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A numerical method for solving fractional optimal control problems using the operational matrix of Mott polynomials

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Abstract

This paper presents a numerical method for solving a class of fractional optimal control problems (FOCPs) based on numerical polynomial approximation. The fractional derivative in the dynamic system is described in the Caputo sense. We used the approach to approximate the state and control functions by the Mott polynomials (M-polynomials). We introduced the operational matrix of fractional Riemann-Liouville integration and apply it to approximate the fractional derivative of the basis. We investigated the convergence of the new method and some examples are included to demonstrate the validity and applicability of the proposed method.

Keywords. Fractional optimal control problem, Caputo derivative, Mott polynomials basis, Operational matrix.

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1. Introduction

Fractional order dynamics appear in several problems in science and engineering such as viscoelasticity, dynamics of interfaces between nanoparticles and substrates, telecommunications, electronics, computers, robotics, medicine, dynamic systems, and time control [6, 8, 11, 26]. It has also been shown that the materials with memory and hereditary effects and dynamical processes including gas diffusion and heat conduction in fractal porous media can be modeled by fractional order models better than integer models [38]. We use the Hamiltonian function [33], to solve the fractional optimal control problems (FOCPs) but giving the analytic solution of them is usually complicated or, in most cases, impossible, for this reason, numerical methods are used to solve them [1, 9]. In recent years, the use of different methods for solving optimal control problems has been considered by some researchers. The numerical solution of the fractional differential equations is reviewed by Roberto [10]. Ghomanjani and his colleagues presented an article called the numerical method for solving FOCPs and differential equations [12]. A new numerical approach for solving fractional optimal control problems including state and control inequality constraints using new biorthogonal multiwavelets by E. Ashpazzadeh, M. Lakestani, A. Yildirim [5]. For the FOCP problem with boundary conditions, it seems more appropriate to derive the Caputo fraction, which is expressed in 2016 is expressed by Nemati et al. In an article entitled Response to the problems of fractional optimal control using the Ritz method [24]. The FOCPs with the Bernstein polynomial operational matrix has been investigated by Jafari and a researcher [13]. The requirements for the FOCP optimality have been investigated by Sewilman et al [32]. A square FOCP was solved directly and without the Hamilton formula by Yousefi and Lotfi [3, 37]. Numerical solution for fractional optimal control problems by Hermite polynomials by A. Yari [36]. Solving fractional optimal control problems using Genocchi polynomials is presented by M. A. Moghaddam, Y. Edrisi, and M. Lakestani [22].

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The Hamilton formula and the direct numerical method for solving the FOCPs by Agrowall are discussed [2]. A new numerical Bernoulli polynomial method for solving FOCPs by Taherpour et al [34]. Samaneh Soradi-Zeid, presented an article called Solving a class of fractional optimal control problems via a new efficient and accurate method [31].

Since in most cases, applying theoretical methods is usually difficult and time consuming for solving the FOCPs. Therefore, we consider Mott polynomials as the basic functions, based on the theory of best approximation, which can be used to determine all the parameters in FOCPs. Nowadays, various operational matrices for the polynomials have been developed to cover the numerical solution of differential and integral equations [4, 18, 20]. The main advantage of our new method is that by applying only a few number of Mott basis we achieve satisfactory results.

In this paper, we focus on optimal control problems with the quadratic performance index and the dynamic system with the Caputo fractional derivative. We intend to extend the application of polynomials to solve fractional differential equations. Our main aim is to generalize Mott operational matrix to fractional calculus. We refer the interested reader to [7, 15, 19, 28] for more studies on this subject.

We solve the problem directly without using Hamiltonian formulas. Our tools for this aim are the Mott basis and the operational matrix of fractional integration.

The problem formulation is as follows:

$$J=\int_a^t (A(t)x^2(t)+B(t)u^2(t))dt, \tag{1.1}$$

$$M\dot{x}(t)+N{}_a^cD_t^\alpha x(t)=f(t,x(t),u(t)), \tag{1.2}$$

$$x(a)=x_a,$$

here, A(t), B(t) are arbitary functions and $(M,N)\neq(0,0)$.

The paper is organized as follows: In section 2, we present some preliminaries on fractional calculus. Section 3 describes the basic formulation of Mott polynomials (M-polynomials) required for our subsequent development and section 4 is devoted to the function approximation by using the M-polynomials basis. In section 5, we explain the general procedure of forming of the operational matrix of integration. Section 6 describes the convergence of the method. In section 7, the fundamental problem of FOCP is stated. We report our numerical findings and demonstrate the validity, accuracy, and applicability of the operational matrix by considering numerical examples in section 8. Finally, this paper will end with a brief conclusion in section 9.

2. Some preliminaries on fractional calculus

In this section, we give some basic defns and properties of the fractional calculus which are used further in this paper [25, 27].

Definition 2.1. The (left sided) Riemann – Liouville fractional integral of order $\alpha>0$, of A function $f:[a,b]\to R$, is defined as:

$${}_0I_t^\alpha f(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds, & \alpha>0, t>0,\\ f(t), & \alpha=0,\end{cases} \tag{2.1}$$

for example, with $f(t)=t^\gamma$ we have:

$${}_0I_t^\alpha t^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}t^{\gamma+\alpha}, \quad \alpha\in N\cup\{0\}, t>0. \tag{2.2}$$

Definition 2.2. The (left sided) Caputo fractional derivative of a function $f:[a,b]\to R$, is defined as:

$${}_0^cD_t^\alpha f(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}\dfrac{d^n}{ds^n}f(s)ds, & n-<\alpha<n, n\in N,\\ f^{(n)}(t), & \alpha=n, n\in N.\end{cases} \tag{2.3}$$

Consequently, the following relations are straightforward from (2.1) and (2.3),

$$(i) \qquad {}_0I_t^{\alpha}\,{}_0^cD_t^{\alpha}f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0)\frac{t^k}{k!}, \tag{2.4}$$

$$(ii) \qquad {}_0^cD_t^{\beta}\,{}_0I_t^{\alpha}f(t) = \begin{cases} I^{\alpha-\beta}f(t), & \text{if } \alpha > \beta, \\ f(t), & \text{if } \alpha = \beta, \\ D^{\beta-\alpha}f(t), & \text{if } \alpha < \beta. \end{cases} \tag{2.5}$$

3. Mott polynomials and their properties

In 1932, Mott [23] originally introduced the polynomial while monitoring the roaming behaviors of electrons for a problem in the theory of electrons.The Mott polynomials are widely studied in mathematics and physics. Roman [29] obtained an associated Sheffer sequence of the Mott polynomials. For further properties of the polynomial, the reader can refer to [17, 21].

The M-polynomials $s_n(t)$, $n \geq 0$, is defined by :

$$s_n(t) = (-1)^n \left(\frac{t}{2}\right)^n (n-1)! \sum_{k=0}^{h\left(\frac{n}{2}\right)} \binom{n}{k} (-1)^k \frac{t^{-2k}}{(n-2k-1)!}, \tag{3.1}$$

where

$$h\left(\frac{n}{2}\right) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even}, \\ \frac{n}{2} - \frac{1}{2}, & \text{if } n \text{ is odd}. \end{cases}$$

From the above equation, it is obvious that $s_0(t) = 1$, for this reason $s_n(t)$ is a Sheffer set [35]. The first few M-polynomials are

$$\begin{aligned} s_0(t) &= 1, \\ s_1(t) &= -\frac{1}{2}t, \\ s_2(t) &= \frac{1}{4}t^2, \\ s_3(t) &= \frac{3}{4}t - \frac{1}{8}t^3, \\ s_4(t) &= -\frac{3}{2}t^2 + \frac{1}{16}t^4, \\ s_5(t) &= -\frac{15}{2}t + \frac{15}{8}t^3 - \frac{1}{32}t^5. \end{aligned}$$

Now we define the Vector A_r, for $r = 0, 1, \dots, n$, if r is odd then for $i = 0, 1, \dots, n$,

$$A_r = b_{i_r}, \tag{3.2}$$

$$b_{i_r} = \begin{cases} 0, & i = 2l, l \in N \cup \{0\}, \\ c_{i_r} & \text{otherwise}, \end{cases} \tag{3.3}$$

$$c_{i_r} = \left(\frac{-1}{2}\right)^r (r-1)! \frac{\binom{r}{k_i}(-1)^{k_i}}{(r-2k_i-1)!},$$

$$k_i = \beta_j, \beta_j - 1, \dots, 0,$$

$$\beta_j = \frac{r-1}{2} - j, j = 0, 1, \dots, \frac{r-1}{2}. \tag{3.4}$$

If r is even then for $i = 0, 1, \ldots, n$,

$$A_r = b'_{i_r}, \tag{3.5}$$

$$b'_{i_r} = \begin{cases} 0 & i = 2l+1, l \in N \cup \{0\}, \\ c'_{i_r} & \text{otherwise}, \end{cases} \tag{3.6}$$

$$c'_{i_r} = \left(\frac{-1}{2}\right)^r (r-1)! \frac{\binom{r}{k'_i}(-1)^{k'_i}}{(r-2k'_i-1)!},$$

$$k'_i = \beta'_j, \beta'_j - 1, \ldots, 0,$$

$$\beta'_j = \frac{r}{2} - j, j = 0, 1, \ldots, \frac{r}{2}, \tag{3.7}$$

then , $S_r(t) = A_r T_r(t)$ for $(r = 0, 1, \ldots, n)$, here $T_r(t) = [1, t, \ldots, t^r]^T$, then, we define $(r+1)\times(r+1)$ lower triangular matrix A such that $A = [A_0, A_1, \ldots, A_r]^T$ and $A_i (i = 0, 1, \ldots, n)$ is row vector of order r.
As a result,

$$\varphi_n(t) = AT_n(t), \tag{3.8}$$

where

$$\varphi_n(t) = \left[s_0(t), s_1(t), \ldots, s_n(t)\right]^T. \tag{3.9}$$

4. Function approximation

We recall here some theorems and lemma that are stated and proved in Kreyszing (1978). Suppose that $H = L^2[0,1]$ be the Hilbert space and $\{s_0, s_1, \ldots, s_n\}$ the M-polynomials of degree n on the interval $[0,1]$. We define $Y = \text{Span}\{s_0, s_1, \ldots, s_n\}$ Let f be an arbitrary element in H Since Y is a finite dimensional subspace of the space H, the function f has the best unique approximation on Y like $f_n \in Y$, that is, [16]

$$\exists f_n \in Y \text{ s.t. } \forall y \in Y \quad \|f - f_n\|_2 \leq \|f - y\|_2, \tag{4.1}$$

where $\|f\|_2 = \sqrt{\langle f, f\rangle}$ and $\langle , \rangle$ denotes the inner product. Since $f_n \in Y$, therefore f_n is a linear combination of the spanning basis of Y, that's mean, there are $n+1$ coefficients

$$C = [c_0, c_1, \ldots, c_n] \in R \tag{4.2}$$

such that

$$f(t) \simeq f_n(t) = \sum_{j=0}^{n} c_j s_j(t) = C^T \varphi_n(t), \tag{4.3}$$

where

$$\|f - f_n\|_2 \to \min, \tag{4.4}$$

then C can be obtained by:

$$C = Q^{-1}\langle f(t), \varphi_n(t)\rangle, \tag{4.5}$$

where

$$Q = \langle \varphi_n(t), \varphi_n(t)\rangle = \int_0^1 \varphi_n(t)\varphi_n(t)^T dt. \tag{4.6}$$

Theorem 4.1. *Let X be an inner product space and $M \neq \emptyset$ a convex subset which is complete in the metric induced by the inner product. Then for every given $x \in X$ there exists a unique $y \in M$ such that*

$$\delta = \inf_{\tilde{y} \in M} \|x - \tilde{y}\| = \|x - y\|. \tag{4.7}$$

5. Mott operational matrix of the fractional integration

In this section we describe the Mott polynomials operational matrix of fractional integration of the vector φ_n. The operational matrix can be approximated as:

$$I^{\alpha}\varphi_n(t) \simeq P\varphi_n(t), \tag{5.1}$$

where P is the $(n+1)\times(n+1)$ Riemann-Liouville fractional operational matrix of integration. We construct P as follows:

$$\begin{aligned} I^{\alpha}s_i(t) &= (-1)^i\left(\frac{1}{2}\right)^i(i-1)!\sum_{k=0}^{h\left(\frac{i}{2}\right)}\binom{i}{k}(-1)^k\frac{I^{\alpha}t^{i-2k}}{(i-2k-1)!} \\ &= (-1)^i\left(\frac{1}{2}\right)^i(i-1)!\sum_{k=0}^{h\left(\frac{i}{2}\right)}\frac{\binom{i}{k}(-1)^k}{(i-2k-1)!}t^{i-2k+\alpha}\frac{\Gamma(i-2k+1)}{\Gamma(i-2k+\alpha+1)}, \end{aligned} \tag{5.2}$$

now we approximate $t^{i-2k+\alpha}$ by $n+1$ terms of the Mott basis

$$t^{i-2k+\alpha} \simeq \sum_{j=0}^{n} b_j s_j, \tag{5.3}$$

where

$$\begin{aligned} b_j &= Q_j^{-1}\int_0^1 t^{i-2k+\alpha}s_j(t)dt \\ &= (-1)^j\left(\frac{1}{2}\right)^j(j-1)!\sum_{L=0}^{h\left(\frac{j}{2}\right)}\frac{\binom{j}{L}(-1)^L}{(j-2L-1)!}\times\frac{1}{i-2k+\alpha+j-2L+1}. \end{aligned} \tag{5.4}$$

Therefore we have

$$I^{\alpha}s_i(t) \simeq \sum_{j=0}^{n} B_{ij}s_j(t), \tag{5.5}$$

where

$$\begin{aligned} B_{ij} = (-1)^{i+j}&\left(\frac{1}{2}\right)^{i+j}(i-1)!(j-1)!\sum_{k=0}^{h\left(\frac{i}{2}\right)}\sum_{L=0}^{h\left(\frac{j}{2}\right)}\binom{j}{L}\binom{i}{k}(-1)^{k+L} \\ &\times\frac{1}{(i-2k-1)!(j-2L-1)!}\times\frac{1}{i+j-2L-2k+\alpha+1}, \end{aligned} \tag{5.6}$$

and

$$Q_j = \langle s_j(t), s_j(t)\rangle = \int_0^1 s_j(t)s_j(t)^T dt. \tag{5.7}$$

Finally, we obtain

$$P = \begin{bmatrix} B_{00} & \dots & B_{0n} \\ \vdots & \ddots & \vdots \\ B_{n0} & \dots & B_{nn} \end{bmatrix}, \tag{5.8}$$

where P is called the Mott polynomials operational matrix of fractional integration.

6. Convergence study

In this section, we discuss the convergence of the method presented in Section 5. First we will find the upper bound error for the operational matrix of the fractional integration P and show that with an increase in the number of Mott polynomials, the error vector eI^{α} tend zero. We restate the following theorems by finding this error.

For any given elements $x_1, x_2, \ldots, x_n$ in a Hilbert space H, the Gram determinant of these elements is defined as follows [16]:

$$G(x_1, x_2, \ldots, x_n) = \begin{vmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle & \ldots & \langle x_1, x_n\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle & \ldots & \langle x_2, x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1\rangle & \langle x_n, x_2\rangle & \ldots & \langle x_n, x_n\rangle \end{vmatrix}. \tag{6.1}$$

Theorem 6.1. *Suppose that H is a Hilbert space and a closed subspace of H such that $\dim Y < \infty$ and $y_1, y_2, \ldots, y_n$ is any basis for Y. Let x be an arbitrary element in H and y_0 be the unique best approximation to x out of Y. Then [16]*

$$\|x - y_0\|_2^2 = \frac{G(x, y_1, y_2, \ldots, y_n)}{G(y_1, y_2, \ldots, y_n)}, \tag{6.2}$$

where

$$G(x, y_1, y_2, \ldots, y_n) = \begin{vmatrix} \langle x, x\rangle & \langle x, y_1\rangle & \ldots & \langle x, y_n\rangle \\ \langle y_1, x\rangle & \langle y_1, y_1\rangle & \ldots & \langle y_1, y_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle y_n, x\rangle & \langle y_n, y_1\rangle & \ldots & \langle y_n, y_n\rangle \end{vmatrix}. \tag{6.3}$$

Theorem 6.2. *Suppose $f(t) \in L^2[0,1]$ is approximated by $f_n(t)$ as [30]*

$$f_n(t) = \sum_{i=0}^{n} c_i \beta_i(t) = C^T \varphi_n(t), \tag{6.4}$$

where $C, \varphi_n(t)$ are defined in Equations (3.9) and (4.5), then we have

$$\lim_{m\to\infty} \|f(t) - f_n(t)\|_{L^2[0,1]} = 0. \tag{6.5}$$

The error vector eI^{α} of the operational matrix is given by

$$eI^{\alpha} = [eI_0^{\alpha}, eI_1^{\alpha}, \ldots, eI_n^{\alpha}]^T = P\varphi_n(t) - I^{\alpha}\varphi_n(t) \tag{6.6}$$

From Equation (6.2) and Theorem 4.1, we have

$$\left\| t^{i-2k+\alpha} - \sum_{j=0}^{n} b_j s_j \right\|_2 = \left(\frac{G(t^{i-2k+\alpha}, s_i(t), s_1(t), \ldots, s_n(t))}{G(s_0(t), s_1(t), \ldots, s_n(t))} \right). \tag{6.7}$$

Hence, according to Equations (6.6) and (5.5) we have

$$\begin{aligned}\|eI_i^\alpha\|_2 &= \left| I^\alpha s_i(t) - \sum_{j=0}^{n} B_{ij}s_j(t)\right|_2 \le \left[\left(-\frac{1}{2}\right)^{i+j}(i-1)!(j-1)!\right] \\ &\times \sum_{k=0}^{h\left(\frac{i}{2}\right)}\sum_{L=0}^{h\left(\frac{j}{2}\right)}\binom{j}{L}\binom{i}{k}(-1)^{k+L}\frac{1}{(i-2k-1)!(j-2L-1)!} \\ &\times \frac{1}{i+j-2L-2k+\alpha+1}\left|t^{i-2k+\alpha} - \sum_{j=0}^{n} b_j s_j\right|_2 \\ &\le \left[\left(-\frac{1}{2}\right)^{i+j}(i-1)!(j-1)!\right] \\ &\times \sum_{k=0}^{h\left(\frac{i}{2}\right)}\sum_{L=0}^{h\left(\frac{j}{2}\right)}\binom{j}{L}\binom{i}{k}(-1)^{k+L}\frac{1}{(i-2k-1)!(j-2L-1)!} \\ &\times \frac{1}{i+j-2L-2k+\alpha+1} \\ &\times \left(\frac{G(t^{i-2k+\alpha}, s_0(t), s_1(t), \ldots, s_n(t))}{G(s_0(t), s_1(t), \ldots, s_n(t))}\right)^{\frac{1}{2}}. \end{aligned} \tag{6.8}$$

By considering Theorem 6.2 and Equation (6.8), we can conclude that by increasing the number of the Mott bases the vector eI^α tends to zero.

7. Problem Statement And Approximate Method

Considering the following fractional optimal control problem

$$\mathrm{Min}J = \int_0^1 (A(t)x^2(t) + B(t)u^2(t) + Kx(t)u(t))dt, \tag{7.1}$$

$$M\dot{x}(t) + N{}_0^cD_t^\alpha x(t) = Gx(t) + Hu(t) + f(t), \tag{7.2}$$

$$x(0) = x_0, x(1) = x_1, \tag{7.3}$$

here A(t), B(t) and f(t) are arbitary functions, $(M, N) \neq (0, 0)$, G, H are real number and $0 < \alpha \le 1$.
We expand the fractional derivative of the state variable by the Mott basis $\varphi_n(t)$:

$${}_0^cD_t^\alpha x(t) \simeq C\varphi_n(t), \tag{7.4}$$

$$u(t) \simeq U\varphi_n(t), \tag{7.5}$$

$$f(t) \simeq F\varphi_n(t), \tag{7.6}$$

where

$$C = [c_0, c_1, \ldots, c_n], \tag{7.7}$$

$$U = [u_0, u_1, \ldots, u_n], \tag{7.8}$$

$$F = [f_0, f_1, \ldots, f_n]. \tag{7.9}$$

Using (2.1),(2.4), and (7.3), $x(t)$ can be represented as

$$x(t) = {}_0I_t^\alpha {}_0^cD_t^\alpha x(t) + x(0) \simeq (CP + d_0)\varphi_n(t), \tag{7.10}$$

where P is the fractional operational matrix of integration of order α and

$$d_0 = [x_0, 0, \dots, 0]. \tag{7.11}$$

We convert the dinamical system (7.2) to the linear system of algebraic equations, for this work derivating of (7.10) we can expand $\dot{x}(t)$ by $\varphi_n(t)$

$$\dot{x}(t) = (CP + d_0)\dot{\varphi_n}(t) = (CP + d_0)D_\varphi \varphi_n(t), \tag{7.12}$$

where $\dot{\varphi_n}(t)$=$D_\varphi \varphi_n(t)$ and D_φ is the (n+1)(n+1) matrix of derivative for mott polynomials.

Using Eqs. (7.4), (7.5), (7.6), (7.10), and (7.12) the dynamical system (7.2) can also be approximated as

$$[M(CP + d_0)D_\varphi + NC - G(CP + d_0) - HU - F]\varphi_n(t) = 0, \tag{7.13}$$

because Eq. (7.13) satisfy for any $t \in [0, 1]$, therefor we can rewrite it as following

$$M(CP + d_0)D_\varphi + NC - G(CP + d_0) - HU - F = 0. \tag{7.14}$$

The boundary conditions $x(0) = x_0$, $x(1) = x_1$ can be expressed as

$$(CP + d_0)\varphi_n(0) - x_0 = 0, (CP + d_0)\varphi_n(1) - x_1 = 0. \tag{7.15}$$

Using Eqs. (7.5) and (7.10), the performance index J can be approximated as

$$\begin{aligned} J \simeq J(C, U) = \int_0^1 & A[(CP + d_0)\varphi_n(t)]^T [(CP + d_0)\varphi_n(t)] \\ & + B[U\varphi_n(t)]^T [U\varphi_n(t)] + K(CP + d_0)\varphi_n(t)U\varphi_n(t)dt. \end{aligned} \tag{7.16}$$

We find the extremum of Eq. (7.16) subject to Eqs. (7.14) and (7.15) using the Lagrange multiplier technique. Let

$$\begin{aligned} J^*[C, U, \lambda] = J[C, U] + (M(CP + d_0)D_\varphi + NC - G(CP + d_0) - HU - F)\lambda_1 + \\ ((CP + d_0)\varphi_n(0) - x_0)\lambda_2 + ((CP + d_0)\varphi_n(1) - x_1)\lambda_3 \end{aligned} \tag{7.17}$$

Where $\lambda = (\lambda_1, \lambda_2, \lambda_3)^T$, is the unknown Lagrange multiplier. Now the necessary conditions for the extremum are:

$$\frac{\partial J^*}{\partial C} = 0, \frac{\partial J^*}{\partial U} = 0, \frac{\partial J^*}{\partial \lambda} = 0. \tag{7.18}$$

By solving the above equations we can obtain C, U, λ, therefore we determine the approximate solution of u(t), x(t) and J (C, U) from Eqs. (7.5), (7.10), and (7.16) respectively.

8. Illustrative examples

Example 1. Consider the following time invariant problem [13]

$$\begin{aligned} \text{Min} \quad & J[u, x] = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt, \\ \text{s.t.} \quad & M\dot{x}(t) + ND^\alpha x(t) = -x(t) + u(t), \\ & x(0) = 1, x(1) = \cosh(\sqrt{2}) + \beta \sinh(\sqrt{2}). \end{aligned}$$

Our aim is to find $u(t)$ which minimizes the performance index J. In the above example, $M, N \in \mathbb{Z}$. For this problem if $M = N = \frac{1}{2}$ we have the exact solution in the case when $\alpha = 1$ given by

$$\begin{aligned} x(t) &= \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t), \\ u(t) &= (1 + \sqrt{2}\beta) \cosh(\sqrt{2}t) + (\sqrt{2} + \beta) \sinh(\sqrt{2}t), \\ \beta &= -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})}. \end{aligned}$$

We approximate ${}_0^cD_t^\alpha x(t)$ and $u(t)$. Table 1 shows the comparison between the approximation of J obtained using the proposed method. Table 2 gives absolute error of J for different values of n and α. Table 3 shows the absolute error of J, x and u at $\alpha = 1$ for different values of n. We give the absolute errors of $x(t)$ and $u(t)$ for different points of time and various n, in Table 4, 5 respectively. In Fig.1, the state variable $x(t)$ and the control variable $u(t)$ are plotted for α=1 and n=7. In Fig.2, the state variable $x(t)$ and the control variable $u(t)$ are plotted for n=7 and different values of α. It is obvious that with an increase in the number of the Mott basis, the approximate values of $x(t)$ and $u(t)$ converge to the exact solutions. The above example has been examined in [5, 22] and we compared the results of our proposed method in addition to the boundary condition $x(1) = \cosh(\sqrt{2}) + \beta \sinh(\sqrt{2})$ with them in Tables 6, 7.

Table 1. Approximation values of J for different values of n and $\alpha = 1$

J_{app} \ J_{exa}	**0.1929092980931**69395510456535 81209
$n = 3$	**0.1929**31605837041991914805182078 03
$n = 4$	**0.192909**445024110019492050703471 35
$n = 5$	**0.192909298**957682019565192664473 79
$n = 6$	**0.19290929809**5708337055838116249 37
$n = 7$	**0.1929092980931**77028472297381972 73

Table 2. Absolute error of J for different values of n and α

α \ n	3	4	5	6	7
0.8	$1.1681e-2$	$1.2691e-2$	$1.2791e-2$	$1.2932e-2$	$1.2916e-2$
0.9	$6.1291e-3$	$6.5737e-3$	$6.61e-3$	$6.6677e-3$	$6.6605e-3$
0.99	$6.4157e-4$	$6.8641e-4$	$6.8791e-4$	$6.8879e-4$	$6.8869e-4$
1	$2.2308e-5$	$1.4693e-7$	$8.6451e-10$	$2.5389e-12$	$7.633e-15$

Table 3. Absolute errors of J, x and u at $\alpha = 1$ for different values of n

n \ error	$\lvert J_e - J_a\rvert$	$\lvert x_e - x_a\rvert$	$\lvert u_e - u_a\rvert$
4	$1.469e-7$	$4.295e-5$	$5.404e-4$
6	$2.539e-12$	$1.232e-7$	$2.25e-6$
8	$7.633e-15$	$5.851e-9$	$1.235e-7$

Our results that are achieved with much less computational work, are agreement with the results obtained in [5, 22]

TABLE 4. Absolute error of $x(t)$ for different points of time and various n

t \ n	3	4	5	6	7
0	0	0	0	0	0
0.1	$9.2597e-4$	$4.4768e-5$	$9.6889e-7$	$5.5214e-8$	$6.2087e-9$
0.2	$5.695e-4$	$2.113e-5$	$3.7412e-6$	$1.486e-7$	$3.9971e-10$
0.3	$2.0442e-4$	$6.502e-5$	$2.437e-6$	$9.6583e-8$	$9.0355e-9$
0.4	$8.3651e-4$	$4.9899e-5$	$2.103e-6$	$1.8505e-7$	$8.8922e-10$
0.5	$1.033e-3$	$4.2355e-6$	$4.2358e-6$	$9.3461e-9$	$9.1735e-9$
0.6	$7.3287e-4$	$5.3733e-5$	$1.7316e-6$	$1.8648e-7$	$1.9123e-10$
0.7	$7.9429e-5$	$5.997e-5$	$2.6123e-6$	$7.8992e-8$	$8.9139e-9$
0.8	$6.0473e-4$	$1.3943e-5$	$3.4451e-6$	$1.4712e-7$	$7.0845e-10$
0.9	$8.3847e-4$	$4.2732e-5$	$1.0677e-6$	$4.3106e-8$	$5.4989e-9$
1	$1.6307e-10$	$1.5307e-10$	$4.7693e-10$	$1.8907e-10$	$2.4493e-10$

TABLE 5. Absolute error of $u(t)$ for different points of time and various n

t \ n	3	4	5	6	7
0	$1.9402e-2$	$1.7698e-3$	$1.4643e-4$	$8.6165e-6$	$5.0019e-7$
0.1	$2.0581e-3$	$3.8303e-4$	$5.7045e-5$	$3.3456e-6$	$1.2129e-7$
0.2	$6.2553e-3$	$6.9762e-4$	$2.3373e-5$	$1.3901e-6$	$1.5768e-7$
0.3	$7.9343e-3$	$2.1108e-4$	$3.6761e-5$	$2.448e-6$	$3.3571e-9$
0.4	$5.2644e-3$	$3.5971e-4$	$4.3298e-5$	$6.0057e-7$	$1.413e-7$
0.5	$4.3739e-4$	$5.998e-4$	$1.724e-6$	$2.5209e-6$	$3.66e-9$
0.6	$4.4305e-3$	$3.7966e-4$	$4.0756e-5$	$6.9307e-7$	$1.3835e-7$
0.7	$7.2848e-3$	$1.5971e-4$	$3.7064e-5$	$2.2861e-6$	$1.0064e-8$
0.8	$6.1187e-3$	$6.2051e-4$	$1.8956e-5$	$1.3784e-6$	$1.5025e-7$
0.9	$1.0396e-3$	$3.6085e-4$	$5.273e-5$	$3.0178e-6$	$1.0864e-7$
1	$1.6139e-2$	$1.4932e-3$	$1.3015e-4$	$7.6575e-6$	$4.5872e-7$

TABLE 6. Comparison of the value of J for $\alpha = 1$

n	Present Method		Method[22]	
	J	error of J	J[22]	error of J[22]
7	0.1929092980931770284722971	7.6e-15	0.1929092980957083369557418	2.5e-12
8	0.1929092980931770284432020	7.6e-15	0.1929092980931770283741513	7.6e-15
9	0.1929092980931694000830916	4.5e-18	0.1929092980931694000830882	1.3e-17

Example 2. Consider the following fractional optimal control problem (FOCP)[14]:

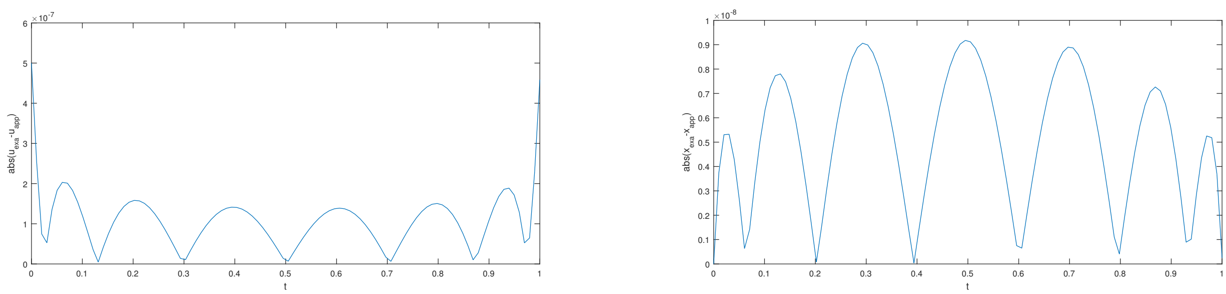
$$\mathrm{Min} J[u, x] = \int_0^1 (tu(t) - (\alpha + 2)x(t))^2 dt,$$

subject to the dynamical fractional control system

$$\dot{x}(t) + {}_0^c D_t^\alpha x(t) = u(t) + t^2,$$

TABLE 7. Absolute error of $x(t)$ with comparison to Ref.[5]

t	Present Method			Method[5]		
	n=6	n=7	n=8	J=6	J=7	J=8
0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	5.52e-8	6.20e-9	6.23e-09	2.05e-5	5.07e-6	1.26e-6
0.2	1.48e-7	3.99e-10	4.60e-10	1.67e-5	4.14e-6	1.03e-6
0.3	9.65e-8	9.03e-9	8.94e-09	1.36e-5	3.36e-6	8.35e-7
0.4	1.85e-7	8.89e-10	1.01e-09	1.09e-6	2.69e-6	6.68e-7
0.5	2.52e-6	3.66e-9	9.33e-09	8.54e-6	2.11e-6	5.23e-7
0.6	6.93e-7	1.38e-7	3.89e-10	6.54e-6	1.60e-6	3.99e-7
0.7	7.89e-8	8.91e-9	8.67e-09	4.75e-6	1.17e-6	2.90e-7
0.8	1.47e-7	7.08e-10	9.96e-10	3.24e-6	7.87e-7	1.94e-7
0.9	4.31e-8	5.49e-9	5.83e-09	1.84e-6	4.44e-7	1.09e-7



(a) error of u(t)

(b) error of x(t)

FIGURE 1. Absolute errors of u(t) , x(t) for Example1 with n=7 and α=1 .

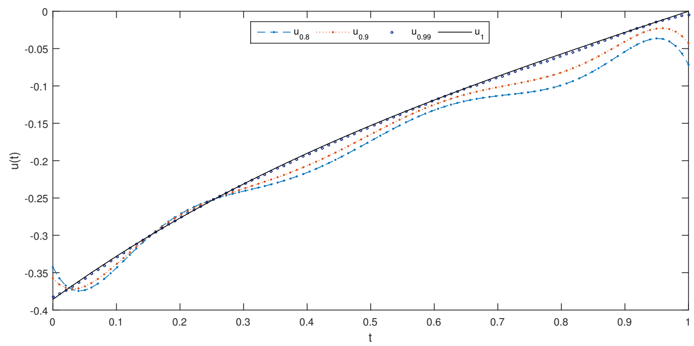
and the boundary conditions

$$x(0) = 0, x(1) = \frac{2}{\Gamma(\alpha+3)},$$

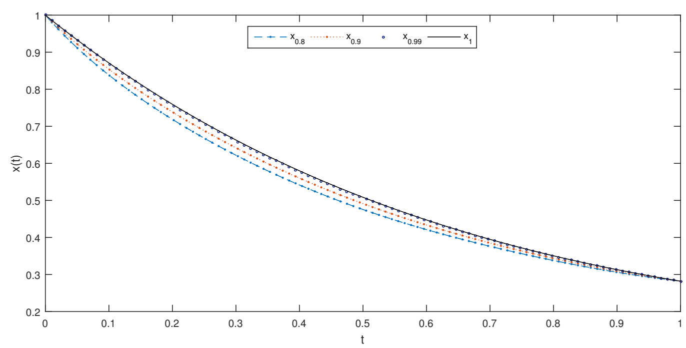
the exact solution is given by

$$(x(t), u(t)) = (\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}, \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)}).$$

The results for $\alpha = 0.5$ and n = 3 are plotted in Figure 3. In Figure 4, we give absolute errors of the exact and approximate control (left) and state (right) for α=1 and n=8. Table 8 shows the error of J for different values of α and n. Absolute errors of x(t), u(t) are listed for different points of time and α in Tables 9 and 10. Table 11 shows the errors of x(t), u(t) for different values of n when α=0.5 .



(a) Approximate of u(t), α=0.8,0.9,0.99,1



(b) Approximate of x(t) , α=0.8,0.9,0.99,1

FIGURE 2. Approximate solution of u(t) , x(t) for Example1 with n=7 and different values of α .

TABLE 8. Absolute error of J for different values of n and α

α \ n	3	4	5	6	7	8
0.3	$9.5161e-6$	$5.2402e-7$	$5.6001e-8$	$9.1729e-9$	$1.9609e-9$	$5.6001e-10$
0.5	$1.0997e-5$	$4.4238e-7$	$3.2496e-8$	$5.0758e-9$	$1.0237e-9$	$2.5684e-10$
0.7	$7.1661e-6$	$2.1146e-7$	$8.5937e-9$	$1.4777e-9$	$2.7575e-10$	$5.1312e-11$
0.9	$1.5386e-6$	$3.4931e-8$	$6.009e-10$	$1.897e-10$	$4.6145e-11$	$6.993e-12$
0.99	$2.2185e-8$	$4.5829e-10$	$4.8944e-12$	$3.0198e-12$	$2.2075e-12$	$3.2225e-12$
1	$2.2041e-39$	$8.2652e-40$	$3.8571e-39$	$4.4999e-39$	$4.5918e-39$	$6.2448e-39$

TABLE 9. Absolute error of $x(t)$ for different points of time and $n = 8$

t \ α	0.3	0.5	0.7	0.9	0.99	1
0	0	0	0	0	0	0
0.1	$2.26e-4$	$1.881e-4$	$1.497e-4$	$1.904e-4$	$3.85e-5$	$5.421e-20$
0.2	$8.612e-4$	$7.042e-4$	$5.123e-4$	$5.668e-4$	$1.08e-4$	$4.337e-19$
0.3	$1.805e-3$	$1.438e-3$	$9.745e-4$	$9.701e-4$	$1.753e-4$	0
0.4	$2.982e-3$	$2.301e-3$	$1.418e-4$	$1.169e-3$	$1.867e-4$	$3.469e-18$
0.5	$4.318e-3$	$3.2e-3$	$1.702e-3$	$8.526e-4$	$6.4e-5$	0
0.6	$5.773e-3$	$4.08e-3$	$1.722e-3$	$2.394e-4$	$2.618e-4$	$1.388e-17$
0.7	$7.31e-3$	$4.918e-3$	$1.474e-3$	$2.085e-3$	$7.863e-4$	0
0.8	$8.873e-3$	$5.733e-3$	$1.141e-3$	$4.086e-3$	$1.347e-3$	$2.776e-17$
0.9	$1.038e-2$	$6.608e-3$	$1.195e-3$	$4.717e-3$	$1.518e-3$	0
1	$1.172e-2$	$7.71e-3$	$2.504e-3$	$1.134e-3$	$4.953e-4$	0

TABLE 10. Absolute error of $u(t)$ for different points of time and $n = 8$

t \ α	0.3	0.5	0.7	0.9	0.99	1
0	$4.018e-3$	$2.525e-3$	$1.091e-3$	$2.291e-4$	$1.702e-4$	$4.744e-21$
0.1	$8.123e-2$	$4.796e-3$	$4.096e-3$	$5.563e-3$	$1.183e-3$	0
0.2	$1.813e-1$	$8.693e-3$	$6.866e-3$	$8.207e-3$	$1.61e-3$	0
0.3	$2.823e-1$	$1.202e-2$	$8.784e-3$	$9.374e-3$	$1.739e-3$	0
0.4	$3.781e-1$	$1.440e-2$	$9.581e-3$	$8.485e-3$	$1.399e-3$	0
0.5	$4.66e-1$	$1.598e-2$	$9.182e-3$	$4.943e-3$	$3.866e-3$	0
0.6	$5.445e-1$	$1.7e-2$	$7.745e-3$	$1.161e-3$	$1.307e-3$	0
0.7	$6.122e-1$	$1.758e-2$	$5.693e-3$	$8.635e-3$	$3.361e-3$	0
0.8	$6.681e-1$	$1.791e-2$	$3.848e-3$	$1.481e-2$	$5.03e-3$	0
0.9	$7.113e-1$	$1.835e-2$	$3.584e-3$	$1.52e-2$	$5.043e-3$	0
1	$7.411e-1$	$1.924e-2$	$6.741e-3$	$3.302e-3$	$1.476e-3$	0

TABLE 11. Errors of the state and the control for different values of n, in α=0.5

Table a. n=2

error \ *Methods*	presented method	method[14]
Error of stste	$1.1e-4$	$1.2e-4$
Error of control	$6.6e-3$	$7.1e-3$

Table b. n=3

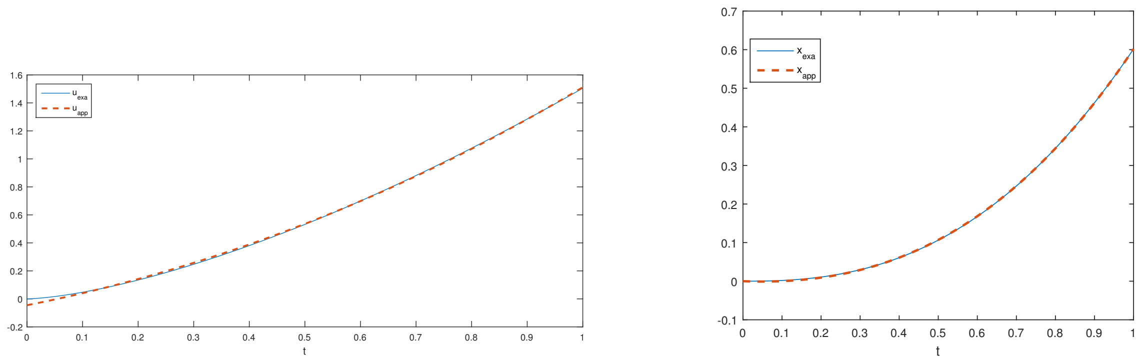
error \ *Methods*	presented method	method[14]
Error of stste	$6.7e-7$	$7.1e-7$
Error of control	$9.2e-5$	$9.6e-5$

Example 3. Consider the following FOCP [14]:

$$\text{Min} J[u, x] = \int_0^1 (u(t) - x(t))^2 dt,$$

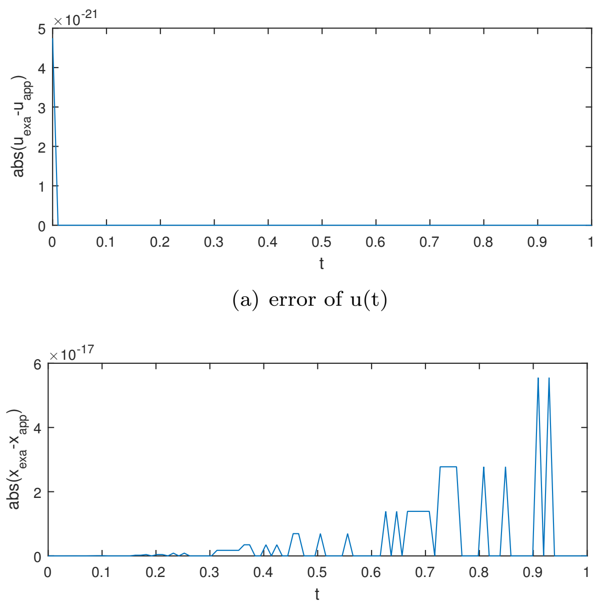
Table c. n=4

Methods / *error*	presented method	method[14]
Error of stste	$2.7e-8$	$2.8e-8$
Error of control	$7.8e-6$	$7.9e-6$



(a) Compare of u(t) (b) Compare of x(t)

FIGURE 3. Comparison between exact and approximate solutions u(t) , x(t) for Example 2 with α=0.5 and n=3.



(a) error of u(t)

(b) error of x(t)

FIGURE 4. Absolute errors of u(t) and x(t) for Example 2 with α=1 and n=8.

subject to the fractional dynamical control system

$$\dot{x}(t) + {}_0^cD_t^\alpha x(t) = u(t) - x(t) + \frac{6t^{\alpha+2}}{\Gamma(\alpha+3)} + t^3,$$

and the boundary conditions

$$x(0) = 0, x(1) = \frac{6}{\Gamma(\alpha + 4)},$$

the exact solution is given by

$$(x(t), u(t)) = (\frac{6t^{\alpha+3}}{\Gamma(\alpha + 4)}, \frac{6t^{\alpha+3}}{\Gamma(\alpha + 4)}).$$

The results for different values of α and n are plotted in Figures 5, 6. We show Approximate solutions of u(t) and x(t) In Figure 7. Tables 12-14 show the errors for different values of α and n. Table 15 shows the errors of x(t), u(t) for $\alpha = 0.5$ and n = 2, 3, 4.

Table 12. Absolute error of J for different values of n and α

α \ n	3	4	5	6	7	8
0.3	$1.4518e-4$	$1.7149e-6$	$5.6001e-8$	$9.1729e-9$	$1.9609e-9$	$5.6001e-10$
0.5	$3.4757e-4$	$1.9903e-6$	$3.2496e-8$	$5.0758e-9$	$1.0237e-9$	$2.5684e-10$
0.7	$6.2451e-4$	$1.2449e-6$	$8.5937e-9$	$1.4777e-9$	$2.7575e-10$	$5.1312e-11$
0.9	$1.0689e-3$	$2.3598e-7$	$6.009e-10$	$1.897e-10$	$4.6145e-11$	$6.993e-12$
0.99	$2.2185e-8$	$3.0979e-9$	$4.8944e-12$	$3.0198e-12$	$2.2075e-12$	$3.2225e-12$
1	$2.2041e-39$	$3.5117e-74$	$3.8571e-39$	$4.4999e-39$	$4.5918e-39$	$6.2448e-39$

Table 13. Absolute error of $x(t)$ for different points of time and $n = 8$

t \ α	0.3	0.5	0.7	0.9	0.99	1
0	0	0	0	0	0	0
0.1	$2.26e-4$	$1.881e-4$	$1.497e-4$	$1.904e-4$	$3.85e-5$	$5.421e-20$
0.2	$8.612e-4$	$7.042e-4$	$5.123e-4$	$5.668e-4$	$1.08e-4$	$4.337e-19$
0.3	$1.805e-3$	$1.438e-3$	$9.745e-4$	$9.701e-4$	$1.753e-4$	0
0.4	$2.982e-3$	$2.301e-3$	$1.418e-4$	$1.169e-3$	$1.867e-4$	$3.469e-18$
0.5	$4.318e-3$	$3.2e-3$	$1.702e-3$	$8.526e-4$	$6.4e-5$	0
0.6	$5.773e-3$	$4.08e-3$	$1.722e-3$	$2.394e-4$	$2.618e-4$	$1.388e-17$
0.7	$7.31e-3$	$4.918e-3$	$1.474e-3$	$2.085e-3$	$7.863e-4$	0
0.8	$8.873e-3$	$5.733e-3$	$1.141e-3$	$4.086e-3$	$1.347e-3$	$2.776e-17$
0.9	$1.038e-2$	$6.608e-3$	$1.195e-3$	$4.717e-3$	$1.518e-3$	0
1	$1.172e-2$	$7.71e-3$	$2.504e-3$	$1.134e-3$	$4.953e-4$	0

TABLE 14. Absolute error of $u(t)$ for different points of time and $n = 8$

t \ α	0.3	0.5	0.7	0.9	0.99	1
0	$4.018e-3$	$2.525e-3$	$1.091e-3$	$2.291e-4$	$1.702e-4$	$4.744e-21$
0.1	$8.123e-2$	$4.796e-3$	$4.096e-3$	$5.563e-3$	$1.183e-3$	0
0.2	$1.813e-1$	$8.693e-3$	$6.866e-3$	$8.207e-3$	$1.61e-3$	0
0.3	$2.823e-1$	$1.202e-2$	$8.784e-3$	$9.374e-3$	$1.739e-3$	0
0.4	$3.781e-1$	$1.440e-2$	$9.581e-3$	$8.485e-3$	$1.399e-3$	0
0.5	$4.66e-1$	$1.598e-2$	$9.182e-3$	$4.943e-3$	$3.866e-3$	0
0.6	$5.445e-1$	$1.7e-2$	$7.745e-3$	$1.161e-3$	$1.307e-3$	0
0.7	$6.122e-1$	$1.758e-2$	$5.693e-3$	$8.635e-3$	$3.361e-3$	0
0.8	$6.681e-1$	$1.791e-2$	$3.848e-3$	$1.481e-2$	$5.03e-3$	0
0.9	$7.113e-1$	$1.835e-2$	$3.584e-3$	$1.52e-2$	$5.043e-3$	0
1	$7.411e-1$	$1.924e-2$	$6.741e-3$	$3.302e-3$	$1.476e-3$	0

TABLE 15. Errors of the state and the control for different values of n, in α=0.5.

Table a. n=2

error \ *Methods*	presented method	method[14]
Error of stste	$7.3e-4$	$7.3e-4$
Error of control	$4.4e-2$	$5.2e-2$

Table b. n=3

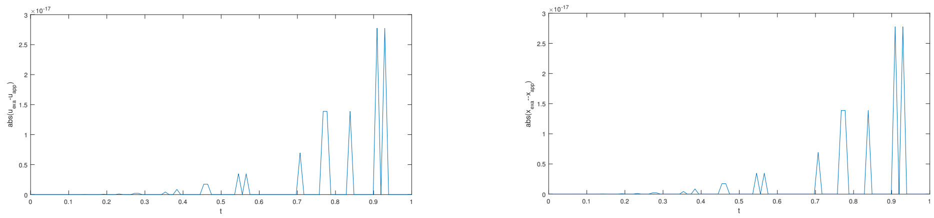
error \ *Methods*	presented method	method[14]
Error of stste	$2.5e-6$	$2.5e-6$
Error of control	$3.6e-4$	$3.9e-4$

Table c. n=4

error \ *Methods*	presented method	method[14]
Error of stste	$8.2e-9$	$8.4e-9$
Error of control	$2.1e-6$	$2.2e-6$

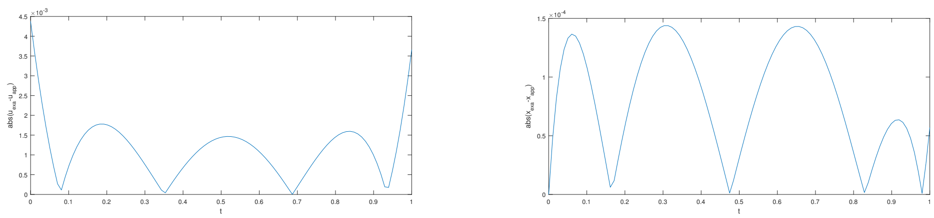
9. CONCLUSION

In the present paper, we developed an efficient and accurate method for solving a class of fractional optimal control problems. The Mott polynomials operational matrices of fractional integration were derived for constrained optimization and applied to reduce the problem to the problem of solving a system of algebraic equations. A general procedure of forming this matrix was given. Illustrative examples demonstrate the validity and applicability of the new method. We exert Matlab for computations in this paper.



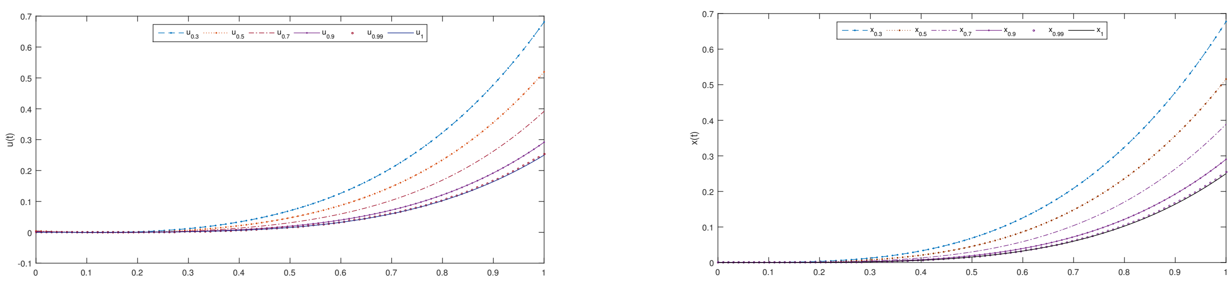
(a) error of u(t) (b) error of x(t)

FIGURE 5. Absolute errors of u(t) and x(t) for Example 3 with α=1 and n=4.



(a) error of u(t) (b) error of x(t)

FIGURE 6. Absolute errors of u(t) and x(t) for Example 3 with α=0.3 and n=4.



(a) $u_{app}(t)$, α=0.3,0.5,0.7,0.9,0.99,1 (b) $x_{app}(t)$, α=0.3,0.5,0.7,0.9,0.99,1

FIGURE 7. Approximate solutions of u(t) and x(t) of Example 3 with n=4 and different values of α.

References

[1] O. P. Agrawal, *General formulation for the numerical solution of optimal control problems*, Int. J. Control, *50*(2) (1989), 627–638 .

[2] O. P. Agrawal, *A general formulation and solution scheme for fractional optimal control problem*, Nonlinear Dynam, *38* (2004), 323–337.

[3] O. P. Agrawal, *A quadratic numerical scheme for fractional optimal control problems*, Trans. ASME, J. Dyn. Syst. Meas. Control, *130*(1) (2008), 011010-011016.

[4] M. Ahmadi Darani and A. Saadatmandi, *The operational matrix of fractional derivative of the fractional-order Chebyshev functions and its applications*, Computational Methods for Differential Equations, *5*(1) (2017), 67-87.

[5] E. Ashpazzadeh and M. Lakestani, *Biorthogonal cubic Hermite spline multiwavelets on the interval for solving the fractional optimal control problems*, Computational Methods for Differential Equations, *4*(2) (2016), 99-115. DOI: 10.1002/oca.2615.

[6] R. L. Bagley and P. J. Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity*, J. Rheol., *27* (1983), 201-210.

[7] D. Baleanu, *About fractional quantization and fractional variational principles*, Commun. Nonlinear Sci. Numer. Simul. *14* (2009), 2520-2523.

[8] T. S. Chow, *Fractional dynamics of interfaces between soft-nanoparticles and rough substrates*, Phys. Lett. A, *342* (2005), 148-155.

[9] S. S. Ezz-Eldien , E. H. Doha, D. Baleanu, and A. H. Bhrawy, *A numerical approach based on Legendre orthonormal polynomials for numerical solutions of fractional optimal control problems*, Journal of Vibration and Control, *23*(1) (2017), 16–30.

[10] R. Garrappa, *Numerical Solution of Fractional Differential Equations: A Survey and a Software Tutorial*, https://www.mdpi.com/2227-7390/6/2/16/htm, 2018.

[11] I. M. Gelfand and S. V. Fomin, *Calculus of Variation (R.A. Silverman, Trans.)*, PrenticeHall, 1963.

[12] F. Ghomanjani, *A numerical technique for solving fractional optimal control problems and fractional Riccati differential equation*, Journal of the Egyptian Mathematical Society, *24* (2016), 638–643.

[13] H. Jafari and H. Tajadodi, *Fractional order optimal control problems via the operational matrices of bernstein polynomials*, U.P.B. Sci. Bull., Series A, *76*(3) (2014), 115-128.

[14] S. Jahanshahi and D. F. M. Torres, *A Simple Accurate Method for Solving Fractional Variational and Optimal Control Problems*, J. Optim Theory Appl, *174* (2017), 156–175 .

[15] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Application of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[16] E. Kreyszig, *Introductory Functional Analysis with Applications*, New York: John Wiley and sons, (1978).

[17] D. V. Kruchinin, *Explicit Formulas for Some Generalized Polynomials*, Advenced Studies in Contemporary Mathematics, *24*(3) (2014), 327-332.

[18] A. Lotfi, M. Dehghan, and S. A. Yousefi, *A numerical technique for solving fractional optimal control problems*, Comput. Math. Appl., *62* (2011), 1055-1067.

[19] R. L. Magin, *Fractional calculus in bioengineering*, Crit. Rev. Biomed. Eng., *32* (2004), 1-104.

[20] K. Maleknejad and H. Almasieh, *Optimal control of Volterra integral equations via triangular functions*, Math. Comput. Model., *53* (2011), 1902-1909.

[21] A. Maheswaran and Elango, *Characterization of delta operator for euler, Bernoulli of second kind and moot polynomials*, International Journal of Pure and Applied Mathematics, *109*(2) (2016), 371-384.

[22] M. A. Moghaddam, Y. Edrisi, and M. Lakestani, *Solving fractional optimal control problems using Genocchi polynomials*, Computational Methods for Differential Equations, *9*(1) (2021), 79-93. DOI: 10.22034/cmde.2020.40292.1759.

[23] N. F. Mott, *The polarization of electrons by double scattering*, Proc.R.Soc.Lond.A., *135* (1932), 429-458.

[24] A. Nemati and S. A. Yousefi, *A Numerical Method for Solving Fractional Optimal Control Problems Using Ritz Method*, Journal of Computational and Nonlinear Dynamics September, *11/051015*(1) (2016).

[25] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Application of Differentiation and Integration to Arbitrary Order*, Acad. Press, N. York and London, (1974).

[26] N. Özdemir, O. P. Agrawal, D. Karadeniz, and B. B. Iskender, *Fractional optimal control problem of an axis-symmetric diffusion-wave propagation*, Phys. Scr, *136* (2009).

[27] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[28] E. M. Rabei, K. I. Nawafleh, R. S. Hijjawi, S. I. Muslih, and D. Baleanu, *The Hamilton formalism with fractional derivatives*, J. Math. Anal. Appl., *327* (2007), 891-897.

[29] S. Roman, *The Umbral Calculus*, Academic Press, (1984).

[30] T. J. Rivlin, *An Introduction to the Approximation of Functions*, New York: Dover Publications, 1981.

[31] S. Soradi-Zeid, *Solving a class of fractional optimal control problems via a new efficient and accurate method*, Computational Methods for Differential Equations, DOI: 10.22034/cmde.2020.35875.1620.

[32] N. H. Sweilam, T. M. AL-Ajmi, and R. H. W. Hoppe, *Numerical solution of some types of fractional optimal control problems*, Department of Mathematics University of Houston November, 2013.

[33] S. S. Tabatabaei and M. J. Yazdanpanah, *Formulation and Numerical Solution for Fractional Order Time Optimal Control Problem Using Pontryagin's Minimum principle*, IFAC PapersOnLine, *50*(1) (2017), 9224-9229.

[34] V. Taherpour, M. Nazari, and A. Nemati, *A new numerical Bernoulli polynomial method for solving fractional optimal control problems with vector components*, Computational Methods for Differential Equations, *9*(2) (2021), 446-466. DOI: 10.22034/cmde.2020.34992.1598.

[35] E. Weisstein, *Mott Polynomial*, From MathWorld–A Wolfram Web Resource, http://mathworld.wolfram.com.

[36] A. Yari, *Numerical solution for fractional optimal control problems by Hermite polynomials*, Journal of Vibration and Control (2020). DOI: 10.1177/1077546320933129.

[37] S. A. Yousefi, A. Lotfi, and M. Dehghan, *The use of a Legendre multiwavelet collocation method for solving the fractional optimal control problems*, Journal of Vibration and Control Received, *17*(13) (2010), 2059–2065.

[38] M. Zamani, M. Karimi-Ghartemani, and N. Sadati, *FOPID controller design for robust performance using particle swarm optimization*, J. Fract. Calc. Appl. Anal., *10* (2007), 169-188.