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Local fractal Fourier transform and applications

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Abstract

In this manuscript, we review fractal calculus and the analogues of both local Fourier transform with its related properties and Fourier convolution theorem are proposed with proofs in fractal calculus. The fractal Dirac delta with its derivative and the fractal Fourier transform of the Dirac delta is also defined. In addition, some important applications of the local fractal Fourier transform are presented in this paper such as the fractal electric current in a simple circuit, the fractal second order ordinary differential equation, and the fractal Bernoulli-Euler beam equation. All discussed applications are closely related to the fact that, in fractal calculus, a useful local fractal derivative is a generalized local derivative in the standard calculus sense. In addition, a comparative analysis is also carried out to explain the benefits of this fractal calculus parameter on the basis of the additional alpha parameter, which is the dimension of the fractal set, such that when $\alpha = 1$, we obtain the same results in the standard calculus.

Keywords. Fractal calculus, Fractal local Fourier transform, Fractal differential equation, Fractal Fourier convolution theorem, Fractal Dirac delta function.

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1. Introduction

In the 1970s, the mathematician Mandelbrot explored a new geometry of nature that accepts the irregular structures of items including coastlines, lightning bolts, clouds, and molecular trajectories. Then, mathematicians had started studying in the late 19th century. The key characteristic of these objects, which Mandelbrot named fractals, is that their borders are so irregular that is not easy to comprehend how simple metric concepts and operations can be applied to them [25]. Mandelbrot has guided us to think in a new scientific manner about the edges of the clouds, the shapes of the tops of the forest on the horizon, and the complex shifting structure of the feathers on the wings of a bird as it flies [3, 25].

The concept of complex dimensions of fractal strings has been established by Lapidus et al. [24]. Such complex dimensions are identified as the poles of the corresponding zeta function [24]. Under an explicit formula, the oscillations in the geometry or frequency spectrum of the fractal string are represented. These oscillations are not found in smooth geometry [24]. Fractal geometry offers a strong method for the quantitative description of complex, highly irregular and random structures [7, 24, 32]. Besides, it can be used to define the processes that contribute to the creation and physical behavior of these structures [7, 32, 33]. The analysis on fractals has a key role in applications and modeling of processes with fractal structures. Scientists have developed various techniques to study fractal analysis,

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like harmonic analysis, probabilistic approaches, metric theory, fractional calculus, fractional spaces, and time-scale calculus [1, 2, 4, 8, 23, 27, 36, 37]. The methods of ordinary calculus are inadequate or not applicable to fractals. Some examples of fractals that have been seen in calculus are the Weierstrass functions and Cantor staircase functions. However, ordinary calculus is unable to deal with problems such as fractal dynamics and fractal time phenomena [35, 38, 39].

In the seminal papers [29–31], a calculus based on fractal subsets of the real line has been introduced by Parvate and Gangal, which is appropriate to integrate functions with fractal support. In addition, the fractional-order derivatives have been specified, which are local, in contrast to the classical fractional derivative. This has a very significant role in physics: first, so as not to violate causality, second, since measurement in physics is local. The classical non-local fractional integrals and derivatives based on fractal sets have been generalized in [12], and non-local derivatives are appropriate for modeling processes with memory effects. The benefits of this method are, firstly, the order of the fractal derivative has a geometric meaning, being equal to the dimension of support for the function, and secondly, the order of the fractal derivative also has a physical meaning by providing a relationship with the spectral dimension [15, 22]. This method is based on the Riemann-like approach which is beneficial from an algorithmic viewpoint. The line of reasoning of using the Riemann method owing to its simplicity among others is not uncommon, e.g. [34].

Recently, several other applications of fractal calculus are addressed in the literature. For instance, a review and summary of applications in classical mechanics, quantum mechanics, and optics can be found in [18]; the fractal Euler method was used to solve fractal differential equations in [10], integrals and derivatives of functions on Cantor tartan spaces of different dimensions were defined in [13], analogues of Laplace and Sumudu transforms in fractal calculus were introduced in [11], and also other relevant studies can be seen in [14, 16, 19, 20]. Fractal calculus has found a connection with some new types of local fractional derivative such as conformable derivative.

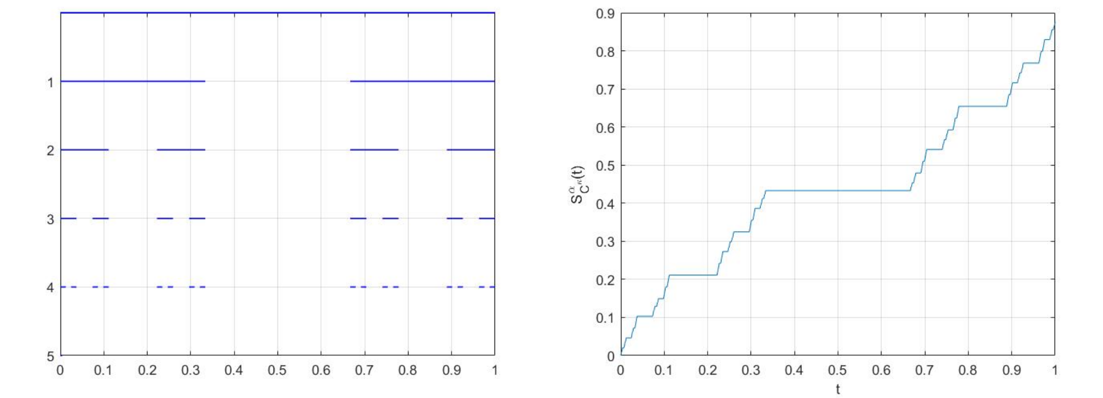
Integral transformations have been widely used to solve various problems in applied mathematics, mathematical physics, and engineering [5, 26, 28]. Fourier transforms can be traced back to Joseph Fourier's seminal dissertation [5]. Fourier's essay offered applications for the modern mathematical theory of heat conduction. In his essay, Fourier reported a remarkable result, now widely known as the Fourier Integral Theorem. In an attempt to extend his ideas to functions defined on an infinite interval, Fourier defined the integral transform and its inversion formula, which are nowadays known as the Fourier transform and the inverse Fourier transform, respectively [5]. The pros of our work are to shed the light on the significant need for a powerful tool with the help of fractal calculus, particulary local fractal derivative, to solve equations that can be encountered in various phenomena of physics and engineering. However, some possible cons of this work can be argued that working with non-local fractional derivatives can explain the dynamics of solutions better than the local ones due to the property of nonlocality and some systems pose memory effects. However, fractal calculus can be applied to some interesting models [9] over their corresponding classical versions with an advantage that is the presence of one arbitrary order of derivatives in the local fractal sense that has a physical meaning, and two arbitrary orders in the non-local fractal sense that allow taking the benefits of memory effect.

This paper is organized as follows: In section 2, the basic definitions are provided. In section 3, the local fractal Fourier transform is defined, and some of its properties are presented with all their proofs. In section 4, the main results of this paper are presented, which contain the fundamental fractal equation that defines derivatives of the fractal delta function, and the application of the local fractal Fourier transform to some suggested equations is also presented. In section 5, the conclusion is given.

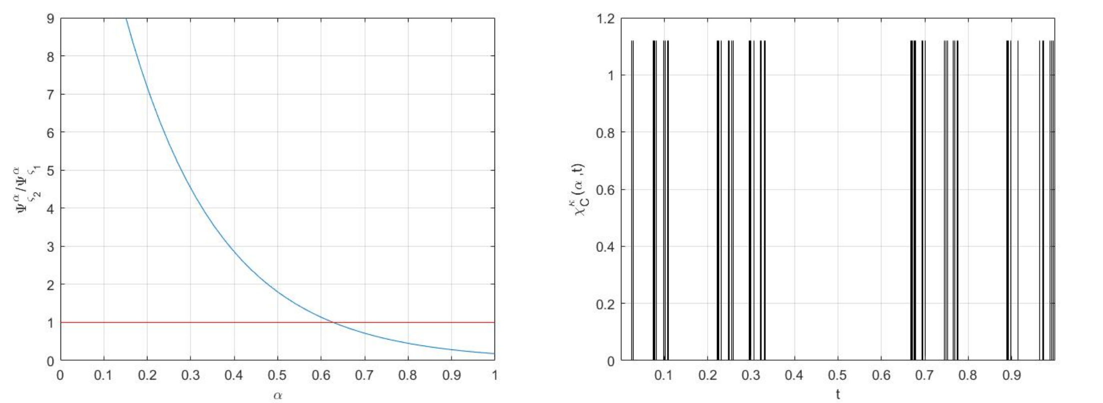
2. Some fundamental tools

2.1. **Staircase functions.** In this subsection, we present some basic tools of fractal calculus on thin Cantor-like set C^{κ} which is shown in Figure 2.1(a) [6, 21, 29–31].

Definition 2.1. ([29–31]) Let $p[a_1, a_2]$ be a subdivision of an interval $I = [a_1, a_2]$ which is a collection of points $\{a_1 = t_0, t_1, ..., t_n = a_2\}$, such that $t_i < t_{i+1}$.



(a) The thin Cantor-like set C^κ ($\kappa = 1/3$) by iteration. (b) The integral staircase function for the thin Cantor set C^κ for the case of ($\kappa = 1/3$).



(c) $\Psi-$dimension of the thin Cantor set C^κ ($\kappa = 1/3$). (d) Characteristic function thin Cantor set C^κ with ($\kappa = 1/3$).

Figure 1. Graphs corresponding to thin Cantor set C^κ with ($\kappa = 1/3$).

Definition 2.2. ([29–31]) Assume that $C^\kappa \subset \mathbf{R}$, is a fractal set and $p[a_1, a_2]$ is a subdivision. The mass function is given by

$$\Psi^\alpha\left(C^\kappa, a_1, a_2\right) = \lim_{\varsigma \to 0} \Psi^\alpha_\varsigma, \tag{2.1}$$

where

$$\Psi^\alpha_\varsigma = \inf_{\{p[a_1,a_2]:|p|\leq\varsigma\}} \sum_{j=0}^{m-1} \Gamma\left(\alpha+1\right)\left(t_{j+1}-t_j\right)^\alpha \phi\left(C^\kappa, [t_{j+1}-t_j]\right) \tag{2.2}$$

and

$$\phi\left(C^\kappa, [t_{j+1}-t_j]\right) = \begin{cases} 1, & C^\kappa \cap [t_{j+1}-t_j] \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

$$|p| = \max_{0\leq j\leq m}\left(t_{j+1}-t_j\right).$$

Definition 2.3. ([29–31]) Assume that $c_0 \in \mathbf{R}$. The staircase function of order α is given by

$$S^{\alpha}_{C^{\kappa}}(t) = \begin{cases} \Psi^{\alpha}(C^{\kappa}, c_0, t), & \text{if t} \geq c_0, \\ -\Psi^{\alpha}(C^{\kappa}, c_0, t), & \text{otherwise.} \end{cases} \tag{2.4}$$

The figure of the integral staircase function is presented in Figure 2.1(b).

Definition 2.4. ([6, 21, 29–31]) The $\Psi-$dimension is defined using the mass function, which is given by

$$\begin{aligned} \dim_{\Psi}(C^{\kappa} \cap [a_1, a_2]) &= \inf\{\alpha : \Psi^{\alpha}(C^{\kappa}, a_1, a_2) = 0\} \\ &= \sup\{\alpha : \Psi^{\alpha}(C^{\kappa}, a_1, a_2) = \infty\}. \end{aligned} \tag{2.5}$$

Figure 2.1(c) presents $\Psi-$dimension which is the intersection point of the red line with the blue line.

Definition 2.5. ([6, 21, 29–31]) The characteristic function $\chi_{C^{\kappa}}(\alpha, t)$ for a given thin Cantor set C^{κ} is defined by

$$\chi_{C^{\kappa}}(\alpha, t) = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)}, & t \in C^{\kappa}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

2.2. **Local Fractal Calculus.** In this subsection, the definitions of the local fractal derivative and fractal integral are presented.

Definition 2.6. ([29]) A function $f : \mathbf{R} \to \mathbf{R}$ is said to be C^{α}_{-}continuous (fractal continuity) at $x \in C^{\kappa}$ if $f(x) = \mathrm{C}^{\alpha}_{-}\lim_{y\to x} f(y)$.

Definition 2.7. ([6, 21, 29–31]) If C^{κ} is $\alpha-$ perfect set, then the C^{α}-derivative (fractal derivative) of $f(t)$ at t is defined by

$$D^{\alpha}_{C^{\kappa}} f(t) = \begin{cases} \mathrm{C}^{\alpha}_{-}\lim_{y\to t} \dfrac{f(y) - f(t)}{S^{\alpha}_{C^{\kappa}}(y) - S^{\alpha}_{C^{\kappa}}(t)}, & \text{if } t \in C^{\kappa}, \\ 0, \text{ otherwise,} \end{cases} \tag{2.7}$$

if the limit exists.

Definition 2.8. ([29–31]) The C^{α}-integral (fractal integral) of $f(t)$ on $[a_1, a_2]$ is defined by

$$\int_{a_1}^{a_2} f(t)\, d^{\alpha}_{C^{\kappa}} t \approx \sum_{j=1}^{n} f_j(t)\,(S^{\alpha}_{C^{\kappa}}(t_j) - S^{\alpha}_{C^{\kappa}}(t_{j-1})). \tag{2.8}$$

Definition 2.9. ([17, 29–31]) The fractal Dirac delta function on a thin Cantor set C^{κ} is defined by

$$\delta_{C^{\kappa}}(t - t_0) = 0, \text{ if } t \neq t_0, \tag{2.9}$$

and its fractal integral is given by

$$\int_{a-\epsilon}^{a+\epsilon} f(t)\delta_{C^{\kappa}}(t - t_0) d_{C^{\kappa}} t = \Gamma(\alpha+1) f(t_0). \tag{2.10}$$

The fractal step function $u_{C^{\kappa}, t_0}(t)$ is related to the fractal Dirac delta function as follows:

$$\int_{-\infty}^{t} \delta_{C^{\kappa}}(t - t_0) d_{C^{\kappa}} t = u_{C^{\kappa}, t_0}(t) \tag{2.11}$$

where

$$u_{C^{\kappa}, t_0}(t) = \begin{cases} 0, & t < t_0 \\ \frac{1}{\Gamma(\alpha+1)}, & t \leq t_0. \end{cases} \tag{2.12}$$

By recalling the **Fundamental Theorem of Fractal Calculus** [6, 21, 29–31], we have the following :

$$D^{\alpha}_{C^{\kappa}} u_{C^{\kappa}, t_0}(t) = \delta_{C^{\kappa}}(t - t_0). \tag{2.13}$$

2.3. **Non-Local Fractal Calculus.**

Definition 2.10. ([12, 17]) For a function $f(t),\ t \in C^{\kappa}$, the **fractal left-sided Riemann-Liouville integral** is defined by

$$ {}_{a}\mathbb{J}_{t}^{\beta} f(t) = \frac{1}{\Gamma_{C^{\kappa}}^{\alpha}(\beta)} \int_{a}^{t} \frac{f(x)}{\left(S_{C^{\kappa}}^{\alpha}(t) - S_{C^{\kappa}}^{\alpha}(x)\right)^{\alpha-\beta}} d_{C^{\kappa}}^{\alpha} x, \tag{2.14} $$

where $t > a$.

Definition 2.11. ([12, 17]) The **fractal left-sided Caputo derivative** is given by

$$ {}_{a}^{C}\mathbb{D}_{t}^{\beta} f(t) = \frac{1}{\Gamma_{C^{\kappa}}^{\alpha}(n-\beta)} \int_{a}^{t} \frac{(D_{C^{\kappa}}^{\alpha})^{n} f(x)}{\left(S_{C^{\kappa}}^{\alpha}(t) - S_{C^{\kappa}}^{\alpha}(x)\right)^{-n\alpha+\beta+\alpha}} d_{C^{\kappa}}^{\alpha} x, \tag{2.15} $$

where $n\alpha - \alpha < \beta \leq n\alpha,\ n \in \mathbf{N}$.

Definition 2.12. ([12, 17]) The **fractal left-sided Riemann–Liouville derivative** is given by

$$ {}_{a}\mathbb{D}_{t}^{\beta} f(t) = \frac{1}{\Gamma_{C^{\kappa}}^{\alpha}(n-\beta)} (D_{C^{\kappa}}^{\alpha})^{n} \int_{a}^{t} \frac{f(x)}{\left(S_{C^{\kappa}}^{\alpha}(t) - S_{C^{\kappa}}^{\alpha}(x)\right)^{-n\alpha+\beta+\alpha}} d_{C^{\kappa}}^{\alpha} x, \tag{2.16} $$

where $n\alpha - \alpha \leq \beta < n\alpha,\ n \in \mathbf{N}$.

Some important formulas of local and non-local fractal calculus are listed as follows [12, 17]:

$$ \begin{aligned} D_{C^{\kappa}}^{\alpha} a\chi_{C^{\kappa}} &= 0,\ a \text{ is constant}, && (2.17)\\ {}_{0}\mathbb{D}_{t}^{\gamma} a\chi_{C^{\kappa}} &= \frac{a S_{C^{\kappa}}^{\alpha}(t)^{-\gamma}}{\Gamma_{C^{\kappa}}^{\alpha}(1-\gamma)} && (2.18)\\ D_{C^{\kappa}}^{\alpha} S_{C^{\kappa}}^{\alpha}(t) &= \chi_{C^{\kappa}}(\alpha, t), && (2.19)\\ D_{C^{\kappa}}^{\alpha} S_{C^{\kappa}}^{\alpha}(t)^{m} &= m S_{C^{\kappa}}^{\alpha}(t)^{m-1} && (2.20)\\ D_{C^{\kappa}}^{\alpha} \cos(S_{C^{\kappa}}^{\alpha}(t)) &= -\chi_{C^{\kappa}}(\alpha, t)\sin(S_{C^{\kappa}}^{\alpha}(t)) && (2.21)\\ D_{C^{\kappa}}^{\alpha}(f(t)g(t)) &= D_{C^{\kappa}}^{\alpha}(f(t))g(t) + f(t)D_{C^{\kappa}}^{\alpha}(g(t)), && (2.22)\\ {}_{0}\mathbb{D}_{t}^{\gamma} f(t)g(t) &= \sum_{n=0}^{\infty} \binom{\gamma}{n} {}_{0}\mathbb{D}_{t}^{n} f(t) {}_{0}\mathbb{D}_{t}^{\gamma-n} g(t), && (2.23)\\ \int S_{C^{\kappa}}^{\alpha}(t)^{n} d_{C^{\kappa}}^{\alpha} t &= \frac{S_{C^{\kappa}}^{\alpha}(t)^{n+1}}{n+1} + c, && (2.24)\\ f(t) &= \sum_{n=1}^{\infty} \frac{D_{C^{\kappa}}^{n\alpha} f(t)|_{t=a}}{n!} (S_{C^{\kappa}}^{\alpha}(t) - S_{C^{\kappa}}^{\alpha}(a)), && (2.25) \end{aligned} $$

3. The local fractal Fourier transform

In this section, we define the local fractal Fourier transform to apply it for the fractal differential equations [5].

Definition 3.1. Let $f(t)$ be a function defined on a thin Cantor-like set C^{κ}, then $f(t)$ is a fractal piecewise continuous if and only if there exists a finite subdivision $[a_0, a_1]$,...,$[a_{n-1}, a_n]$ of $C^{\kappa} \subset [a, b]$ where $a_0 = a$ and $a_n = b$, such that $\forall i \in \{1, 2, 3, ..., n\}$; $f(t)$ is a fractal continuous function on (a_{i-1}, a_i) .

Definition 3.2. Let $f(t)$ be a function defined on a thin Cantor-like set C^{κ}, then $f(t)$ is called absolutely fractal integrable if and only if its absolute value of $|f(t)|$ has a fractal integral.

Definition 3.3. If $f(t)$ is a fractal continuous, fractal piecewise smooth, and fractal absolutely integrable function, then the fractal Fourier of $f(t)$ with respect to $t \in C^{\kappa}$ is denoted by $G(w)$ and is given by

$$F^{\alpha}_{C^{\kappa}}\{f(t)\} = \mathbf{F}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(iS^{\alpha}_{C^{\kappa}}(w)\, S^{\alpha}_{C^{\kappa}}(t)\right) f(t)\, d^{\alpha}_{C^{\kappa}} t, \tag{3.1}$$

where w is called the fractal Fourier transform variable and $\exp\left(-iS^{\alpha}_{C^{\kappa}}(w)\, S^{\alpha}_{C^{\kappa}}(t)\right)$ is called the fractal kernel of the transform. Then, $\forall\, w \in C^{\kappa}$, the inverse local fractal Fourier transform of $\mathbf{F}(w)$ is defined by

$$F^{\alpha,\,-1}_{C^{\kappa}}\{\mathbf{F}(w)\} = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-iS^{\alpha}_{C^{\kappa}}(w)\, S^{\alpha}_{C^{\kappa}}(t)\right) \mathbf{F}(w)\, d^{\alpha}_{C^{\kappa}} w, \tag{3.2}$$

3.1. **Properties of the local fractal Fourier transform.** Some properties of the local fractal Fourier series are presented. Their proofs are also given in this section.

Theorem 3.4. *(Linearity). The local fractal Fourier transformation is linear.*

Proof. We have

$$F^{\alpha}_{C^{\kappa}}\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(iS^{\alpha}_{C^{\kappa}}(w)\, S^{\alpha}_{C^{\kappa}}(t)\right) f(t)\, d^{\alpha}_{C^{\kappa}} t. \tag{3.3}$$

Then, for any constants γ and β,

$$\begin{aligned} F^{\alpha}_{C^{\kappa}}\{\gamma f(t) + \beta g(t)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\gamma f(t) + \beta g(t)] \exp\left(iS^{\alpha}_{C^{\kappa}}(w)\, S^{\alpha}_{C^{\kappa}}(t)\right) d^{\alpha}_{C^{\kappa}} t, \\ &= \frac{\gamma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \exp\left(iS^{\alpha}_{C^{\kappa}}(w)\, S^{\alpha}_{C^{\kappa}}(t)\right) d^{\alpha}_{C^{\kappa}} t + \frac{\beta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) \exp\left(iS^{\alpha}_{C^{\kappa}}(w)\, S^{\alpha}_{C^{\kappa}}(t)\right) d^{\alpha}_{C^{\kappa}} t, \\ &= \gamma F^{\alpha}_{C^{\kappa}}\{f(t)\} + \beta F^{\alpha}_{C^{\kappa}}\{g(t)\}. \end{aligned} \tag{3.4}$$

□

Theorem 3.5. *(Shifting). Let $F^{\alpha}_{C^{\kappa}}\{f(t)\}$ be a local fractal Fourier transform of $f(t)$. Then, we have*

$$F^{\alpha}_{C^{\kappa}}[f(t-a)] = \exp\left(iaS^{\alpha}_{C^{\kappa}}(t)\right) F^{\alpha}_{C^{\kappa}}(f(t)), \tag{3.5}$$

where a is a real constant.

Proof. According to the definition, we have, for $a > 0$,

$$\begin{aligned} F^{\alpha}_{C^{\kappa}}[f(t-a)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(iS^{\alpha}_{C^{\kappa}}(w)\, S^{\alpha}_{C^{\kappa}}(t)\right) f(t-a)\, d^{\alpha}_{C^{\kappa}} t, \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(iS^{\alpha}_{C^{\kappa}}(w)\, S^{\alpha}_{C^{\kappa}}(t)\right) f(\eta)\, d^{\alpha}_{C^{\kappa}} \eta, \;\; \eta = x - a \\ &= \exp\left(iS^{\alpha}_{C^{\kappa}}(w)\, S^{\alpha}_{C^{\kappa}}(t)\right) F^{\alpha}_{C^{\kappa}}\{f(t)\}. \end{aligned} \tag{3.6}$$

□

Theorem 3.6. *(Scaling). If $F^{\alpha}_{C^{\kappa}}$ is a local fractal Fourier transform of f, then we obtain*

$$F^{\alpha}_{C^{\kappa}}\left\{f\left(S^{\alpha}_{C^{\kappa}}(\lambda t)\right)\right\}=\frac{1}{|\lambda^{\alpha}|}F^{\alpha}_{C^{\kappa}}\left(\frac{S^{\alpha}_{C^{\kappa}}(w)}{\lambda^{\alpha}}\right), \tag{3.7}$$

where λ is a real nonzero constant.

Proof. For $\lambda\neq 0$,

$$F^{\alpha}_{C^{\kappa}}\left\{f\left(S^{\alpha}_{C^{\kappa}}(\lambda t)\right)\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(iS^{\alpha}_{C^{\kappa}}(w)\,S^{\alpha}_{C^{\kappa}}(t)\right)f\left(S^{\alpha}_{C^{\kappa}}(\lambda t)\right)d^{\alpha}_{C^{\kappa}}t. \tag{3.8}$$

If we let $\eta=\lambda^{\alpha}S^{\alpha}_{C^{\kappa}}(t)$, then we have

$$\begin{aligned}F^{\alpha}_{C^{\kappa}}\left\{f\left(S^{\alpha}_{C^{\kappa}}(\lambda t)\right)\right\}&=\frac{1}{|\lambda^{\alpha}|}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(i\left(\frac{S^{\alpha}_{C^{\kappa}}(w)}{\lambda^{\alpha}}\right)\eta\right)f(\eta)\,d^{\alpha}_{C^{\kappa}}\eta\\&=\frac{1}{|\lambda^{\alpha}|}F^{\alpha}_{C^{\kappa}}\left(\frac{S^{\alpha}_{C^{\kappa}}(w)}{\lambda^{\alpha}}\right).\end{aligned} \tag{3.9}$$

□

Remark 3.7. An example is the middle $\frac{1}{3}$ thin Cantor set C, with $a=0,\ b=1,\ \alpha=\frac{\ln 2}{\ln 3}$ and $\lambda=\frac{1}{3^n}$ for any positive integer n.

Theorem 3.8. *(differentiation). Let f be fractal continuous and fractal piecewise smooth in $(-\infty,\infty)$. Let $f(t)$ approache zero as $|t|\to\infty$. If f and $D^{\alpha}_{C^{\kappa}}f$ are fractal absolutely integrable, then we have*

$$F^{\alpha}_{\kappa}\left\{D^{\alpha}_{C^{\kappa}}f\right\}=-iS^{\alpha}_{C^{\kappa}}(w)\,F^{\alpha}_{C^{\kappa}}\left\{f(t)\right\}=-iS^{\alpha}_{C^{\kappa}}(w)\,\boldsymbol{F}(w). \tag{3.10}$$

Proof.

$$\begin{aligned}F^{\alpha}_{C^{\kappa}}\left\{D^{\alpha}_{C^{\kappa},t}f(t)\right\}&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}D^{\alpha}_{C^{\kappa},t}f(t)\exp\left(iS^{\alpha}_{C^{\kappa}}(w)\,S^{\alpha}_{C^{\kappa}}(t)\right)d^{\alpha}_{C^{\kappa}}t\\&=\frac{1}{\sqrt{2\pi}}\Big[\,f(t)\exp\left(iS^{\alpha}_{C^{\kappa}}(w)\,S^{\alpha}_{C^{\kappa}}(t)\right)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}f(t)\exp\left(iS^{\alpha}_{C^{\kappa}}(w)\,S^{\alpha}_{C^{\kappa}}(t)\right)d^{\alpha}_{C^{\kappa}}t\Big]\\&=-iS^{\alpha}_{C^{\kappa}}(w)\,F^{\alpha}_{C^{\kappa}}\left\{f(t)\right\}=-iS^{\alpha}_{C^{\kappa}}(w)\,\mathbf{F}(w).\end{aligned} \tag{3.11}$$

In general, if f and its first $(n-1)$ derivatives are fractal continuous, and if its αth derivatives are fractal piecewise continuous, then the local fractal Fourier transform of order $n-1<\alpha\leq n,\ \ n\in\mathbb{N}$ can be expressed as follows:

$$F^{\alpha}_{C^{\kappa}}\left\{\left(D^{\alpha}_{C^{\kappa},t}\right)^{n}f(t)\right\}=\left(-iS^{\alpha}_{C^{\kappa}}(w)\right)^{n}F^{\alpha}_{C^{\kappa}}\left\{f(t)\right\}=\left(-iS^{\alpha}_{\kappa}(w)\right)^{n}\mathbf{F}(w),\quad n=0,1,2,..., \tag{3.12}$$

provided that f and its derivatives are fractal absolutely integrable. Furthermore, we suppose that f and its first $(n-1)$ derivatives tend to zero as $|t|$ approaches to infinity. □

Theorem 3.9. *(Convolution Theorem). If $\boldsymbol{F}(w)$ and $G(w)$ are the local fractal Fourier transforms of $f(t)$and $g(t)$, respectively, then the local fractal Fourier transform of the convolution $(f*g)$ is the product $\boldsymbol{F}(w)\,G(w)$. That is,*

$$F^{\alpha}_{C^{\kappa}}\left\{f(t)*g(t)\right\}=\boldsymbol{F}(w)\,G(w). \tag{3.13}$$

Or, equivalently,

$$F^{\alpha,-1}_{C^{\kappa}}\left\{\boldsymbol{F}(w)\,G(w)\right\}=f(t)*g(t). \tag{3.14}$$

More explicitly,

$$\begin{aligned}(f*g)(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \boldsymbol{F}(w)\, G(w) \exp\left(iS_{C^\kappa}^{\alpha}(w)\, S_{C^\kappa}^{\alpha}(t)\right) d_{C^\kappa}^{\alpha} t \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-\xi) g(\xi)\, d_{C^\kappa}^{\alpha}\xi.\end{aligned} \tag{3.15}$$

Proof. By definition, we have

$$\begin{aligned}F_{C^\kappa}^{\alpha}\{(f*g)(t)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(iS_{C^\kappa}^{\alpha}(w)\, S_{C^\kappa}^{\alpha}(t)\right) d_{C C^\kappa}^{\alpha} t \int_{-\infty}^{\infty} f(t-\xi)\, g(\xi)\, d_{C^\kappa}^{\alpha}\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi) \exp\left(iS_{C^\kappa}^{\alpha}(w)\, S_{C^\kappa}^{\alpha}(\xi)\right) d_{\kappa}^{\alpha}\xi * \int_{-\infty}^{\infty} f(t-\xi) \exp\left(iS_{C^\kappa}^{\alpha}(w)\left(S_{C^\kappa}^{\alpha}(t) - S_{C^\kappa}^{\alpha}(\xi)\right)\right) d_{C^\kappa}^{\alpha} t.\end{aligned} \tag{3.16}$$

With the change of variable $S_{C^\kappa}^{\alpha}(\eta) = S_{C^\kappa}^{\alpha}(t) - S_{C^\kappa}^{\alpha}(\xi)$, we get

$$\begin{aligned}F_{C^\kappa}^{\alpha}\{(f*g)(t)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\xi) \exp t(iS_{C^\kappa}^{\alpha}(w) S_{C^\kappa}^{\alpha}(\xi)) d_{C^\kappa}^{\alpha}\xi \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\eta) \exp(iS_{C^\kappa}^{\alpha}(w) S_{C^\kappa}^{\alpha}(\eta)) d_{C^\kappa}^{\alpha}\eta \\ &= \mathbf{F}(w) G(w).\end{aligned} \tag{3.17}$$

□

TABLE 1. Some analogies of the Fourier transform of both C^α- calculus and the ordinary calculus.

Function	Fourier transform	Local fractal Fourier transform
$f(t)$	$\boldsymbol{f}(w)$	$\mathbf{F}(w)$
$f(t) = \begin{cases} 1, & \lvert t\rvert < b \\ 0, & \lvert t\rvert > b. \end{cases}$	$\frac{2\sin w}{w}$	$\frac{2\sin S_{C^\kappa}^{\alpha}(w)}{S_{C^\kappa}^{\alpha}(w)}$
$\frac{1}{t^2+b^2}$	$\frac{\pi\exp(-bw)}{b}$	$\frac{\pi\exp\left(-bS_{C^\kappa}^{\alpha}(w)\right)}{b}$
$\frac{t}{t^2+b^2}$	$-i\pi\exp(bw)$	$-i\pi\exp\left(bS_{C^\kappa}^{\alpha}(w)\right)$
$f^{(n)}(t)$	$(iw)^n \boldsymbol{f}(w)$	$\left(-iS_{C^\kappa}^{\alpha}(w)\right)^n \mathbf{F}(w)$
$t^n f(t)$	$i^n \frac{d^n \boldsymbol{f}}{dw^n}$	$i^n D_{C^\kappa,w}^{n\alpha}(\mathbf{F}(w))$
$f(bt)\exp(ixt)$	$\frac{1}{b}\boldsymbol{f}\left(\frac{w-x}{b}\right)$	$\frac{1}{b}\mathbf{F}\left(\frac{S_{C^\kappa}^{\alpha}(w) - S_{C^\kappa}^{\alpha}(x)}{b}\right)$

In Table 1, we have presented the fractal Fourier transform of some functions.

4. Main results

In this section, the fundamental fractal equation that defines the fractal derivatives of the fractal Dirac delta function, and the application of the local fractal Fourier transform to some proposed physical equations are presented.

Theorem 4.1. *The fractal nth order derivative of the Dirac delta function is*

$$S^{\alpha}_{C^{\kappa}}(t)^{n} D^{\alpha n}_{C^{k}} \delta_{C^{k}}(t) = (-1)^{n} n! \delta_{C^{k}}(t). \tag{4.1}$$

Using **the fractal integration by parts** [29], we have

$$\int_{-\infty}^{\infty} f(t) D^{n\alpha}_{C^{k}} \delta_{C^{k}}(t)\, d_{C^{\kappa}} t = f(t)\, D^{(n-1)\alpha}_{C^{k}} \delta_{C^{k}}(t)\,|^{\infty}_{-\infty} - \int_{-\infty}^{\infty} D^{\alpha}_{C^{\kappa},t} f(t)\, D^{(n-1)\alpha}_{C^{k}} \delta_{C^{k}}(t)\, d_{C^{\kappa}} t. \tag{4.2}$$

According to the definition of the Dirac delta function $f(t)\, D^{\alpha(n-1)}_{C^{\kappa}} \delta_{C^{\kappa}}(t)\,|^{\infty}_{-\infty} = 0$, Eq. (4.2) reduces to the following form:

$$\int_{-\infty}^{\infty} f(t) D^{n\alpha}_{C^{k}} \delta_{C^{k}}(t)\, d_{C^{\kappa}} t = -\int_{-\infty}^{\infty} D^{\alpha}_{C^{\kappa},t} f(t)\, D^{(n-1)\alpha}_{C^{k}} \delta_{C^{k}}(t)\, d_{C^{\kappa}} t, \tag{4.3}$$

By letting $f(t) = S^{\alpha}_{C^{\kappa}}(t)\, g(t)$ and $n = 1$, then Eq. (4.2) reduces to the following form:

$$\begin{aligned} \int_{-\infty}^{\infty} S^{\alpha}_{C^{\kappa}}(t)\, g(t)\, D^{\alpha}_{C^{k}} \delta_{C^{k}}(t)\, d_{C^{\kappa}} t &= -\int_{-\infty}^{\infty} \delta_{C^{k}}(t) D^{\alpha}_{C^{\kappa},t} \left[S^{\alpha}_{C^{\kappa}}(t)\, g(t)\right] d_{C^{\kappa}} t \\ &= -\int_{-\infty}^{\infty} \delta_{C^{k}}(t) \left(g(t)\, \chi_{C^{\kappa}}(\alpha, t) + S^{\alpha}_{C^{\kappa}}(t)\, D^{\alpha}_{C^{\kappa},t}(g(t))\right) d_{C^{\kappa}} t \\ &= -\int_{-\infty}^{\infty} \delta_{C^{k}}(t)\, g(t) \chi_{C^{\kappa}}(\alpha, t) d_{C^{\kappa}} t, \end{aligned} \tag{4.4}$$

since $\int_{-\infty}^{\infty} \delta_{C^{k}}(t)\, S^{\alpha}_{C^{\kappa}}(t) D^{\alpha}_{C^{\kappa},t} g(t)\, d_{C^{\kappa}} t = 0$, then Eq. (4.2) takes the following form:

$$S^{\alpha}_{C^{\kappa}}(t)\, D^{\alpha}_{C^{\kappa}} \delta_{C^{\kappa}}(t) = -\delta_{C^{k}}(t)\, \chi_{C^{\kappa}}(\alpha, t). \tag{4.5}$$

In general, the same procedure gives

$$\int_{-\infty}^{\infty} [S^{\alpha}_{C^{\kappa}}(t)^{n} g(t)] D^{\alpha n}_{C^{k}} \delta_{C^{k}}(t)\, d_{C^{\kappa}} t = (-1)^{n} \int_{-\infty}^{\infty} \delta_{C^{k}}(t) D^{\alpha n}_{C^{\kappa},t} \left(S^{\alpha}_{C^{\kappa}}(t)^{n} g(t)\right) d_{C^{\kappa}} t, \tag{4.6}$$

but because of any power of $S^{\alpha}_{C^{\kappa}}(t)$ times $\delta_{C^{\kappa}}(t)$ integrates to it which implies that only the constant term contributes. So, all terms multiplied by the derivative of $g(t)$ disappear, leaving $n! g(t)$, so we have

$$\int_{-\infty}^{\infty} [S^{\alpha}_{C^{\kappa}}(t)^{n} g(t)] D^{\alpha n}_{C^{\kappa}} \delta_{C^{\kappa}}(t)\, d_{C^{\kappa}} t = (-1)^{n} n! \int_{-\infty}^{\infty} g(t) \delta_{C^{k}}(t)\, d_{C^{\kappa}} t, \tag{4.7}$$

which means that

$$S^{\alpha}_{C^{\kappa}}(t)^{n} D^{\alpha n}_{C^{k}} \delta_{C^{k}}(t) = (-1)^{n} n! \delta_{C^{k}}(t). \tag{4.8}$$

The Dirac delta function is given as the Local fractal Fourier transform which can be expressed as follows:

$$F^{\alpha}_{C^{\kappa}}\{1\} = \delta_{C^{\kappa}}(t) = \int_{-\infty}^{\infty} \exp\left(-2\pi i S^{\alpha}_{C^{\kappa}}(w) S^{\alpha}_{C^{\kappa}}(t)\right) d_{C^{\kappa}} w. \tag{4.9}$$

Similarly, we have

$$F^{\alpha;-1}_{C^\kappa}\{\delta_{C^\kappa}(t)\} = \int_{-\infty}^{\infty} \delta_{C^\kappa}(t)\exp\left(2\pi i S^\alpha_{C^\kappa}(t) S^\alpha_{C^\kappa}(w)\right) d_{C^\kappa} w = 1. \tag{4.10}$$

The local fractal Fourier transform of the fractal Dirac delta is given by

$$\begin{aligned} F^\alpha_{C^\kappa}\{\delta_{C^\kappa}(t-t_0)\} &= \int_{-\infty}^{\infty} \delta_{C^\kappa}(t-t_0)\exp(-2\pi i S^\alpha_{C^\kappa}(w) S^\alpha_{C^\kappa}(t)) d^\alpha_{C^\kappa} t \\ &= \exp\left(-2\pi i S^\alpha_{C^\kappa}(w) S^\alpha_{C^\kappa}(t_0)\right). \end{aligned} \tag{4.11}$$

5. The application of local fractal Fourier transform to some suggested equations

1. The fractal electric current in a simple circuit. The present time (t) in a simple circuit including the resistance R and the inductance L follows the following equation:

$$LD^\alpha_{C^\kappa,t}(I) + RI = E(t), \tag{5.1}$$

where $E(t)$ represents the applied electromagnetic force and R and L are constants. Using $E(t) = \cos(t)$, then by applying the local fractal Fourier transform to Eq. (5.1), we obtain

$$\left(iS^\alpha_{C^\kappa}(w)L + R\right) I(w) = \sqrt{\frac{\pi}{2}}\left(\delta_{C^\kappa}\left(-1 + S^\alpha_{C^\kappa}(w)\right) + \delta_{C^\kappa}\left(1 + S^\alpha_{C^\kappa}(w)\right)\right).$$

Or,

$$I(w) = \frac{\sqrt{\frac{\pi}{2}}\left(\delta_{C^\kappa}\left(-1 + S^\alpha_{C^\kappa}(w)\right) + \delta_{C^\kappa}\left(1 + S^\alpha_{C^\kappa}(w)\right)\right)}{\left(iS^\alpha_{C^\kappa}(w)L + R\right)}. \tag{5.2}$$

Taking the inverse local fractal Fourier transform to both sides of Eq. (5.2), we get the following solution:

$$I(t) = \frac{R\cos\left(S^\alpha_{C^\kappa}(t)\right) + L\sin\left(S^\alpha_{C^\kappa}(t)\right)}{L^2 + R^2}. \tag{5.3}$$

By $(a_1 t^\alpha \leq S^\alpha_{C^\kappa}(t) \leq a_2 t^\alpha)$ [29], we have

$$I(t) \approx \frac{R\cos(t^\alpha) + L\sin(t^\alpha)}{L^2 + R^2}. \tag{5.4}$$

In Figure 2, we have illustrated Eqs. (5.3) and (5.4).

2. The fractal second order ordinary differential equation.

$$D^{2\alpha}_{C^\kappa,t}(u) + b^2 u = f(t), \quad -\infty < t < \infty. \tag{5.5}$$

Applying the local fractal Fourier transform method gives

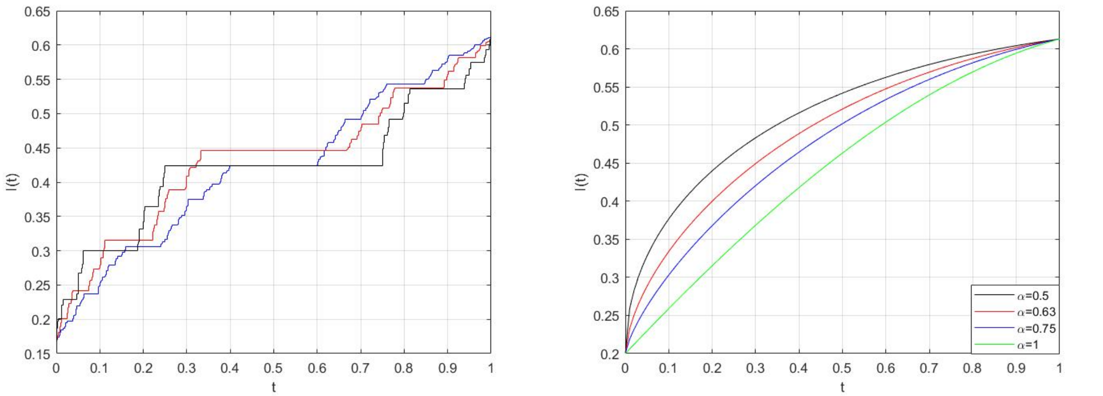
$$U(w) = \frac{F\left(S^\alpha_{C^\kappa}(w)\right)}{S^\alpha_{C^\kappa}(w)^2 + b^2}. \tag{5.6}$$

Using the convolution **Theorem 5**, we obtain the following solution:

$$u(t) = \frac{1}{2a}\int_{-\infty}^{\infty} f(\xi)\exp\left(-b\left(\left|S^\alpha_{C^\kappa}(t) - S^\alpha_{C^\kappa}(\xi)\right|\right)\right) d^\alpha_{C^\kappa}\xi. \tag{5.7}$$

3. The fractal Bernoulli-Euler beam equation. Consider the vertical deflection specified vertical load of $Q(t)$. The deflection of $u(t)$ follows the following standard differential equation:

$$EID^{4\alpha}_{C^\kappa,t}(u) + Ku = Q(t), \quad -\infty < t < \infty \tag{5.8}$$



(a) We have sketched Eq. (5.3) setting $R = 0.5$, $L = 1.5$, $\kappa = 1/2$ (black color), $\kappa = 2/3$ (red color) and $\kappa = 1/5$ (blue color)

(b) We have plotted Eq. (5.4) using $R = 0.5$, and $L = 1.5$.

FIGURE 2. Graphs of the fractal electric current

where EI represents the flexural rigidity and K represents the foundation modulus of the beam. By setting up $Q(t) = t^2 sin(t)$, then Eq. (5.8) takes the following form:

$$D^{4\alpha}_{C^{\kappa},t}(u) + b^4 u = t^2 sin(t), \tag{5.9}$$

where $b^4 = \frac{K}{EI}$. Applying the local fractal Fourier transform to Eq. (5.9), we get

$$U(w) = \frac{\mathrm{i}\sqrt{\frac{\pi}{2}} D^{2\alpha}_{C^{\kappa}} \delta_{C^{\kappa}}\left(-1 + S^{\alpha}_{C^{\kappa}}(w)\right) + \mathrm{i}\sqrt{\frac{\pi}{2}} D^{2\alpha}_{C^{\kappa}} \delta_{C^{\kappa}}\left(1 + S^{\alpha}_{C^{\kappa}}(w)\right)}{S^{\alpha}_{\kappa}(w)^4 + a^4}. \tag{5.10}$$

Taking the inverse local fractal Fourier transform to both sides of Eq. (5.10), we have

$$u(t) = \frac{8(1+b^4) S^{\alpha}_{C^{\kappa}}(t) \cos(S^{\alpha}_{C^{\kappa}}(t))}{(1+b^4)^3} + \frac{(4(-5+3b^4) + (1+b^4)^2 S^{\alpha}_{C^{\kappa}}(t)^2) \sin(S^{\alpha}_{C^{\kappa}}(t))}{(1+b^4)^3}. \tag{5.11}$$

In view of $(a_1 t^{\alpha} \leq S^{\alpha}_{C^{\kappa}}(t) \leq a_2 t^{\alpha})$ [29], we have

$$u(t) \approx \frac{8\left(1+b^4\right) t^{\alpha} \cos\left(t^{\alpha}\right) + \left(4\left(-5+3b^4\right) + \left(1+b^4\right)^2 t^{2\alpha}\right) \sin\left(t^{\alpha}\right)}{\left(1+b^4\right)^3}. \tag{5.12}$$

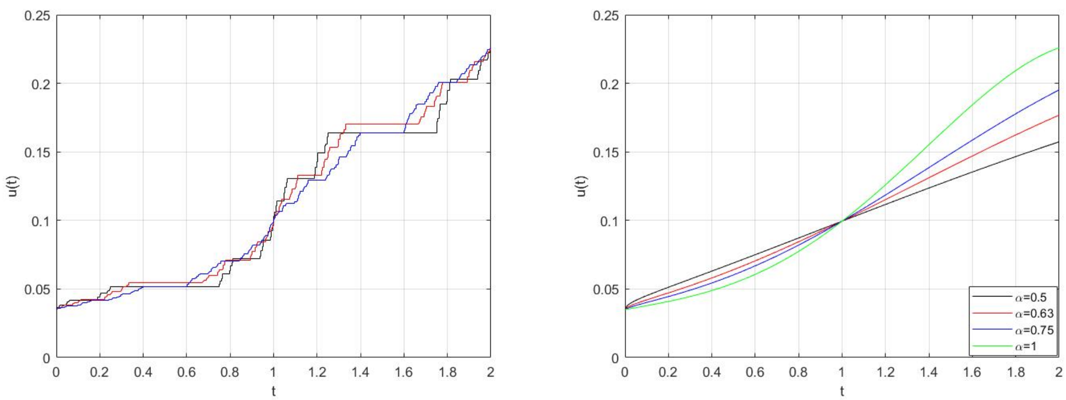
In Figure 3, we have shown Eqs. (5.11) and (5.12).

Remark 5.1. We note that one can obtain the ordinary (classical) result by choosing $\alpha = 1$ in this research paper.

For some additional examples that consist of non-smooth solution, we can refer to the example 7 and example 8 in [11] by similarly applying the local fractal Fourier transform method and using the convolution Theorem 5 to obtain their solutions. Regarding the discrete transformation and how our work can be possibly parallelized in this case, we refer to [10].

6. Conclusion

In this article, we have introduced the local fractal Fourier transform and proved some of its properties. The fractal Dirac delta with its derivative and the fractal Fourier transform of the Dirac delta are defined. Besides, some models in the local fractal calculus are investigated. In addition, we perform a comparative analysis by solving the given equations in the standard version and in the local fractal calculus, and we also demonstrate that when $\alpha = 1$, we



(a) We have sketched Eq. (5.11) setting $b = 2$, $\kappa = 1/2$ (black color), $\kappa = 2/3$ (red color) and $\kappa = 1/5$ (blue color).

(b) We have presented Eq. (5.12) using $b = 2$.

FIGURE 3. Graphs of the fractal Bernoulli-Euler beam equation.

get the same results in the standard version. In order to study the effect of a local fractal order derivative, we have changed the values of α. Simulation analysis has been performed in order to explain the physical characteristics of some given models.

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