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Uniformly convergent fitted operator method for singularly perturbed delay differential equations

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Abstract

This paper deals with the numerical treatment of singularly perturbed delay differential equations having a delay on the first derivative term. The solution of the considered problem exhibits boundary layer behavior on the left or right side of the domain depending on the sign of the convective term. The term with the delay is approximated using Taylor series approximation, resulting in an asymptotically equivalent singularly perturbed boundary value problem. The uniformly convergent numerical scheme is developed using exponentially fitted finite difference method. The stability of the scheme is investigated using solution bound. The uniform convergence of the scheme is discussed and proved. Numerical examples are considered to validate the theoretical analysis.

Keywords. Fitted operator, Singularly perturbed problem, Uniform convergence.
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1. Introduction

A number of mathematical models appear in different application areas of science and engineering such as in epidemiology, laser optics, control theory and etc. that considers not only the current state of a physical system but also it includes its past history [14]. Such types of models are described as functional differential equations in general cases or delay differential equations. It is known that time delays are natural components of the dynamic processes of mechanics, economics, physiology, ecology and epidemiology [9] and to ignore them is to ignore reality [3]. Solution methods for different types of delay differential equations are studied by different researchers [9]. For the behavior of solution of delay differential equations, interested reader can refer [17] and the references cited there. The stability estimate of numerical methods for delay parabolic differential and difference equations are discussed in [2].

A singularly perturbed delay differential equations (SPDDEs) are defined as differential equations in which its highest order derivative is multiplied by a small perturbation parameter and having at least one delay term. Singularly perturbed delay differential equations exists in several research areas of applied mathematics [23], to list few of them: in water quality problem in river networks, in simulation of oil extraction from under-ground reservoirs, in convective heat transport problem with large Peclet numbers, in atmospheric pollution, in fluid flow at high Reynolds number and so on.

In general, when the perturbation parameter tends to zero the smoothness of the solution of the singularly perturbed problems deteriorates and it forms boundary layer [16]. So, numerical methods developed for solving regular problems turn out inapplicable for singular perturbation problem as the solution profile in this case depend up on the value of the singular perturbation parameter. It is well-known that the standard numerical method in FDM, FEM and Collocation methods on uniform meshes fail to converge uniformly with respect to the singular perturbation parameter [28]. The efficiency of a numerical method is determined by its accuracy, simplicity in computing the discrete solution and

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its sensitivity to the parameters of the given problem. It is desirable to develop numerical methods which converges uniformly. This motivates the researcher for developing simple, easy to use, parameter uniformly convergent numerical methods for solving singularly perturbed differential equations.

The literature in singularly perturbed problems are very large to address all. We only mention few of the recently published papers. Differently many numerical methods and techniques are discussed and highlighted in the review papers [10, 11] for numerical treatment of different classes of singularly perturbed problems that involves boundary layer phenomenon. Shah et al. in [24, 25] used collocation method based on Haar wavelets for the numerical solutions of singularly perturbed boundary value problems. Ai-qing and Xu-ming in [1] apply the Haar wavelets collocation method for singularly perturbed 2-D reaction diffusion problem.

The numerical treatments of singularly perturbed delay differential equations have received a great attention in the present past because of their wide applications. It gives theoretical and practical interest to consider numerical method for such problems. Owing to this, here we present some of the literatures pertinent on solution of SPDDEs. Phaneendra and Lulu [19] consider singularly perturbed delay differential equation and approximate the problem to equivalent singularly perturbed boundary value problem. They used Gaussian quadrature two-point formula for treating the problem. Kumar [15] first approximating the delay term and applied a simple finite difference method using mesh ratio. Kadalbajoo and Ramesh [12] after treating the delay term they used simple upwind , midpoint upwind and a hybrid scheme on Shishkin mesh. The authors discuss the uniform convergence of the schemes. Reddy et al. [21] first convert the problem to first order neutral type delay differential equation and then used the numerical integration method.

Kanth and Kumar [20] after converting to equivalent BVPs, they proposed hybrid numerical scheme comprises with the tension spline scheme in the boundary layer region and the midpoint approximation in the outer region on piecewise uniform mesh. File and Reddy [5] proposed terminal boundary-value method for solving the problem, by imposing a terminal point, the problem is decomposed into inner region and outer region problems. Both the inner and outer region problems are solved using central difference method. In [6–8] the authors first replaced the problem by equivalent first order neutral delay differential equations and they used Trapezoidal and Simpson integration formula for treating the resulting problems. Reza and Khan [22] used Taylor series approximation for the terms with delay and advance parameters and applied the Haar wavelats collocation method for solving the resulting singularly perturbed problem. Woldaregay and Duressa [28] solved singularly perturbed problem that involves small delay and advance parameters on the reaction terms. The authors applied the Richardson extrapolation technique to extend the rate of convergence of the scheme. Daba and Duressa [4] considered a time dependent form of singularly perturbed delay differential equation that arises from modeling of neuronal variability. The authors proposed implicit Euler method in temporal discretization and exponentially fitted cubic spline method in spatial discretization. Their scheme gives linear order uniform convergence.

Notations: The symbol N denoted for the number of mesh interval in the discretization. The symbol C denotes positive constant independent of ε and N. The norm $\|.\|$ denotes suprimum norm.

2. Statement of the Problem

We consider a class of singularly perturbed delay differential equations of the form

$$-\varepsilon u''(x) + a(x)u'(x-\delta) + b(x)u(x) = f(x),\ x \in \Omega = (0,1), \tag{2.1}$$

with interval-boundary conditions

$$u(x) = \phi(x),\ -\delta \le x \le 0\ , u(1) = \psi(1), \tag{2.2}$$

where, $\varepsilon, 0 < \varepsilon \ll 1$ is singular perturbation parameter and δ is delay parameter satisfying $\delta < \varepsilon$. We assume the coefficient functions $a(x), b(x)$ and the source function $f(x)$ are sufficiently smooth and bounded for guaranteeing the existence of unique solution. The coefficient function $b(x)$ assumed to satisfy

$$b(x) \ge \beta > 0,\ x \in \bar{\Omega},$$

for β is lower bound of $b(x)$.

In case $\delta = 0$, equation (2.1)-(2.2) reduces to singularly perturbed boundary value problem, in which for small ε it exhibits boundary layer. The layer is maintained for $\delta \neq 0$ but sufficiently small.

2.1. **Approximation for the delay term.** For the case $\delta < \varepsilon$, using Taylor's series approximation for treating the delay term is appropriate [26]. So, we approximate $u'(x-\delta)$ as

$$u'(x-\delta) \approx u'(x) - \delta u''(x) + O(\delta^2). \tag{2.3}$$

Substituting (2.3) into (2.1), we obtain

$$-(\varepsilon - \delta\alpha)u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \tag{2.4}$$

with the boundary conditions

$$u(0) = \phi(0),\ u(1) = \psi(1), \tag{2.5}$$

where $\alpha > 0$ is lower bound of $a(x)$. For small values of δ, (2.1)-(2.2) and (2.4)-(2.5) are asymptotically equivalent, since the difference between the two equations is $O(\delta^2)$. The differential operator L is denoted for the differential equation in (2.4)–(2.5) and defined as

$$Lu(x) = -(\varepsilon - \delta\alpha)u''(x) + a(x)u'(x) + b(x)u(x).$$

The problem in (2.4)-(2.5) exhibits regular boundary layer of thickness $O(\varepsilon - \delta\alpha)$ and the position of the boundary layer depends on the conditions: If $a(x) < 0$ left boundary layer exist and for $a(x) > 0$ right boundary layer exist. In case of $a(x), x \in \bar{\Omega}$ change sign interior layer will exist [13].

The problem obtained by setting $\varepsilon - \delta\alpha = 0$ in (2.4)-(2.5) is called reduced problem and given as

$$\begin{aligned} &a(x)u_0'(x) + b(x)u_0(x) = f(x),\ x \in \Omega, \\ &\quad u_0(0) = \phi(0),\ u_0(1) \neq \psi(1), \end{aligned} \tag{2.6}$$

for right boundary layer case it does not satisfy the right boundary condition. For left boundary layer case it does not satisfy the left boundary condition(i.e. $u_0(0) \neq \phi(0),\ u_0(1) = \psi(1)$). For small values of ε the solution $u(x)$ of (2.4)-(2.5) is very close to the solution $u_0(x)$ of (2.6).

2.2. **Properties of the continuous solution.** The continuous solution of the problem in (2.4)-(2.5) satisfies the maximum principle, stability estimate or solution bound given in next Lemmas.

Lemma 2.1. *(The maximum principle.) Let z be a sufficiently smooth function defined on Ω which satisfies $z(0) \geq 0$ and $z(1) \geq 0$. Then, $Lz(x) \geq 0,\ x \in \Omega$ implies that $z(x) \geq 0,\ \forall x \in \bar{\Omega}$.*

Proof. Suppose that there exist $x^* \in \bar{\Omega}$ such that $z(x^*) = \min_{x\in\bar{\Omega}} z(x) < 0$. It is clear that $x^* \notin \{0, 1\}$ i.e. $x^* \in \Omega$. Since $z(x^*) = \min_{(x)\in\bar{\Omega}} z(x)$ using elementary calculus, which implies $z'(x^*) = 0$ and $z''(x^*) \geq 0$. Giving that $Lz(x^*) = -(\varepsilon - \delta\alpha)z''(x^*) + a(x^*)z'(x^*) + b(x^*)z(x^*) < 0$ which is contradiction to the assumption made $Lz(x^*) \geq 0, x \in \Omega$. Therefore, $z(x) \geq 0, \forall x \in \bar{\Omega}$. □

Lemma 2.2. *(Stability estimate.) The solution $u(x)$ of the continuous problem in (2.4)-(2.5) is bounded as*

$$|u(x)| \leq \beta^{-1}\|f\| + \max\{|\phi(0)|, |\psi(1)|\}. \tag{2.7}$$

Proof. Define two barrier functions $\vartheta^{\pm}$ as $\vartheta^{\pm}(x) = \beta^{-1}\|f\| + \max\{|\phi(0)|, |\psi(1)|\} \pm u(x)$. On the boundary points, we obtain

$$\vartheta^{\pm}(0) = \beta^{-1}\|f\| + \max\{|\phi(0)|, |\psi(1)|\} \pm u(0) \geq 0,$$
$$\vartheta^{\pm}(1) = \beta^{-1}\|f\| + \max\{|\phi(0)|, |\psi(1)|\} \pm u(1) \geq 0.$$

On the domain $x \in \Omega$, we have

$$\begin{aligned} L\vartheta^{\pm}(x) &= -(\varepsilon-\delta\alpha)\vartheta''_{\pm}(x) + a(x)\vartheta'_{\pm}(x) + b(x)\vartheta_{\pm}(x) \\ &= -(\varepsilon-\delta\alpha)(0 \pm u''(x)) + a(x)\big(0 \pm u'(x)\big) + b(x)\big(\beta^{-1}\|f\| + \max\{|\phi(0)|, |\psi(1)|\} \pm u(x)\big) \\ &= b(x)\big(\beta^{-1}\|f\| + \max\{|\phi(0)|, |\psi(1)|\}\big) \pm [-(\varepsilon-\delta\alpha)u''(x) + a(x)u'(x) + b(x)u(x)] \\ &= b(x)\big(\beta^{-1}\|f\| + \max\{|\phi(0)|, |\psi(1)|\}\big) \pm f(x) \\ &\geq 0, \text{ since } b(x) \geq \beta > 0. \end{aligned}$$

With the hypothesis of the maximum principle $\vartheta^{\pm}(x) \geq 0, \forall x \in \bar{\Omega}$, implies the required bound. □

The next lemma gives a bound for the derivatives of solution.

Lemma 2.3. *The derivative of the solution $u(x)$ of the problem in (2.4)-(2.5) is bounded as*

$$|u^{(k)}| \leq \begin{cases} C\big(1 + (\varepsilon-\delta\alpha)^{-k} \exp\big(\frac{-\alpha x}{\varepsilon-\delta\alpha}\big), & \text{for left layer,} \\ C\big(1 + (\varepsilon-\delta\alpha)^{-k} \exp\big(\frac{-\alpha(1-x)}{\varepsilon-\delta\alpha}\big), & \text{for right layer,} \end{cases}$$

for $0 \leq k \leq 4$, where α is lower bound of $a(x)$.

Proof. See in [16]. □

3. Numerical Scheme Formulation

Generally, there are two strategies for designing numerical methods which have small truncation errors in the boundary layer. The first approach is the class of fitted mesh methods which uses fine mesh in the layer region and coarse mesh in outer layer. The convergence analysis of this approach is well developed. The second approach is the fitted operator methods in which it uses uniform mesh and an exponentially fitting factor is determined for stabilizing the term containing the singular perturbation parameter. In this approach the difference schemes reflect the qualitative behaviour of the solution inside the layer. In this article, we formulate fitted operator finite difference method to find the solution of the problem in (2.4)-(2.5). Using the solution techniques developed in asymptotic method for treating singularly perturbed boundary value problems. We consider and treat separately the left and the right boundary layer cases.

Let us discretize the domain $[0, 1]$ as $x_i = ih,\ i = 0, 1, 2, ..., N$ with $x_0 = 0$, $x_N = 1$ and h is mesh length defined as $h = 1/N$ where N number of subintervals.

3.1. **Case I: Right boundary layer problem.** For singularly perturbed boundary value problem of the form in (2.4)-(2.5) having right boundary layer, the asymptotic solution up to zero order approximation is given as [18]:

$$u(x) = u_0(x) + \frac{a(1)}{a(x)}(\psi(1) - u_0(1)) \exp\Big(-\int_x^1 \big(\frac{a(x)}{\varepsilon-\delta\alpha} - \frac{b(x)}{a(x)}\big)dx\Big) + O(\varepsilon-\delta\alpha), \tag{3.1}$$

where u_0 is the solution of the reduced problem.
Using Taylor series approximation for $a(x)$ and $b(x)$ centred $x = 1$ and simplifying, we obtain

$$u(x) = u_0(x) + (\psi(1) - u_0(1)) \exp\Big(-\frac{a(1)}{\varepsilon-\delta\alpha}(1-x)\Big). \tag{3.2}$$

Considering h is small enough for each i, the discretized form of (3.2) becomes

$$u(x_i) = u_0(ih) + (\psi(1) - u_0(1)) \exp\big(-a(1)(1/(\varepsilon-\delta\alpha) - i\rho)\big), \tag{3.3}$$

where $\rho = h/(\varepsilon-\delta\alpha)$, $h = 1/N$.
Using Taylor's series approximation for $u_0((i+1)h)$ and $u_0((i-1)h)$ up to first order, we obtain

$$\begin{aligned} u(x_{i+1}) &= u_0(ih) + (\psi(1) - u_0(1)) \exp\big(-a(1)(1/(\varepsilon-\delta\alpha) - (i-1)\rho)\big), \\ u(x_{i-1}) &= u_0(ih) + (\psi(1) - u_0(1)) \exp\big(-a(1)(1/(\varepsilon-\delta\alpha) - (i+1)\rho)\big). \end{aligned} \tag{3.4}$$

We multiply exponentially fitting factor $\sigma_1(\rho)$ for the term having the perturbation parameter to handle the disturbance of the perturbation parameter as

Next, on discretized domain $\bar{\Omega}^N = \{x_i\}_{i=0}^N$, using the difference approximations

$$u'(x) = \frac{u(x_{i+1}) - u(x_{i-1})}{2h} + \tau_1$$

and

$$u''(x) = \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} + \tau_2,$$

where $\tau_1 = -\frac{h^2}{6}u^{(3)}(x_i)$ and $\tau_2 = -\frac{h^2}{12}u^{(4)}(x_i)$.

We multiply exponentially fitting factor 1( ) for the term having the perturbation parameter to handle the disturbance of the perturbation parameter.

Let U_i be an approximate solution for $u(x)$ at grid point x_i, then we write the numerical scheme for (**??**) in operator form as

$$L_R^h U_i = f_i, \ i = 1, 2, ..., N-1, \tag{3.5}$$

where $L_R^h U_i = -(\varepsilon - \delta\alpha)\sigma_1(\rho)\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + a(x_i)\frac{U_{i+1} - U_{i-1}}{2h} + b(x_i)U_i$.

Since h is small and $(f_i - b(x_i)U_i)$ is bounded, multiplying (3.5) by h and truncating the term $h(f_i - b(x_i)U_i)$, results to

$$-\frac{\sigma_1(\rho)}{\rho}\big(U_{i-1} - 2U_i + U_{i+1}\big) + \frac{a(x_i)}{2}\big(U_{i+1} - U_{i-1}\big) = 0. \tag{3.6}$$

Substituting the results in (3.3) and (3.4) into (3.6) and simplifying, the fitting factor is obtained as

$$\sigma_1(\rho) = \frac{\rho a(x_i)}{2} \coth\big(a(1)\frac{\rho}{2}\big). \tag{3.7}$$

Hence, the required finite difference scheme becomes

$$L_R^h U_i = f_i, \ i = 1, 2, ..., N-1, \tag{3.8}$$

with the boundary conditions $U_0 = \phi(0)$ and $U_N = \psi(1)$.

3.2. **Case II: Left boundary layer problem.** For the case of left boundary layer problem, the asymptotic solution up to zeroth order approximation is given as [18]

$$u(x) = u_0(x) + \frac{a(0)}{a(x)}(\phi(0) - u_0(0)) \exp\big(-\int_0^x \big(\frac{a(x)}{\varepsilon - \delta\alpha} - \frac{b(x)}{a(x)}\big)dx\big) + O(\varepsilon - \delta\alpha), \tag{3.9}$$

by using Taylor series at $x = 0$ for $a(x)$ and $b(x)$ then simplifying we obtain

$$u(x) = u_0(x) + (\phi(0) - u_0(0)) \exp\big(-\frac{a(0)}{\varepsilon - \delta\alpha}x\big), \tag{3.10}$$

where u_0 is the solution of the reduced problems.

Using the same procedure as the right boundary layer case, the fitting factor is obtained as

$$\sigma_2(\rho) = \frac{\rho a(x_i)}{2} \coth\big(a(0)\frac{\rho}{2}\big). \tag{3.11}$$

and the required finite difference scheme becomes

$$L_L^h U_i = f_i, \ i = 1, 2, ..., N-1, \tag{3.12}$$

with the boundary conditions $U_0 = \phi(0)$ and $U_N = \psi(1)$, where

$$L_L^h U_i = -(\varepsilon - \delta\alpha)\sigma_2(\rho)\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + a(x_i)\frac{U_{i+1} - U_{i-1}}{2h} + b(x_i)U_i.$$

3.3. **Uniform convergence analysis.** Here we show the convergence analysis for right boundary layer case and the left boundary layer case follows in similar manner. First, we need to prove the discrete maximum principle for the discrete scheme in (3.8).

Lemma 3.1. *(Discrete maximum principle.) Assume that the mesh function $z(x_i)$ satisfies $z(x_0) \geq 0$ and $z(x_N) \geq 0$. If $L^h z(x_i) \geq 0$ for $1 \leq i \leq N-1$, then $z(x_i) \geq 0$ $\forall i, 0 \leq i \leq N$.*

Proof. Let choose k such that $z(x_k) = \min_{x_i} z(x_i) < 0$, $1 \leq i \leq N-1$. If $z(x_k) \geq 0$, the the proof completed. We can see that $z(x_{k+1}) - z(x_k) \geq 0$ and $z(x_k) - z(x_{k-1}) \leq 0$. Now from (3.5), we obtain

$$\begin{aligned}
L^h z(x_k) =& -(\varepsilon - \delta\alpha)\sigma(\rho)\frac{z(x_{k-1}) - 2z(x_k) + z(x_{k+1})}{h^2} + a(x_k)\frac{z(x_{k+1}) - z(x_{k-1})}{2h} + b(x_k)z(x_k) \\
=& -(\varepsilon - \delta\alpha)\sigma(\rho)\frac{(z(x_{k+1}) - z(x_k)) + (z(x_{k-1}) - z(x_k))}{h^2} \\
&+ a(x_k)\frac{(z(x_{k+1}) - z(x_k)) + (z(x_k) - z(x_{k-1}))}{2h} + b(x_k)z(x_k) \\
<& 0,
\end{aligned}$$

which contradicts $L^h z(x_k) \geq 0$. Hence, the assumption is wrong. So, we conclude that $z(x_i) \geq 0$, $\forall i$, $0 \leq i \leq N$. ☐

Lemma 3.2. *(Discrete uniform stability estimate.) The solution U_i of the discrete scheme in (3.8) satisfy the following bound.*

$$|U_i| \leq \beta^{-1}\|L^h U_i\| + \max\{|U_0|, |U_N|\}. \tag{3.13}$$

Proof. Let $p = \beta^{-1}\|L^h U_i\| + \max\{|U_0|, |U_N|\}$ and define the barrier function $\vartheta_i^{\pm}$ by $\vartheta_i^{\pm} = p \pm U_i$. On the boundary points, we obtain

$$\begin{aligned}
\vartheta_0^{\pm} &= p \pm U_0 = \beta^{-1}\|L^h U_i\| + \max\{|U_0|, |U_N|\} \pm \phi(0) \geq 0, \\
\vartheta_N^{\pm} &= p \pm U_N = \beta^{-1}\|L^h U_i\| + \max\{|U_0|, |U_N|\} \pm \psi(1) \geq 0.
\end{aligned}$$

On the discretized spatial domain x_i, $0 < i < N$, we obtain

$$\begin{aligned}
L^h \vartheta_i^{\pm} =& -c_\varepsilon \sigma(\rho)\Big(\frac{p \pm U_{i+1} - 2(p \pm U_i) + p \pm U_{i-1}}{h^2}\Big) + a(x_i)\Big(\frac{p \pm U_{i+1} - p \pm U_{i-1}}{2h}\Big) + b(x_i)(p \pm U_i) \\
=& b(x_i)p \pm L^h U_i \\
=& b(x_i)\big(\beta^{-1}\|L^h U_i\| + \max\{|U_0|, |U_N|\}\big) \pm f(x_i) \\
\geq& 0, \ \ \text{since} \ \ b(x_i) \geq \beta.
\end{aligned}$$

Using the discrete maximum principle in Lemma 3.1 gives $\vartheta_i^{\pm} \geq 0$, $\forall x_i \in \bar{\Omega}^N$. Hence the required bound is obtained. ☐

Let us define the difference operators for approximating the derivatives as

$$\begin{aligned}
D^- u(x_i) =& \frac{u_i - u_{i-1}}{h}, \ D^+ u(x_i) = \frac{u_{i+1} - u_i}{h}, \\
D^0 u(x_i) =& \frac{u_{i+1} - u_{i-1}}{2h}, \ \text{and} \ D^+ D^- u(x_i) = \frac{u_{i-1} - u_i + u_{i+1}}{h^2}.
\end{aligned} \tag{3.14}$$

The next theorem gives the error bound of the developed scheme.

Theorem 3.3. *Let $u(x_i)$ and U_i be respectively the exact solution of (2.4)-(2.5) and computed solution of (3.8), then for sufficiently large N, the following error bound satisfied:*

$$\big|L^h(u(x_i) - U_i)\big| \leq \frac{CN^{-2}}{N^{-1} + (\varepsilon - \delta\alpha)}\big(1 + (\varepsilon - \delta\alpha)^{-3}\exp\big(-\frac{\alpha(1 - x_i)}{(\varepsilon - \delta\alpha)}\big)\big). \tag{3.15}$$

Proof. Consider the discretization error as

$$\begin{aligned} L^h(u(x_i)-U_i) =&(\varepsilon-\delta\alpha)\big(u''(x_i)-\sigma(\rho)D^+D^-u(x_i)\big)+a(x_i)\big(u'(x_i)-D^0u(x_i)\big),\\ =&(\varepsilon-\delta\alpha)\big[a(x_i)\frac{\rho}{2}\coth\big(a(1)\frac{\rho}{2}\big)-1\big]D^+D^-u(x_i)+(\varepsilon-\delta\alpha)\big(u''(x_i)-D^+D^-u(x_i)\big)\\ &+a(x_i)\big(u'(x_i)-D^0u(x_i)\big), \end{aligned}$$

where $\sigma(\rho)=a(x_i)\frac{\rho}{2}\coth\big(a(1)\frac{\rho}{2}\big)$, and $\rho=\frac{N^{-1}}{\varepsilon-\delta\alpha}$. Now for $z>0$ and for C_1 and C_2 are constants, then we have the following relation

$$C_1\frac{z^2}{z+1}\le z\coth(z)-1\le C_2\frac{z^2}{z+1}, \tag{3.16}$$

$$(\varepsilon-\delta\alpha)\frac{(N^{-1}/(\varepsilon-\delta\alpha))^2}{N^{-1}/(\varepsilon-\delta\alpha)+1}=\frac{N^{-2}}{N^{-1}+(\varepsilon-\delta\alpha)}. \tag{3.17}$$

Using Taylor series expansion for $u(x_{i-1})$ and $u(x_{i+1})$ at x_i as,

$$u(x_{i\pm1})=u(x_i)\pm hu'(x_i)+\frac{h^2}{2!}u''(x_i)\pm\frac{h^3}{3!}u^{(3)}(x_i)+\frac{h^4}{4!}u^{(4)}(x_i)\pm\ldots$$

we obtain the bound for

$$\begin{aligned} \big|D^+D^-u(x_i)\big|\le& C\big\|u^{(2)}(x_i)\big\|,\\ \big|u''(x_i)-D^+D^-u(x_i)\big|\le& CN^{-2}\big\|u^{(4)}(x_i)\big\|. \end{aligned} \tag{3.18}$$

Similarly for first derivative approximation error,

$$\big|u'(x_i)-D^0u(x_i)\big|\le CN^{-2}\big\|u^{(3)}(x_i)\big\|. \tag{3.19}$$

where $\big\|u^{(k)}(x_i)\big\|=sup_{x_i\in(x_0,x_N)}|u^{(k)}(x_i)|, k=2,3,4.$
Using the bounds in (3.18) and (3.19), and the results in (3.16) and (3.17), we obtain

$$\begin{aligned} \big|L^h(u(x_i)-U_i)\big|\le& C\frac{N^{-2}}{N^{-1}+(\varepsilon-\delta\alpha)}\big\|u''(x_i)\big\|+(\varepsilon-\delta\alpha)CN^{-2}\big\|u^{(4)}(x_i)\big\|+CN^{-2}\big\|u^{(3)}(x_i)\big\|\\ \le&\frac{N^{-2}}{N^{-1}+(\varepsilon-\delta\alpha)}\big\|u''(x_i)\big\|+CN^{-2}\big[(\varepsilon-\delta\alpha)\big\|u^{(4)}(x_i)\big\|+\big\|u^{(3)}(x_i)\big\|\big]. \end{aligned}$$

Using the results in Lemma 2.3 for bounds of the derivatives of the solution, we obtain

$$\begin{aligned} \big|L^h(u(x_i)-U_i)\big|\le&\frac{CN^{-2}}{N^{-1}+(\varepsilon-\delta\alpha)}\big(1+(\varepsilon-\delta\alpha)^{-2}\exp\big(-\frac{\alpha(1-x_i)}{\varepsilon-\delta\alpha}\big)\big)\\ &+CN^{-2}\Big[(\varepsilon-\delta\alpha)\big(1+(\varepsilon-\delta\alpha)^{-4}\exp\big(-\frac{\alpha(1-x_i)}{\varepsilon-\delta\alpha}\big)\big)\\ &+\big(1+(\varepsilon-\delta\alpha)^{-3}\exp\big(-\frac{\alpha(1-x_i)}{\varepsilon-\delta\alpha}\big)\big)\Big]\\ \le&\frac{CN^{-2}}{N^{-1}+(\varepsilon-\delta\alpha)}\big(1+(\varepsilon-\delta\alpha)^{-2}\exp\big(-\frac{\alpha(1-x_i)}{\varepsilon-\delta\alpha}\big)\big)\\ &+CN^{-2}\Big[\big((\varepsilon-\delta\alpha)+(\varepsilon-\delta\alpha)^{-3}\exp\big(-\frac{\alpha(1-x_i)}{\varepsilon-\delta\alpha}\big)\big)\\ &+\big(1+(\varepsilon-\delta\alpha)^{-3}\exp\big(-\frac{\alpha(1-x_i)}{\varepsilon-\delta\alpha}\big)\big)\Big] \end{aligned}$$

which simplifies to

$$\big|L^h(u(x_i)-U_i)\big|\le\frac{CN^{-2}}{N^{-1}+\varepsilon-\delta\alpha}\big(1+(\varepsilon-\delta\alpha)^{-3}\exp\big(-\frac{\alpha(1-x_i)}{\varepsilon-\delta\alpha}\big)\big), \tag{3.20}$$

since $(\varepsilon-\delta\alpha)^{-3} \geq (\varepsilon-\delta\alpha)^{-2}$ and $\frac{CN^{-2}}{N^{-1}+\varepsilon-\delta\alpha} \geq CN^{-2}$. □

Lemma 3.4. *For a fixed number of mesh numbers N and for $\varepsilon-\delta\alpha \to 0$, it holds*

$$\lim_{(\varepsilon-\delta\alpha)\to 0} \max_{1\leq i\leq N-1} (\varepsilon-\delta\alpha)^m \exp\Big(\frac{-\alpha x_i}{\varepsilon-\delta\alpha}\Big) = 0,\ m=1,2,3,... \tag{3.21}$$

$$\lim_{(\varepsilon-\delta\alpha)\to 0} \max_{1\leq i\leq N-1} (\varepsilon-\delta\alpha)^m \exp\Big(\frac{-\alpha(1-x_i)}{\varepsilon-\delta\alpha}\Big) = 0,\ m=1,2,3,... \tag{3.22}$$

where $x_i = ih, h = 1/N, i = 1,2,...,N-1$.

Proof. See in [27]. □

Theorem 3.5. *Let $u(x_i)$ and U_i be the exact solution of (2.4)-(2.5)and discrete solution of (3.8) respectively. Then the following error bound holds*

$$\sup_{0<\varepsilon-\delta\alpha\leq 1} \big\|u(x_i)-U_i\big\| \leq \frac{CN^{-2}}{N^{-1}+\varepsilon-\delta\alpha}. \tag{3.23}$$

Proof. Substituting the results in Lemma 3.4 into Theorem 3.3 and applying the discrete maximum principle, gives the required bound. □

Remark 3.6. As one sees the error bound in (3.23), for the case $\varepsilon-\delta\alpha > N^{-1}$ the scheme secures second order convergence. This means that for the case mesh size less than perturbation parameter the scheme is second order convergent. We expect to lose an order of convergence for the case $\varepsilon-\delta\alpha \leq N$, and in fact it turns out that the scheme is first-order uniformly convergent.

4. Numerical Results and Discussion

In this section, we consider numerical examples to illustrate the theoretical findings of the developed schemes.

Example 4.1. We consider an example of constant coefficient problem with right boundary layer

$$-\varepsilon u''(x) + u'(x-\delta) + u(x) = 0$$

with interval-boundary conditions $u(x) = 1,\ -\delta \leq x \leq 0,\ u(1) = -1$. The exact solution is given as

$$u(x) = \frac{(1+e^{m_2})e^{m_1 x} - (1+e^{m_1})e^{m_2 x}}{e^{m_2}-e^{m_1}},$$

where $m_1 = \frac{1-\sqrt{(1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}$ and $m_2 = \frac{1+\sqrt{(1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}$.

Example 4.2. Now we consider variable coefficient problem with right boundary layer

$$-\varepsilon u''(x) + \exp(x)u'(x-\delta) + xu(x) = 0$$

with interval-boundary conditions $u(x) = 1,\ -\delta \leq x \leq 0,\ u(1) = 1$.

Example 4.3. Now we consider a constant coefficient problem with left boundary layer

$$-\varepsilon u''(x) - u'(x-\delta) + u(x) = 0$$

with interval-boundary conditions $u(x) = 1,\ -\delta \leq x \leq 0,\ u(1) = 1$.
The exact solution is given as

$$u(x) = \frac{(1-e^{m_2})e^{m_1 x} + (e^{m_1}-1)e^{m_2 x}}{e^{m_1}-e^{m_2}}$$

where $m_1 = \frac{-1-\sqrt{(1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}$ and $m_2 = \frac{-1+\sqrt{(1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}$.

Example 4.4. Now we consider a variable coefficient problem with left boundary layer

$$-\varepsilon u''(x) - \exp(-0.5x)u'(x-\delta) + u(x) = 0$$

with interval-boundary conditions $u(x) = 1, \ -\delta \leq x \leq 0, \ u(1) = 1$.

The exact solution of the variable coefficient problems is not known. So, we use the double mesh technique to calculate maximum absolute error. Let U_i^N denoted for the computed solution of the problem where N is the number of mesh points and U_i^{2N} is used to denote the computed solution on double number of mesh points $2N$ by including the mid-points $x_{i+1/2} = \frac{x_{i+1}+x_i}{2}$ into the mesh points. The maximum absolute error is given by

$$E^N_{\varepsilon,\delta} = \max_i |U_i^N - u(x_i)|, \text{ or } E^N_{\varepsilon,\delta} = \max_i |U_i^N - U_i^{2N}|,$$

and the ε-uniform error is calculated using

$$E^N = \max_{\varepsilon,\delta} |E^N_{\varepsilon,\delta}|.$$

The rate of convergence of the scheme is calculated using

$$r^N_{\varepsilon,\delta} = \log_2 \left(E^N_{\varepsilon,\delta} / E^{2N}_{\varepsilon,\delta} \right).$$

and the ε- uniform rate of convergence is calculated using

$$r^N = \log_2 \left(E^N / E^{2N} \right).$$

Table 1 Maximum absolute error and rate of convergence of Example 1 for $\delta = 0.5\varepsilon$.

$\varepsilon \downarrow$	N $\rightarrow 2^6$	2^7	2^8	2^9	2^{10}
2^{-2}	1.5959e-06	3.9901e-07	9.9754e-08	2.4940e-08	6.2334e-09
	1.9999	2.0000	1.9999	2.0004	-
2^{-4}	4.1786e-05	1.0453e-05	2.6137e-06	6.5345e-07	1.6336e-07
	1.9991	1.9998	1.9999	2.0000	-
2^{-6}	2.2228e-04	5.5990e-05	1.4025e-05	3.5078e-06	8.7706e-07
	1.9891	1.9972	1.9994	1.9998	-
2^{-8}	8.2386e-04	2.2774e-04	5.8521e-05	1.4733e-05	3.6899e-06
	1.8550	1.9604	1.9899	1.9974	-
2^{-10}	1.4084e-03	6.2171e-04	2.0857e-04	5.7563e-05	1.4785e-05
	1.1797	1.5757	1.8573	1.9610	-
2^{-12}	1.4222e-03	7.1487e-04	3.5492e-04	1.5608e-04	5.2294e-05
	0.9924	1.0102	1.1852	1.5776	-
2^{-14}	1.4222e-03	7.1490e-04	3.5837e-04	1.7940e-04	8.8896e-05
	0. 9923	0.9963	0.9983	1.0130	-
2^{-16}	1.4222e-03	7.1490e-04	3.5837e-04	1.7941e-04	8.9759e-05
	0.9923	0.9963	0.9982	0.9991	-
2^{-18}	1.4222e-03	7.1490e-04	3.5837e-04	1.7941e-04	8.9759e-05
	0.9923	0.9963	0.9982	0.9991	-
2^{-20}	1.4222e-03	7.1490e-04	3.5837e-04	1.7941e-04	8.9759e-05
	0.9923	0.9963	0.9982	0.9991	-
E^N	1.4222e-03	7.1490e-04	3.5837e-04	1.7941e-04	8.9759e-05
r^N	0.9923	0.9963	0.9982	0.9991	-

The solution of the problems in Examples 1 and 2 exhibits right boundary layer behaviour and in Example 3 and 4 exhibits left boundary layer behaviour. In Figure 2 we observe, as the perturbation parameter ε goes small the

Table 2 Maximum absolute error and rate of convergence of Example 2 for $\delta = 0.5\varepsilon$.

$\varepsilon \downarrow$	N $\rightarrow 2^6$	2^7	2^8	2^9	2^{10}
2^{-2}	6.8034e-06	1.7016e-06	4.2542e-07	1.0636e-07	2.6593e-08
	1.9994	1.9999	1.9999	1.9998	-
2^{-4}	4.5971e-05	1.1505e-05	2.8771e-06	7.1933e-07	1.7984e-07
	1.9985	1.9996	1.9999	1.9999	-
2^{-6}	1.9210e-04	4.8570e-05	1.2177e-05	3.0463e-06	7.6172e-07
	1.9837	1.9959	1.9990	1.9997	-
2^{-8}	6.5653e-04	1.8758e-04	4.8706e-05	1.2296e-05	3.0817e-06
	1.8074	1.9453	1.9859	1.9964	-
2^{-10}	1.0133e-03	4.6602e-04	1.6568e-04	4.7120e-05	1.2218e-05
	1.1206	1.4920	1.8140	1.9473	-
2^{-12}	1.0184e-03	5.1326e-04	2.5636e-04	1.1715e-04	4.1517e-05
	0.9885	1.0015	1.1298	1.4966	-
2^{-14}	1.0184e-03	5.1327e-04	2.5766e-04	1.2908e-04	6.4280e-05
	0. 9885	0.9943	0.9972	1.0058	-
2^{-16}	1.0184e-03	5.1327e-04	2.5766e-04	1.2909e-04	6.4608e-05
	0.9885	0.9943	0.9971	0.9986	-
2^{-18}	1.0184e-03	5.1327e-04	2.5766e-04	1.2909e-04	6.4608e-05
	0.9885	0.9943	0.9971	0.9986	-
2^{-20}	1.0184e-03	5.1327e-04	2.5766e-04	1.2909e-04	6.4608e-05
	0.9885	0.9943	0.9971	0.9986	-
E^N	1.0184e-03	5.1327e-04	2.5766e-04	1.2909e-04	6.4608e-05
r^N	0.9885	0.9943	0.9971	0.9986	-

boundary layer formation becomes visible. In this figure we plot for $\varepsilon = 2^{-4}, 2^{-5}$ and 2^{-6} for $\delta = 0.3\varepsilon$. In Figure 1 one can see the effect of the delay parameter on the solution of the problems. In this figure we plot for $\delta = 0, 0.2\varepsilon, 0.4\varepsilon$ and 0.6ε for $\varepsilon = 0.1$. As one see on the figures for right boundary layer problems as values of δ increases the thickness of the boundary layer increases. For the case of left boundary layer problem as the values of the δ increases the thickness of the boundary layer decreases.

In Tables 1-4, the maximum absolute error, rate of convergence of Examples 1 - 4 are given. As one see the results in these tables the proposed scheme have second order of convergence for the case ε is greater than mesh size h, and for the case ε goes small the scheme reduces to first order convergence as stated in Remark 3.6. As the perturbation parameter goes small for each N, the maximum absolute error becomes stable and uniform. This shows that the proposed scheme is uniformly convergent(converges independent of the perturbation parameter as the perturbation parameter goes small) with rate of convergence one.

In Tables 5-7, the comparison of maximum absolute error of the proposed scheme and scheme in [21] are given. In these tables the result are calculated for $\varepsilon = 0.1$ and different values of the delay parameter. In Table 8, the comparison of the maximum absolute error of the proposed scheme and the hybrid Tension spline method with mid-point upwind scheme on Shishkin mesh in [20] are given. As one observes the results in these tables the proposed scheme is more accurate that the scheme in [20, 21].

Table 3 Maximum absolute error and rate of convergence of Example 3 for $\delta = 0.5\varepsilon$.

$\varepsilon \downarrow$	N $\rightarrow 2^6$	2^7	2^8	2^9	2^{10}
2^{-2}	3.0150e-05	7.5403e-06	1.8853e-06	4.7136e-07	1.1804e-07
	1.9995	1.9998	1.9999	1.9976	-
2^{-4}	1.6393e-04	4.1165e-05	1.0303e-05	2.5764e-06	6.4414e-07
	1.9936	1.9984	1.9996	1.9999	-
2^{-6}	6.5049e-04	1.7267e-04	4.3857e-05	1.1008e-05	2.7548e-06
	1.9135	1.9771	1.9943	1.9985	-
2^{-8}	1.3709e-03	5.4382e-04	1.6524e-04	4.3805e-05	1.1122e-05
	1.3339	1.7186	1.9154	1.9777	-
2^{-10}	1.4228e-03	7.1450e-04	3.4546e-04	1.3660e-04	4.1457e-05
	0.9937	1.0484	1.3386	1.7203	-
2^{-12}	1.4228e-03	7.1498e-04	3.5838e-04	1.7929e-04	8.6529e-05
	0.9928	0.9964	0.9992	1.0510	-
2^{-14}	1.4228e-03	7.1498e-04	3.5838e-04	1.7941e-04	8.9759e-05
	0.9928	0.9964	0.9982	0.9991	-
2^{-16}	1.4228e-03	7.1498e-04	3.5838e-04	1.7941e-04	8.9759e-05
	0.9928	0.9964	0.9982	0.9991	-
2^{-18}	1.4228e-03	7.1498e-04	3.5838e-04	1.7941e-04	8.9759e-05
	0.9928	0.9964	0.9982	0.9991	-
2^{-20}	1.4228e-03	7.1498e-04	3.5838e-04	1.7941e-04	8.9759e-05
	0.9928	0.9964	0.9982	0.9991	-
E^N	1.4228e-03	7.1498e-04	3.5838e-04	1.7941e-04	8.9759e-05
r^N	0.9928	0.9964	0.9982	0.9991	-

Table 4 Example 4, maximum absolute error for $\delta = 0.5\varepsilon$.

$\varepsilon \downarrow$	N $\rightarrow 2^6$	2^7	2^8	2^9	2^{10}
2^{-2}	5.3033e-05	1.3263e-05	3.3167e-06	8.2919e-07	2.0729e-07
	1.9995	1.9996	2.0000	2.0001	-
2^{-4}	2.5937e-04	6.5064e-05	1.6280e-05	4.0719e-06	1.0181e-06
	1.9951	1.9988	1.9993	1.9998	-
2^{-6}	1.0006e-03	2.6041e-04	6.7107e-05	1.6825e-05	4.2093e-06
	1.9420	1.9562	1.9959	1.9990	-
2^{-8}	2.1551e-03	7.5387e-04	2.5253e-04	6.5640e-05	1.6909e-05
	1.5154	1.5779	1.9438	1.9568	-
2^{-10}	2.7057e-03	1.3242e-03	5.4455e-04	1.8896e-04	6.3279e-05
	1.0309	1.2820	1.5270	1.5783	-
2^{-12}	2.7072e-03	1.3642e-03	6.8440e-04	3.3296e-04	1.3651e-04
	0.9887	0.9951	1.0395	1.2863	-
2^{-14}	2.7072e-03	1.3642e-03	6.8480e-04	3.4308e-04	1.7161e-04
	0.9887	0.9943	0.9971	0.9994	-
2^{-16}	2.7072e-03	1.3642e-03	6.8480e-04	3.4308e-04	1.7171e-04
	0.9887	0.9943	0.9971	0.9986	-
2^{-18}	2.7072e-03	1.3642e-03	6.8480e-04	3.4308e-04	1.7171e-04
	0.9887	0.9943	0.9971	0.9986	-
2^{-20}	2.7072e-03	1.3642e-03	6.8480e-04	3.4308e-04	1.7171e-04
	0.9887	0.9943	0.9971	0.9986	-
E^N	2.7072e-03	1.3642e-03	6.8480e-04	3.4308e-04	1.7171e-04
r^N	0.9887	0.9943	0.9971	0.9986	-

Table 5 Example 2, Comparison of maximum absolute error for $\varepsilon = 0.1$.

$\delta \downarrow$	$N \rightarrow 10^2$	10^3	10^4
Proposed Method			
0.01	1.6856e-05	1.6864e-07	1.6691e-09
0.03	1.3270e-05	1.3275e-07	1.3270e-09
0.06	9.8506e-06	9.8535e-08	1.0152e-09
0.08	8.3261e-06	8.3278e-08	8.3463e-10
Result in [21]			
0.01	0.00575975	0.00050842	5.02478e-05
0.03	0.003932768	0.00036132	3.58384e-05
0.06	0.002702569	0.00025507	2.53643e-05
0.08	0.00224689	0.00021413	2.13134e-05

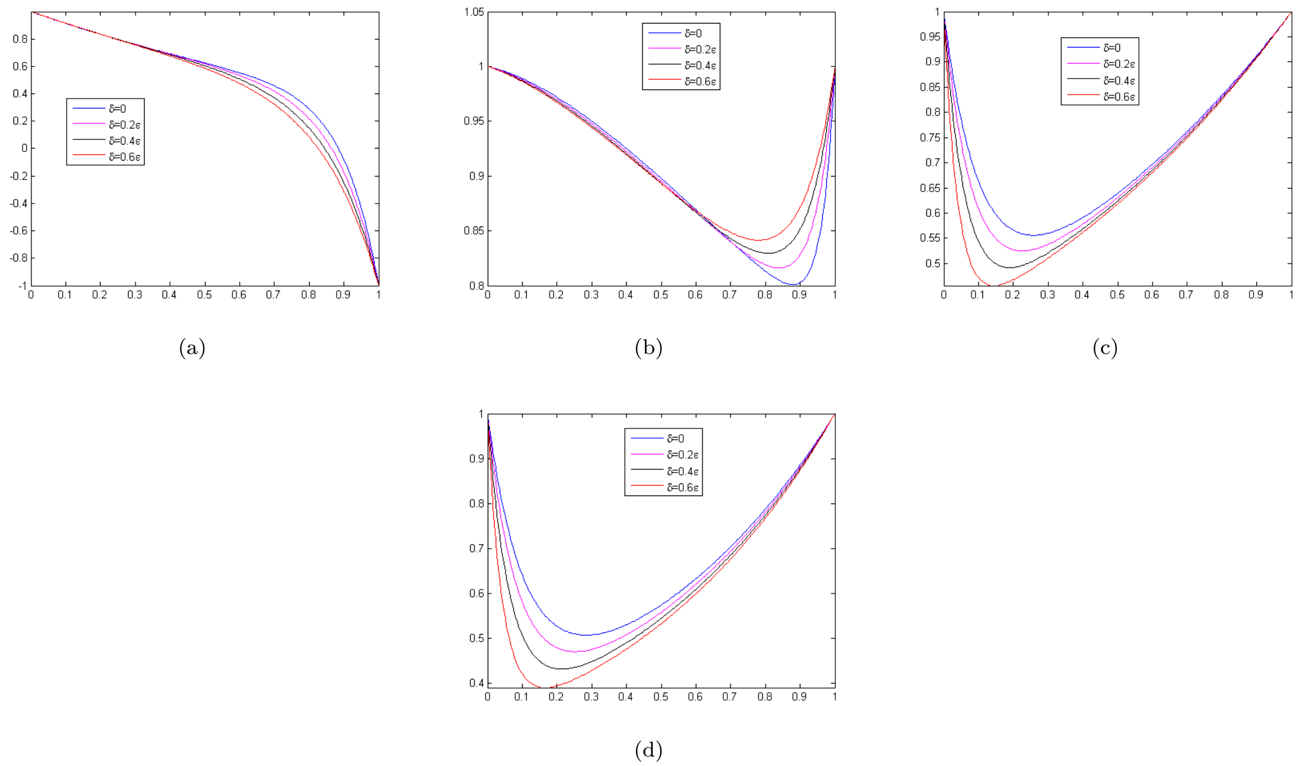
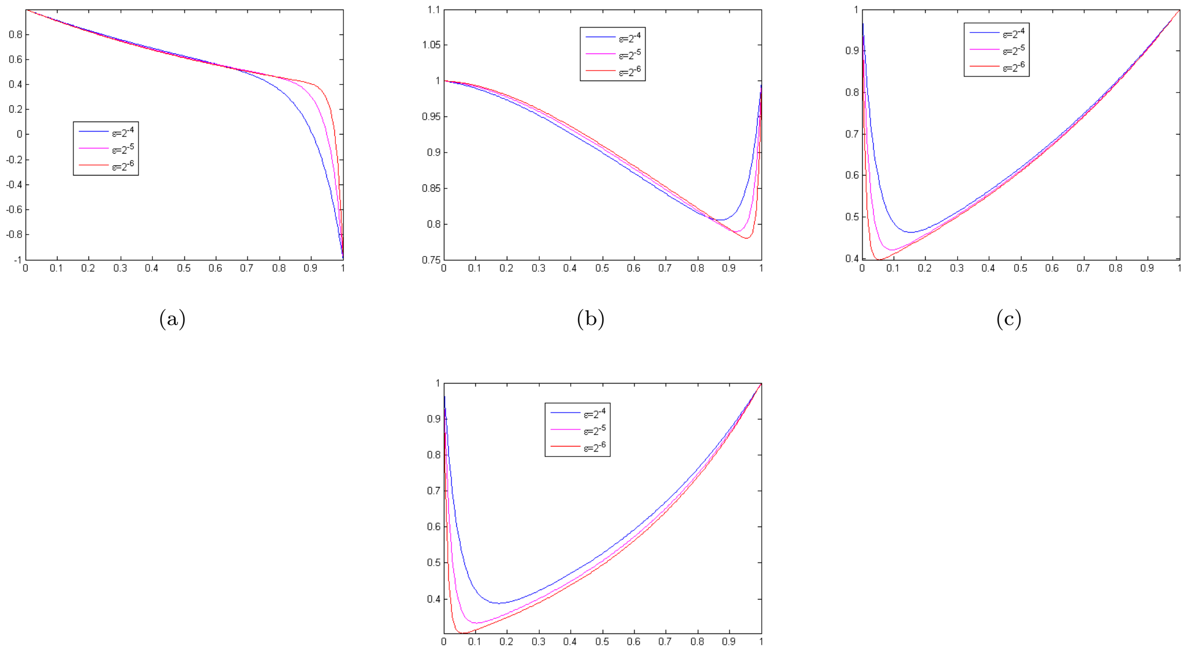


Figure 1 Delay effect on solution (a) Example 1, (b) Example 2, (c) Example 3 and (d) Example 4 for $\varepsilon = 0.1$.

Table 6 Example 3, Comparison of maximum absolute error for $\varepsilon = 0.1$.

$\delta \downarrow$	$N \to 10^2$	10^3	10^4
Proposed Method			
0.01	1.9342e-05	1.9348e-07	2.6703e-07
0.03	2.6592e-05	2.6607e-07	2.2057e-07
0.06	5.1266e-05	5.1346e-07	1.2318e-08
0.08	1.0854e-04	1.0915e-06	1.1049e-08
Result in [21]			
0.01	0.01172	0.00122	1.231e-004
0.03	0.01505	0.00158	1.598e-004
0.06	0.02575	0.00281	2.839e-004
0.08	0.04781	0.00562	5.735e-004



(a) (b) (c) (d)

Figure 2 Effect of ε on solution with boundary layer, (a) Example 1, (b) Example 2, (c) Example 3 and (d) Example 4 for $\delta = 0.3\varepsilon$.

Table 7 Example 4, Comparison of maximum absolute error for $\varepsilon = 0.1$.

$\delta \downarrow$	$N \to 10^2$	10^3	10^4
Proposed Method			
0.01	3.0968e-05	3.0994e-07	7.9967e-08
0.03	4.2151e-05	4.2161e-07	3.4257e-07
0.06	8.7483e-05	8.7707e-07	5.1270e-08
0.08	2.5205e-04	2.5655e-06	4.0077e-08
Result in [21]			
0.01	0.00632996	0.000674268	6.7871251e-05
0.03	0.00815917	0.000882563	8.8986856e-05
0.06	0.01384760	0.001579726	1.6020004e-04
0.08	0.02477158	0.003173235	3.2602775e-04

Table 8 Example 3, Comparison of maximum absolute error for $\delta = 0.5\varepsilon$.

$\varepsilon\downarrow$	$N \rightarrow 2^5$	2^6	2^7	2^8	2^9
Proposed	Scheme				
2^{-4}	3.3570e-04	1.6393e-04	4.1165e-05	1.0303e-05	2.5764e-06
2^{-8}	2.7500e-03	1.3709e-03	5.4382e-04	1.6524e-04	4.3805e-05
2^{-12}	2.8165e-03	1.4228e-03	7.1498e-04	3.5838e-04	1.7929e-04
2^{-16}	2.8165e-03	1.4228e-03	7.1498e-04	3.5838e-04	1.7941e-04
2^{-20}	2.8165e-03	1.4228e-03	7.1498e-04	3.5838e-04	1.7941e-04
Result	in [20]				
2^{-4}	1.52e-02	4.63e-03	1.16e-03	3.05e-04	8.50e-05
2^{-8}	1.67e-02	5.59e-03	1.85e-03	6.01e-04	1.90e-04
2^{-12}	1.67e-02	5.61e-03	1.85e-03	6.02e-04	1.90e-04
2^{-16}	1.67e-02	5.61e-03	1.85e-03	6.02e-04	1.90e-04
2^{-20}	1.67e-02	5.61e-03	1.85e-03	6.02e-04	1.90e-04

5. Conclusion

In this paper, numerical treatment of singularly perturbed delay differential equations is considered. The solution of the considered problem exhibits boundary layer behavior. Using Taylor's series approximation for the delay term asymptotically equivalent singularly perturbed boundary value problem obtained. The numerical schemes are developed using the exponentially fitted finite difference method. The stability of the schemes is investigated using the barrier function for the solution bound and discrete maximum principle is used for the existence of the unique discrete solution. The uniform convergence of the schemes is proved. The proposed scheme was investigated by considering four test examples exhibiting boundary layers. The effect of the perturbation parameter and the delay parameter on the solution is shown using figures and tables. The developed scheme is accurate and uniformly convergent with the rate of convergence one.

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