$\lim_{k\to+\infty} \lambda_k^{\frac{2}{2-\tau_i}} = 0,$$

then the sequences $\{x^k\}$, $\{y^k_N\}$, $\{t^k\}$, $\{s^k\}$ and $\{v^k\}$ generated by Algorithm 3.1 strongly converge to some point $p \in \Omega$ where $p = P_\Omega(x^0)$.

Proof. The Theorem is proved through claims.
Claim 1: The sequences $\{x^k\}$, $\{v^k\}$, $\{y^k_i\}$ $(i \in I)$, $\{t^k\}$ and $\{s^k\}$ converge to some point p in C.
Proof of claim 1: Put $w = P_\Omega(x^0)$ (we note that Ω is closed and convex). From $\Omega \subset C_k \cap D_k$ and $x^k = P_{C_k\cap D_k}(x^0)$ for all $k \geq 0$, we get

$$\|x^k - x^0\| \leq \|w - x^0\|.$$

Also from $x^k = P_{C_k\cap D_k}(x^0)$ and $x^{k+1} \in C_{k+1} \cap D_{k+1} \subset C_k \cap D_k$, we have

$$\|x^k - x^0\| \leq \|x^{k+1} - x^0\|.$$

It follows that the sequence $\{\|x^k - x^0\|\}$ is bounded and nondecreasing. Hence $\lim_{k\to+\infty} \|x^k - x^0\|$ exists. For $m > k$ we have $x^m \in C_m \cap D_m \subset C_k \cap D_k$. Now by applying Lemma 2.2, we have

$$\|x^m - x^k\|^2 \leq \|x^m - x^0\|^2 - \|x^k - x^*\|^2$$

Since $\lim_{k\to+\infty} \|x^k - x^0\|$ exists, it follows that $\{x^k\}$ is a Cauchy sequence, and hence there exists $p \in C$ such that $\lim_{k\to+\infty} x^k = p$. Putting $m = k + 1$, in the above inequality, we have

$$\lim_{k\to+\infty} \|x^{k+1} - x^k\| = 0$$

In view of $x^{k+1} = P_{C_{k+1}\cap D_{k+1}}(x^0) \in C_{k+1} \cap D_{k+1} \subset C_{k+1}$ and $\eta_k \to 0$, we see that

$$\|s^k - x^{k+1}\|^2 \leq \|x^k - x^{k+1}\|^2 + L\eta_k.$$

It follows that $\|s^k - x^{k+1}\| \to 0$. This implies that $s^k \to p$. As a result of the inequalities $\|s^k - x^k\| \le \|s^k - x^{k+1}\| + \|x^k - x^{k+1}\|$, we have

$$\lim_{k\to+\infty} \|s^k - x^k\| = 0 \tag{3.15}$$

Note that

$$\begin{aligned} \|x^k - x^*\|^2 - \|s^k - x^*\|^2 + L\eta_k &\le (\|x^k - x^*\| + \|s^k - x^*\|)(\|x^k - x^*\| - \|s^k - x^*\|) + L\eta_k \\ &\le (\|x^k - x^*\| + \|s^k - x^*\|)\|x^k - s^k\| + L\eta_k. \end{aligned} \tag{3.16}$$

From Lemma 3.2 (ii) and from (3.16) above

$$(1-\alpha_k)\sum_{i=1}^N \|y_{i-1}^k - y_i^k\|^2 \le (\|x^k - x^*\| + \|s^k - x^*\|)\|x^k - s^k\| + L\eta_k, \tag{3.17}$$

$$\delta_k(1-\alpha_k)(1-\delta_k)\|T(v^k) - v^k\|^2 \le (\|x^k - x^*\| + \|s^k - x^*\|)\|x^k - s^k\| + L\eta_k \tag{3.18}$$

and

$$(1-\alpha_k)\|v^k - y_N^k\|^2 \le (\|x^k - x^*\| + \|s^k - x^*\|)\|x^k - s^k\| + L\eta_k. \tag{3.19}$$

Combining (3.15), (3.17), (3.18), and (3.19) together with $\alpha_k \in (0,1)$, $0 < \liminf_{k\to\infty} \delta_k \le \limsup_{k\to\infty} \delta_k < 1$, $\limsup_{k\to\infty} \alpha_k < 1$, $\eta_k \to 0$ and s^k, $x^k \to p$, we get

$$\lim_{k\to+\infty} \|y_{i-1}^k - y_i^k\| = \lim_{k\to+\infty} \|T(v^k) - v^k\| = \lim_{k\to+\infty} \|v^k - y_N^k\| = 0. \tag{3.20}$$

Since $\|x^k - y_i^k\| = \|y_0^k - y_i^k\| \le \sum_{j=1}^i \|y_{j-1}^k - y_j^k\|$ for each $i \in I$, and using (3.20), we have

$$\lim_{k\to+\infty} \|x^k - y_i^k\| = 0, \quad \forall i \in I. \tag{3.21}$$

From (3.20) and (3.21) and x^k, $s^k \to p$, we get that y_i^k, $v^k \to p$.
Since $x^{k+1} = P_{C_{k+1}\cap D_{k+1}}$ we have $x^{k+1} \in C_{k+1} \cap D_{k+1}$. Thus, by definition of D_{k+1}, we have

$$\|t^k - x^{k+1}\| \le \|v^k - x^{k+1}\| \le \|y_N^k - x^{k+1}\|.$$

Using $\|t^k - x^{k+1}\| \le \|v^k - x^{k+1}\|$, $\|v^k - x^{k+1}\| \le \|v^k - p\| + \|x^{k+1} - p\|$ and x^k, $v^k \to p$, we obtain $t^k \to p$.
Claim 2: $p \in \Omega$.
Proof of claim 2: By $v^k = T_{r_k}^f(y_N^k)$

$$f(v^k, y) + \frac{1}{r_k}\langle y - v^k, v^k - y_N^k\rangle \ge 0, \quad \forall y \in C.$$

Since f is monotone on C, we also have

$$\frac{1}{r_k}\langle y - v^k, v^k - y_N^k\rangle \ge f(y, v^k), \ \forall y \in C.$$

Since $v^k - y_N^k \to 0$, $v^k \to p$ and by the lower semicontinuity and the convexity of of $f(y,.)$ we have $f(y,p) \le 0$ for each $y \in C$.
Let $y \in C$ and $\gamma \in (0.1)$. Then, using convexity of $f(y,.)$, we have

$$\begin{aligned} 0 &= f(\gamma y + (1-\gamma)p, \gamma y + (1-\gamma)p) \\ &\le \gamma f(\gamma y + (1-\gamma)p, y) + (1-\gamma) f(\gamma y + (1-\gamma)p, p) \\ &\le \gamma f(\gamma y + (1-\gamma)p, y), \quad \forall y \in C. \end{aligned}$$

Letting $\gamma \to 0$ and using (A2) of Assumption 1 it follows that $f(p,y) \ge 0$, $\forall y \in C$. It means that $p \in SEP(f,C)$. Moreover, (3.20), $v^k \to p$ and demiclosedness of T gives $p \in FixT$. Therefore, $p \in \Omega = \mathrm{SEP}\big(\sum_{i=1}^N f_i, C\big) \bigcap FixT$.
Claim 3: $p = P_\Omega(x^0)$.
Proof of claim 3: Since $x^k = P_{C_k\cap D_k}(x^0)$, by Lemma 2.1, we have

$$\langle x^0 - x^k, y - x^k\rangle \le 0 \ \ \forall y \in C_k \cap D_k.$$

Since $\Omega \subset C_k \cap D_k$, we have

$$\langle x^0 - p, y - p\rangle \le 0 \ \ \forall y \in \Omega.$$

Now by Lemma 2.1, we obtain that $p = P_\Omega(x^0)$.
This completes the proof. □

Remark 3.7. **(i):** *For* $i, j \in I$,

$$1 < \frac{\tau_i}{2-\tau_j} + 1 \le 2 \quad \textit{and} \quad 1 < \frac{2}{2-\tau_i} \le 2.$$

It is easy to choose a sequence $\{\lambda_k\}$ *satisfying conditions (iii) and (iv), for example,* $\lambda_k = \frac{1}{k^\gamma}$ *with* $\gamma \in (0, +\infty)$.

(ii): *If* $\tau_i = \tau$, *then the conditions (iii) and (iv) in Theorem 3.6 is reduced to only one condition*

$$\lim_{k\to+\infty} \lambda_k^{\frac{2}{2-\tau}} = 0.$$

(iii): *If* $\tau_i = \tau$, *alternatively we can also proceed as*

$$\|y_N^k - y\|^2 \le \|x^k - y\|^2 - \sum_{i=2}^{N} \|y_{i-1}^k - y_i^k\|^2 + 2\lambda_k \sum_{i=2}^{N} f_i(x^k, y) + L'\lambda_k^{\frac{2}{2-\tau}}$$

where $L' = \sum_{i=2}^N \sum_{j=1}^{i-1} Q_i Q_j^{\frac{\tau}{2-\tau}} + \sum_{i=1}^N Q_i Q_i^{\frac{2}{2-\tau}}$.

Letting $T = Id_C$ in Algorithm 3.1, then from Theorem 3.6 we obtain the following result solving (1.3) by assuming conditions (a1), (a2) and (a3) are satisfied.

Corollary 3.7.1. *Let* $\{r_k\}$, $\{\alpha_k\}$ *and* $\{\lambda_k\}$ *be real sequences such that* $r_k \ge r{>}0$, $0{<}\alpha_k{<}1$, $0{<}\lambda_k$. *If* $\{\alpha_k\}$ *and* $\{\lambda_k\}$ *satisfy the restrictions (ii), (iii) and (iv) in Theorem 3.6, then the sequences* $\{x^k\}$, $\{y_N^k\}$, $\{s^k\}$ *and* $\{v^k\}$ *generated by iterative algorithm*

$$\begin{cases} x^0 \in C = C_0 = D_0, \\ x^k = y_0^k, \\ y_i^k = \arg\min\{\lambda_k f_i(y_{i-1}^k, y) + \frac{1}{2}\|y_{i-1}^k - y\|^2 : y \in C\}, \quad i \in I, \\ v^k \in T_{r_k}^f(y_N^k), \\ s^k = \alpha_k x^k + (1-\alpha_k) v^k, \\ C_{k+1} = \{y \in C_k : \|s^k - y\|^2 \le \|x^k - y\|^2 + L\eta_k\}, \\ D_{k+1} = \{y \in D_k : \|v^k - y\| \le \|y_N^k - y\|\}, \\ x^{k+1} = P_{C_{k+1}\cap D_{k+1}}(x^0), \end{cases}$$

strongly converge to $p \in SEP\big(\sum_{i=1}^N f_i, C\big)$ *where* $p = P_{SEP\left(\sum_{i=1}^N f_i, C\right)}(x^0)$.

Note that in Corollary 3.7.1

$$\eta_k = \sum_{i=2}^{N} \sum_{j=1}^{i-1} \lambda_k^{\frac{\tau_i}{2-\tau_j}+1} + \sum_{i=1}^{N} \lambda_k^{\frac{2}{2-\tau_i}} \text{ and } L = 2QM$$

where $Q = \max\{Q_i : i \in I\}$ and $M = \max\{Q^{\frac{\tau_i}{2-\tau_j}} : i, j \in I\}$.

3.2. **Shrinking projection and parallel extragradient method.** Assume that

(b1): $f = \sum_{i=1}^N f_i$ satisfy Assumption 1,
(b2): each f_i satisfy Assumption 2 for all $i \in I$,
(b3): f_i is τ_i-Hölder continuous in the first or in second argument with constant Q_i for $i \in I$.

The solution set Ω of problem (1.2) is closed convex subset of H by the same reasoning as in Lemma 3.1.

In the following, we use the computation of intermediate approximations parallelly instead of sequentially.

Algorithm 3.2

Initialization: Choose $x^0 \in C$. Let $C_0 = D_0 = C$ and $\{\lambda_k\}$, $\{r_k\}$, $\{\delta_k\}$, $\{\alpha_k\}$ are real sequences such that $0<\lambda_k$, $r_k \geq r>0$, $0<\delta_k<1$, $0<\alpha_k<1$.

Step 1: Solve N strongly convex optimization programs

$$y_i^k = \arg\min\{\lambda_k f_i(x^k, y) + \frac{1}{2}\|x^k - y\|^2 : y \in C\}, \quad i \in I.$$

If $x^k = y_1^k = \ldots = y_N^k$, then take $v^k = x^k$ and go to Step 4. Otherwise, go to Step 2.

Step 2: Evaluate

$$z^k = \frac{1}{N}\sum_{i=1}^{N} y_i^k.$$

Step 3: Find v^k such that

$$v^k \in T_{r_k}^f(z^k) = \{v \in C : f(v, y) + \frac{1}{r_k}\langle y - v, v - z^k\rangle \geq 0, \ \forall y \in C\}.$$

Step 4: Find $t^k = \delta_k v^k + (1 - \delta_k)T(v^k)$.

If $x^k = y_1^k = \ldots = y_N^k$ and $t^k = T(x^k)$, stop. Otherwise, go to Step 5.

Step 5: Evaluate $s^k = \alpha_k x^k + (1 - \alpha_k)t^k$.

Step 6: Evaluate

$$x^{k+1} = P_{C_{k+1}\cap D_{k+1}}(x^0)$$

where

$$C_{k+1} = \{y \in C_k : \|s^k - y\|^2 \leq \|x^k - y\|^2 + L\omega_k\},$$
$$D_{k+1} = \{y \in D_k : \|t^k - y\| \leq \|v^k - y\| \leq \|z^k - y\|\},$$

for $\omega_k = \sum_{i=1}^{N} \lambda_k^{\frac{2}{2-\tau_i}}$, $L = \frac{2QM}{N}$ such that $Q = \max\{Q_i : i \in I\}$ and $M = \max\{Q^{\frac{\tau_i}{2-\tau_i}} : i \in I\}$.

Step 7: Set $k := k + 1$ and go to step 1.

Lemma 3.8. *For the sequences $\{x^k\}$, $\{z^k\}$, $\{v^k\}$ and $\{s^k\}$ generated by Algorithm 3.2, we have*

(i): $\|z^k - y\|^2 \leq \|x^k - y\|^2 - \frac{1}{N}\sum_{i=1}^{N}\|x^k - y_i^k\|^2 + \frac{2\lambda_k}{N} f(x^k, y) + L\omega_k, \quad \forall y \in C,$

(ii): $\|s^k - x^*\|^2 \leq \|x^k - x^*\|^2 - (1 - \alpha_k)\frac{1}{N}\sum_{i=1}^{N}\|x^k - y_i^k\|^2 - \delta_k(1 - \delta_k)\|T(y_N^k) - y_N^k\|^2 + L\omega_k, \quad \forall x^* \in \Omega,$

(iii): $\|s^k - x^*\|^2 \leq \|x^k - x^*\|^2 - (1 - \alpha_k)\|v^k - z^k\|^2 + L\omega_k, \quad \forall x^* \in \Omega,$

where $\omega_k = \sum_{i=1}^{N} \lambda_k^{\frac{2}{2-\tau_i}}$, $L = \frac{2QM}{N}$ such that $Q = \max\{Q_i : i \in I\}$ and $M = \max\{Q^{\frac{\tau_i}{2-\tau_i}} : i \in I\}$.

Proof. (i) By the defintion of z^k, we obtain

$$\|z^k - x^k\| \leq \frac{1}{N}\sum_{i=1}^{N}\|y_i^k - x^k\|. \tag{3.22}$$

From step 1 of Algorithm 2, we have

$$\langle y_i^k - x^k, x^k - y\rangle \leq \lambda_k(f_i(x^k, y) - f_i(x^k, y_i^k)) - \|x^k - y_i^k\|^2, \quad \forall y \in C, \tag{3.23}$$

for all $i \in I$.
Setting $y = x^k$ in (3.23), we have

$$\|x^k - y_i^k\|^2 \leq -\lambda_k f_i(x^k, y_i^k) \leq |\lambda_k f_i(x^k, y_i^k)| \leq \lambda_k Q_i \|x^k - y_i^k\|^{\tau_i},$$

for all $i \in I$.
Thus,

$$\|x^k - y_i^k\| \leq (\lambda_k Q_i)^{\frac{1}{2-\tau_i}}, \ \ \forall i \in I. \tag{3.24}$$

Using (3.23) and (3.24) and condition (b3), we have for $y \in C$,

$$\begin{aligned}
\sum_{i=1}^{N} \langle y_i^k - x^k, x^k - y\rangle &\leq \lambda_k \sum_{i=1}^{N} (f_i(x^k, y) - f_i(x^k, y_i^k)) \\
&= \lambda_k \Big(f(x^k, y) - \sum_{i=1}^{N} f_i(x^k, y_i^k) \Big) \\
&\leq \lambda_k \Big(f(x^k, y) + \sum_{i=1}^{N} |f_i(x^k, y_i^k)| \Big) \\
&\leq \lambda_k \Big(f(x^k, y) + \sum_{i=1}^{N} Q_i \|x^k - y_i^k\|^{\tau_i} \Big) \\
&\leq \lambda_k f(x^k, y) + QM \sum_{i=1}^{N} \lambda_k^{\frac{2}{2-\tau_i}} \qquad (3.25)
\end{aligned}$$

where $Q = \max\{Q_i : i \in I\}$ and $M = \max\{Q^{\frac{\tau_i}{2-\tau_i}} : i \in I\}$. Hence, the result follows from

$$\langle z^k - x^k, x^k - y\rangle = \frac{1}{N} \sum_{i=1}^{N} \langle y_i^k - x^k, x^k - y\rangle, \ \ \forall y \in C,$$

and

$$\|z^k - y\|^2 = \|z^k - x^k\|^2 + \|x^k - y\|^2 + 2\langle z^k - x^k, x^k - y\rangle, \ \ \forall y \in C,$$

together with (3.22) and (3.25). The results (ii) and (iii) are obtained using the same technique as in Lemma 3.2. ☐

Remark 3.9. (a) *If $x^k = y_1^k = \ldots = y_N^k$, then we have $f_i(y_{i-1}^k, y) \geq 0, \ \forall y \in C$, hence $\sum_{i=1}^{N} f_i(x^k, y) \geq 0, \ \forall y \in C$, implying that $x^k \in SEP\big(\sum_{i=1}^{N} f_i, C\big)$. Moreover,*

$$f(x^k, y) + \frac{1}{r_k}\langle y - x^k, x^k - z^k\rangle = f(x^k, y) \geq 0, \ \ \forall y \in C.$$

Hence, $v^k = T_{r_k}^f(z^k) = x^k$.

(b) *If $x^k = y_1^k = \ldots = y_N^k$ and $t^k = T(x^k)$ then $x^k \in SEP\big(\sum_{i=1}^{N} f_i, C\big)$ and $x^k \in FixT$, i.e., $x^k \in \Omega$. Therefore, Algorithm 3.2 terminates at Step 4 when $x^k = y_1^k = \ldots = y_N^k$ and $t^k = T(x^k)$.*

Theorem 3.10. *If the real sequences $\{\alpha_k\}$, $\{\delta_k\}$, $\{\lambda_k\}$ satisfy the following restrictions:*

(i): $0 < \liminf_{k\to\infty} \delta_k \leq \limsup_{k\to\infty} \delta_k < 1$,

(ii): $\limsup_{k\to\infty} \alpha_k < 1$,

(iii): *for each $i \in I$,* $\lim_{k\to+\infty} \lambda_k^{\frac{2}{2-\tau_i}} = 0$,

then sequences $\{x^k\}$, $\{z^k\}$, $\{v^k\}$, $\{t^k\}$ and $\{s^k\}$ generated by Algorithm 3.2 strongly converge to $p \in \Omega$ where $p = P_\Omega(x^0)$.

Proof. We omitted the proof as it is similar to the proof of Theorem 3.6. □

Letting $T = Id_C$ in Algorithm 3.2, then from Theorem 3.10 we obtain the following algorithm solving (1.3) by assuming conditions (b1), (b2) and (b3) are satisfied.

Corollary 3.10.1. *Let $\{r_k\}$, $\{\alpha_k\}$ and $\{\lambda_k\}$ be real sequences such that $r_k \geq r>0$, $0<\alpha_k<1$, $0<\lambda_k$. If $\{\alpha_k\}$ and $\{\lambda_k\}$ satisfy the restrictions (ii) and (iii) in Theorem 3.10, then the sequences $\{x^k\}$, $\{z^k\}$, $\{v^k\}$ and $\{s^k\}$ generated by iterative algorithm*

$$\begin{cases} x^0 \in C = C_0 = D_0, \\ y_i^k = \arg\min\{\lambda_k f_i(x^k, y) + \frac{1}{2}\|x^k - y\|^2 : y \in C\}, \quad i \in I, \\ z^k = \frac{1}{N}\sum_{i=1}^N y_i^k, \\ v^k \in T_{r_k}^f(z^k), \\ s^k = \alpha_k x^k + (1-\alpha_k)v^k, \\ C_{k+1} = \{y \in C_k : \|s^k - y\|^2 \leq \|x^k - y\|^2 + L\omega_k\}, \\ D_{k+1} = \{y \in D_k : \|v^k - y\| \leq \|z^k - y\|\}, \\ x^{k+1} = P_{C_{k+1}\cap D_{k+1}}(x^0), \end{cases}$$

strongly converge to $p \in SEP(\sum_{i=1}^N f_i, C)$ where $p = P_{SEP(\sum_{i=1}^N f_i, C)}(x^0)$.

Remark 3.11. *Comparing Hölder continuity conditions (a3) in Algorithm 3.1 and (b3) in Algorithm 3.2, we can see that the condition (b3) is a bit more relaxed Hölder continuity condition than (a3), because in condition (a3) at least $N-1$ bifunctions must be Hölder continuous in the first argument out of N Hölder continuous bifunctions as a summand of the main bifunction.*

4. Application

Lemma 4.1. *[14, Proposition 4.34] Suppose C is closed convex subset of a Hilbert space H and $U_j : C \to C$ be nonexpansive mappings for $j \in J = \{1, \dots, N'\}$ such that $\bigcap_{j=1}^{N'}[FixU_j] \neq \emptyset$. Let $U(x) := \sum_{j=1}^{N'} \theta_j U_j(x)$ with $0 < \theta_j \leq 1$ for every $j \in J$ and $\sum_{j=1}^{N'} \theta_j = 1$. Then, U is nonexpansive and $FixU = \bigcap_{i=1}^{N'}[FixU_i]$.*

The following are some typical problems for equilibrium problem and a finite family of nonexpansive mappings in a Hilbert space that can be reformulated as problem (1.2).

4.0.1. ***Equilibrium problems and a finite family of nonexpansive mappings***. Let $I = \{1, \dots, N\}$, $J = \{1, \dots, N'\}$ and H is a real Hilbert spaces. Suppose C be nonempty closed convex subset of H, $f_i : C \times C \to \mathbb{R}$ be bifunctions for each $i \in I$, and $T_j : C \to C$ be nonexpansive operators for each $j \in J$. Consider the problem

$$\text{find } x^* \in C \text{ such that } \begin{cases} x^* \in FixT_j, \quad \forall j \in J, \\ \sum_{i=1}^N f_i(x^*, y) \geq 0, \ \forall y \in C. \end{cases} \tag{4.1}$$

Take $\{\theta_1, \theta_2, \dots, \theta_{N'}\} \subset (0, 1]$ such that $\sum_{j=1}^{N'} \theta_i = 1$. Set $T(x) = \sum_{j=1}^{N'} \theta_i T_j(x)$. By Lemma 4.1, T is nonexpansive and $FixT = \bigcap_{j=1}^{N'}[FixT_j]$. Therefore,

$$\Big(\bigcap_{j=1}^{N'} FixT_j\Big) \bigcap SEP\Big(\sum_{i=1}^N f_i, C\Big) = FixT \bigcap SEP\Big(\sum_{i=1}^N f_i, C\Big) = \Omega,$$

and hence, problem (4.1) is of the form (1.2).

4.0.2. ***Equilibrium problem over the intersection of closed convex sets***. Let $\{C_1,\dots,C_{N'}\}$ be finite collection of closed convex subsets of a real Hilbert space H such that $\bigcap_{j\in J} C_j \neq \emptyset$ where $J=\{1,\dots,N'\}$. Let C be closed convex subset of H containing $\bigcup_{j\in J} C_j$ and $f_i : C\times C \to \mathbb{R}$ be bifunctions for $i \in I \in \{1,\dots,N\}$. Consider the problem

$$\text{find } x^* \in D := \bigcap_{j\in J} C_j \text{ such that } \sum_{i=1}^{N} f_i(x^*,y) \geq 0, \ \ \forall y \in C. \tag{4.2}$$

In this case, we can take $T_j = P_{C_j}$ for each $j \in J$, i.e., T_j is the projection map on C_j which is nonexpansive, hence using Lemma 4.1, we can reformulate the problem (4.2) in to problem (1.2).

4.0.3. ***Common solution of equilibrium problem and maximal monotone operator***. Let $U_j : H \to 2^H$ be maximal monotone operators and $f_i : H\times H \to \mathbb{R}$ be bifunctions for each $j \in J=\{1,\dots,N'\}$, $i\in I=\{1,\dots,N\}$ where H is a real Hilbert space. Consider the problem of finding $x^*\in H$ with a property

$$0 \in U_j(x^*), \ \ \forall j \in J \text{ such that } \sum_{i=1}^{N} f_i(x^*,y) \geq 0, \ \ \forall y \in H. \tag{4.3}$$

It is well-known (see e.g. [21]) that the operator $T_j = (U_j + \varepsilon Id_H)^{-1}$ with $\varepsilon > 0$ is defined everywhere, single-valued, nonexpansive on the whole space and its fixed point set coincides with the solution set of the inclusion $0 \in M_j(x^*)$. Therefore, Algorithm 3.1 and Algorithm 3.2 solves problem (4.1), (4.2), and (4.3) if the required assumptions are satisfied.

4.0.4. ***Example***. Consider the problem (1.2) for $H=\mathbb{R}$, feasible set $C=[-1,1]$, a bifunctions $f_i : C\times C\to\mathbb{R}$, $f=\sum_{i=1}^N f_i$, a nonexpansive mapping $T : C \to C$ given by

$$f_1(x,y) = (2x-x^2)(y-x) \ \text{ and } \ f_i(x,y) = \rho_i x(y-x) \ \text{ for } i \in \{2,\dots,N\},$$

and $T=\sum_{j=1}^{N'} \frac{j}{\zeta} T_j$ where$\rho_i > 0$, $\zeta = 1+\ldots+N'$ and $T_j(x) = \frac{1}{j^2+j+1}x$, $j\in J=\{1,\dots,N'\}$. Note that

$$|f_1(x,y)-f_1(x,z)| = |(2x-x^2)(y-x)-(2x-x^2)(z-x)| \leq Q_1|y-z|$$

and for $i \in \ \{2,\dots,N\}$, we have

$$|f_i(x,y)-f_i(z,y)| = |\rho_i x(y-x)-\rho_i z(y-z)| \leq \rho_i(|y|+|x+z|)|x-z| \leq Q_i|x-z|$$

where $Q_1=\max\{|2x-x^2| : x\in C\}$, $Q_i = \max\{\rho_i(|y|+|x+z|) : x,y,z\in C\}$ for $i\in \ \{2,\dots,N\}$. Thus, f_1 is 1-Hölder continuous in the first argument on C with $Q_1 = 3$ and for $i \in \ \{2,\dots,N\}$, each f_i is 1-Hölder continuous in the first argument on C with $Q_i = 3\rho_i$. It is also easy to show that each f_i is 1-Hölder continuous in the second argument on C for $i\in \ \{2,\dots,N\}$. Note that $FixT=\{0\}$ and $\text{SEP}\big(\sum_{i=1}^N f_i, C\big)=\{0\}$. Therefore, $\Omega=\{0\}$.

5. Conclusion

Using sequential and parallel computation of extragradient-like method combined with shrinking projection we have proposed two algorithms for finding a common element of the set of common fixed points of a nonexpansive mapping and solutions of equilibrium problems for monotone bifunction f where f is the sum of a finite number of Hölder continuous bifunctions. One advantage of our result is that we can also apply it to solve problems that can't be solved by proposed algorithms in [12], i.e., we can apply our result to solve the equilibrium problem EP(f,C) for a bifunction f that can be written as a sum of more than two Hölder continuous bifunctions and can't be written as a sum of two Hölder continuous bifunctions (problem (1.3) for $N>2$).

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