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Mean-square stability of a constructed Third-order stochastic Runge–Kutta schemes for general stochastic differential equations

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Abstract

In this paper, we are interested in the construction of an explicit third-order stochastic Runge–Kutta (SRK3) schemes for the weak approximation of stochastic differential equations (SDEs) with the general diffusion coefficient $b(t, x)$. To this aim, we use the Itô-Taylor method and compare them with the stochastic expansion of the approximation. In this way, the authors encountered a large number of equations and could find to derive four families for SRK3 schemes. Also, we investigate the mean-square stability (MS-stability) properties of SRK3 schemes for a linear SDE. Finally, the proposed families are implemented on some examples to illustrate convergence results.

Keywords. Stochastic differential equations, Stochastic Runge-Kutta schemes, Itô-Taylor expansion, Mean-square stability, Convergence.
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1. Introduction

Stochastic differential equations (SDEs) are widely known as useful tools for describing those phenomena which are influenced by some random factors. Such models have been used with great success in a variety of application areas, including biology, epidemiology, mechanics, economics, and finance (see, e.g., [9, 10, 19, 32]). Since in many cases, analytic solutions to these equations are not available so we are forced to use numerical schemes to approximate them. In recent years many authors presented some efficient numerical schemes for solving different types of SDEs with different properties (see [1, 2, 11, 12, 24, 29, 31]). Recently, the development of numerical schemes for strong and weak approximate solutions of SDEs has focused amongst others on Runge–Kutta type schemes (see [4, 5, 20–22, 25, 28]).

In various works, mean-square and weak numerical schemes have been also derived and extended presentations on this subject are given in Kloeden & Platen [17]. Runge–Kutta schemes in the strong sense have been proposed, for example, by Klauder & Petersen [18]. Furthermore, Guo et al. [13] studied asymptotic strong MS-stability of explicit Runge–Kutta Maruyama methods for stochastic delay differential equations. Also, Haghighi et al. [14] provided a structure of the MS-stability matrix of the strong diagonally drift implicit stochastic Runge–Kutta schemes for the general form of the linear SDEs. On the other hand, Tocino (see [25, 28]) presented stochastic Runge–Kutta schemes in the weak sense for the constant diffusion term.

Based on the proposed papers, the main difficulty in the first objective of constructing Runge–Kutta schemes, arises from the fact that the Itô expansion depends on multiple integrals, and this complicates any kind of matching. In this way, weak approximations of multiple Itô integrals are implemented, and we succeeded to introduce a class of four SRK3 families with stochastic diffusion terms by comparing the coefficients of the Itô-Taylor expansion and approximate solution which yields large quantities of equations. Moreover, MS-stability analysis of the proposed families is implemented on a linear test SDE with specific coefficient numbers.

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The remainder of the paper is organized as follows. Section 2 presents some notations and preliminaries to obtain Itô-Taylor expansions and the third-order weak Taylor scheme. In section 3 we outline SRK3 schemes for the weak approximation of SDEs with the general diffusion coefficient $b(t, x)$ in the Itô sense and derive for SRK3 families. MS-stability properties of the proposed four families are illustrated in section 4. At final, in section 5 convergence results are shown for some examples and they are compared with similar papers.

2. Weak approximations and Itô-Taylor expansions of third order

Let us consider the following scalar Itô stochastic differential equations (SDEs):

$$\begin{aligned} &dX_t = a(t, X_t)dt + b(t, X_t)dW(t), \qquad t_0 \le t \le T, \\ &X_{t_0} = x_0, \end{aligned} \tag{2.1}$$

where $a, b : [t_0, T] \times \mathbb{R} \to \mathbb{R}$ are the drift and diffusion coefficients, and $W(t)$ represents the one-dimensional Wiener process defined on this probability space.

Assumption 2.1. We suppose that the coefficients a and b are measurable functions and satisfy the following conditions:

(i) (Lipschitz condition) There exists a constant $K_1 > 0$ such that:

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \le K_1|x - y|,$$

for all $t \in [t_0, T]$ and $x, y \in \mathbb{R}$.

(ii) (Linear growth condition) There exists a constant $K_2 > 0$ such that:

$$|a(t, x)|^2 + |b(t, x)|^2 \le K_2(1 + |x|^2),$$

for all $t \in [t_0, T]$ and $x \in \mathbb{R}$.

These requirements ensure existence and uniqueness of solution The numerical schemes presented here are all constructed along time discretizations $t_0 \le t_1 \le \ldots \le t_N = T$ with constant stepsize:

$$\Delta = \frac{T - t_0}{N} > 0.$$

Then, $t_n = t_0 + n\Delta,\ n \in \{0, 1, \ldots, N\}$ denotes the nth step point. In the following, we shall use the notation $\bar{X}_n$ to denote the value of the approximation of the exact solution X at time t_n.

Definition 2.1. A sequence of approximation $\bar{X} = \{\bar{X}_0, \bar{X}_1, \ldots, \bar{X}_N\}$ is said to *converges weakly with order* β to the solution $X = \{X_t\}$ of equation (2.1), if for each $u \in l_P^{2(\beta+1)}$ there exist constants $K_u > 0$ (independent on Δ) and $\delta_0 > 0$ such that:

$$\Big|\mathbb{E}\big(u(X_T)\big) - \mathbb{E}\big(u(\bar{X}_N)\big)\Big| \le K_u \Delta^\beta, \tag{2.2}$$

for each $\Delta \in (0, \delta_0]$.

Next to the SDE (2.1) we consider the operators:

$$L^{(0)} = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \frac{1}{2}b^2\frac{\partial^2}{\partial x^2}, \qquad L^{(1)} = b\frac{\partial}{\partial x}. \tag{2.3}$$

Given $\beta \in \mathbb{N}$, we denote by Γ_β the set of all multi-indices $\alpha = (j_1, j_2, \ldots, j_l)$, $j_k \in \{0, 1\}$, of length $l \in \{1, 2, \ldots, \beta\}$. As explained in Kloeden & Platen [17], taking $f(t, x) = x$ and based on the Itô-Taylor expansion of the process $X_t = f(t, X_t)$ one can construct the *weak-order* β Taylor scheme:

$$\bar{X}_{n+1} = \bar{X}_n + \sum_{\alpha\in\Gamma_\beta} (L^\alpha f)(t_n, \bar{X}_n)I_{\alpha,n}, \tag{2.4}$$

where for $\alpha = (j_1, \dots, j_l)$ we have denoted:

$$L^{\alpha} = L^{(j_1)} L^{(j_2)} \dots L^{(j_l)},$$
$$I_{\alpha,n} = \int_t^{t+\Delta} 1 d\alpha = \int_t^{t+\Delta} \int_t^{s_l} \dots \int_t^{s_2} dW_{s_1}^{j_1} \dots dW_{s_l}^{j_l},$$

with $dW^{(0)} = dt$. Once obtained the Taylor approximation (2.4), which has local order $\beta + 1$, simplified schemes of order β can be constructed by changing the variables $I_{\alpha,n}$ by appropriate simpler variables $\hat{I}_{\alpha,n}$. A *simplified order β weak Taylor scheme* is given by:

$$\bar{X}_{n+1} = \bar{X}_n + \sum_{\alpha \in \Gamma_\beta} (L^{\alpha} f)(t_n, \bar{X}_n) \hat{I}_{\alpha,n}, \tag{2.5}$$

where the variables $\hat{I}_{\alpha,n}$, are such that:

$$\Big| \mathbb{E}\Big(\prod_{k=1}^{l} I_{\alpha_k,n} - \prod_{k=1}^{l} \hat{I}_{\alpha_k,n} \Big) \Big| \leq K \Delta^{\beta+1}, \tag{2.6}$$

for a constant $K > 0$ and for all choices of multi-indices $\alpha_k \in \Gamma_\beta$ with $k = 1, \dots, l$ and $l = 1, \dots, 2\beta + 1$. In the sequel, for simplicity of notation we shall abbreviate $I_{\alpha,n}$, $\hat{I}_{\alpha,n}$ and similar integrals to I_α, $\hat{I}_\alpha$, etc. respectively. Simplified third-order weak Taylor schemes are obtained replacing the seven I_α's that appear in the above scheme by new variables $\hat{I}_\alpha$'s satisfying for some constant $K > 0$:

$$\Big| \mathbb{E}\Big(\prod_{k=1}^{l} I_{\alpha_k} - \prod_{k=1}^{l} \hat{I}_{\alpha_k} \Big) \Big| \leq K \Delta^4, \qquad l = 1, 2, \dots, 7. \tag{2.7}$$

If $\beta = 3$, $d = m = 1$ and $\Delta \hat{W}_n$ is any variable satisfying the moment conditions:

$$\begin{aligned} &\big|\mathbb{E}(\Delta \hat{W}_n)\big| + \big|\mathbb{E}\big((\Delta \hat{W}_n)^2\big) - \Delta\big| + \big|\mathbb{E}\big((\Delta \hat{W}_n)^3\big)\big| + \big|\mathbb{E}\big((\Delta \hat{W}_n)^4\big) - 3\Delta^2\big| \\ &\quad + \big|\mathbb{E}\big((\Delta \hat{W}_n)^5\big)\big| + \big|\mathbb{E}\big((\Delta \hat{W}_n)^6\big) - 15\Delta^3\big| + \big|\mathbb{E}\big((\Delta \hat{W}_n)^7\big)\big| \leq K \Delta^4, \end{aligned} \tag{2.8}$$

for some constant $K > 0$, it's easy to prove that the following variables satisfy (2.6):

$$\begin{aligned} I_{(0),n} &= \hat{I}_{(0),n} = \Delta, \\ \hat{I}_{(1),n} &= \Delta \hat{W}_n, \\ I_{(0,0),n} &= \hat{I}_{(0,0),n} = \frac{1}{2} \Delta^2, \\ \hat{I}_{(0,1),n} &= \hat{I}_{(1,0),n} = \frac{1}{2} \Delta \Delta \hat{W}_n, \\ \hat{I}_{(1,1),n} &= \frac{1}{2} \big((\Delta \hat{W}_n)^2 - \Delta\big), \\ I_{(0,0,0),n} &= \hat{I}_{(0,0,0),n} = \frac{1}{6} \Delta^3, \\ \hat{I}_{(0,0,1),n} &= \hat{I}_{(0,1,0),n} = \hat{I}_{(1,0,0),n} = \frac{1}{6} \Delta^2 \Delta \hat{W}_n, \\ \hat{I}_{(1,1,0),n} &= \hat{I}_{(1,0,1),n} = \hat{I}_{(0,1,1),n} = \frac{1}{6} \Delta \Big(\big(\Delta \hat{W}_n\big)^2 - \Delta \Big), \\ \hat{I}_{(1,1,1),n} &= \frac{1}{6} \Delta \hat{W}_n \Big(\big(\Delta \hat{W}_n\big)^2 - 3\Delta \Big). \end{aligned} \tag{2.9}$$

Then, based on (2.5) and relations in (2.9) the third-order weak Taylor scheme is written as follows:

$$\begin{aligned}\bar{X}_{n+1} =&\bar{X}_n + a\Delta + b\Delta\hat{W}_n + \frac{1}{2}bb_{01}\Big((\Delta\hat{W}_n)^2 - \Delta\Big) + ba_{01}\Delta\hat{Z}_n\\ &+ \big(b_{10} + ab_{01} + \frac{1}{2}b^2b_{02}\big)\Big(\Delta\Delta\hat{W}_n - \Delta\hat{Z}_n\Big) + \frac{1}{2}\big(a_{10} + aa_{01} + \frac{1}{2}b^2a_{02}\big)\Delta^2\\ &+ \frac{1}{6}\Big(b_{20} + 2ab_{11} + b^2b_{12} + a_{10}b_{01} + 2aa_{01}b_{01} + a^2b_{02} + \frac{5}{2}b^2a_{02}b_{01} + \frac{3}{2}b^2a_{01}b_{02}\\ &+ ab^2b_{03} + bb_{10}b_{02} + abb_{01}b_{02} + \frac{1}{2}b^2b_{01}^2b_{02} + \frac{1}{2}b^3b_{02}^2 + b^3b_{01}b_{03} + \frac{1}{4}b^4b_{04} + b_{10}a_{01}\\ &+ ba_{01}^2 + 2ba_{11} + 2aba_{02} + b^3a_{03}\Big)\Delta^2\Delta\hat{W}_n + \frac{1}{6}\Big(2ba_{01}b_{01} + b^2a_{02} + 2bb_{11} + 2abb_{02}\\ &+ \frac{5}{2}b^2b_{01}b_{02} + b^3b_{03} + b_{10}b_{01} + ab_{01}^2\Big)\Big((\Delta\hat{W}_n)^2 - \Delta\Big)\Delta + \frac{1}{6}\Big(a_{20} + 2aa_{11} + b^2a_{12}\\ &+ a_{10}a_{01} + aa_{01}^2 + a^2a_{02} + \frac{3}{2}b^2a_{01}a_{02} + ab^2a_{03} + bb_{10}a_{02} + aba_{02}b_{01} + \frac{1}{2}b^2b_{01}^2a_{02}\\ &+ \frac{1}{2}b^3b_{02}a_{02} + b^3b_{01}a_{03} + \frac{1}{4}b^4a_{04}\Big)\Delta^3 + \frac{1}{6}\big(bb_{01}^2 + b^2b_{02}\big)\Big((\Delta\hat{W}_n)^2 - 3\Delta\Big)\Delta\hat{W}_n, \end{aligned}\tag{2.10}$$

where $\Delta\hat{W}_n$ and $\Delta\hat{Z}_n$ are correlated Gaussian random variables with:

$$\Delta\hat{W}_n \sim N\big(0, \Delta\big), \qquad \Delta\hat{Z}_n \sim N\big(0, \frac{1}{3}\Delta^3\big), \qquad \mathbb{E}\big(\Delta\hat{W}_n\Delta\hat{Z}_n\big) = \frac{1}{2}\Delta^2, \tag{2.11}$$

and for a function $g = g(t, x)$ with $t, x \in \mathbb{R}$ we have denoted:

$$g_{ij} = \frac{\partial^{i+j} g}{\partial t^i \partial x^j}(t_n, \bar{X}_n), \qquad i, j = 1, 2, \dots, \tag{2.12}$$

and $g = g_{00} = g(t_n, \bar{X}_n)$. In the sequel, for simplicity of notation we shall often abbreviate $I_{\alpha,n}$, $\Delta\hat{W}_n$, $\Delta\hat{Z}_n$, etc. to I_α, $\Delta\hat{W}_n$ and $\Delta\hat{Z}_n$, respectively. In the multi-dimensional and scalar cases, if $\Delta\hat{W}$ and $\Delta\hat{Z}$ verify (2.11), we have the following 3-equivalences:

$$(\Delta\hat{W})^3 \overset{(2)}{\simeq} 3\Delta\Delta\hat{W}, \tag{2.13}$$

$$\Delta(\Delta\hat{W})^2 \overset{(2)}{\simeq} \Delta^2, \tag{2.14}$$

$$(\Delta\hat{W})^4 \overset{(2)}{\simeq} 3\Delta^2. \tag{2.15}$$

3. Third-order stochastic Runge–Kutta (SRK3) schemes

An important disadvantage of Taylor schemes is that they require us to determine many derivatives. Using the idea of the deterministic and stochastic cases, we obtain SRK schemes. Now, we consider the following an explicit *s-stage* stochastic Runge–Kutta scheme:

$$\bar{X}_{n+1} = \bar{X}_n + \Delta\sum_{j=1}^{s}\alpha_j a\big(t_n + \mu_j\Delta, \eta_j\big) + \Delta\hat{W}_n\sum_{j=1}^{s}\beta_j b\big(t_n + \mu_j\Delta, \eta_j\big) + R, \tag{3.1}$$

where $\mu_1 = 0$, $\eta_1 = X_n$ and

$$\eta_j = \bar{X}_n + \Delta\sum_{i=1}^{j-1}\lambda_{ji}a\big(t_n + \mu_i\Delta, \eta_i\big) + \Delta\hat{W}_n\sum_{i=1}^{j-1}\gamma_{ji}b\big(t_n + \mu_i\Delta, \eta_i\big), \quad j = 2, 3, \dots, s, \tag{3.2}$$

and R is a fit term. The numerical constants α_j, β_j, μ_j, λ_{ji}, γ_{ji} and the term R must be chosen so that the approximation (3.1) is β-equivalent to the order β Taylor scheme. Since the truncated expansion of order β of a process is, under appropriate conditions, β-equivalent to the process, it suffices to choose the parameters and R so that a β-equivalent approximation to the truncated expansion of order β of (3.1) is equal to the order β Taylor scheme. Note that R is free, then it can be chosen so that the required equality is fulfilled. The coefficients of the SRK scheme (3.1) are represented by the following extended Butcher arrays: As in the deterministic and stochastic cases, in order

$$\begin{array}{c|cccc|cccc}
\mu_2 & \lambda_{21} & & & & \gamma_{21} & & & \\
\vdots & \vdots & \ddots & & & \vdots & \ddots & & \\
\mu_s & \lambda_{s1} & \cdots & \lambda_{s,s-1} & & \gamma_{s1} & \cdots & \gamma_{s,s-1} & \\
\hline
R & \alpha_1 & \cdots & \alpha_{s-1} & \alpha_s & \beta_1 & \cdots & \beta_{s-1} & \beta_s
\end{array}$$

to match the truncated expansion of (3.1) with the Taylor scheme we need an expression of the order β truncated expansion of a process $f(t+\Delta, X_t+\Delta X)$ in terms of Δ and $\Delta X = X_{t+\Delta} - X_t$. This expression has been obtained by Tocino & Ardanuy [26] for $\beta = 2$ in the multi-dimensional case and for $\beta = 3$ in the scalar case.

Proposition 3.1. *(Third-order expansion)(see [26]). Let $\{X_t\}_{t\in[t_0,T]}$ be the solution of the SDE (2.1) where the coefficients a and b have continuous partial derivatives of third-order. If $f:[t_0,T]\times\mathbb{R}\to\mathbb{R}$ has continuous sixth partial derivatives, the third-order Itô-Taylor truncated expansion of $f(t,X_t)$ is:*

$$\begin{aligned}
f(t+\Delta, X_t+\Delta X) \overset{(3)}{\simeq} & f_{00} + f_{10}\Delta + f_{01}\Delta X + \Big(f_{20} - \frac{1}{4}b_{00}^4 f_{04}\Big)\frac{\Delta^2}{2} + f_{11}\Delta\Delta X \\
& + f_{02}\frac{(\Delta X)^2}{2} + \Big(f_{30} + \frac{3}{2}b_{00}^2 f_{22} + 3b_{00}^3 b_{01} f_{13} + b_{00}^4\big(4b_{01}^2 + b_{00}b_{02}\big) f_{04} \\
& + \frac{3}{4}b_{00}^4 f_{14} + \frac{3}{2}b_{00}^5 b_{01} f_{05} + \frac{1}{8}b_{00}^6 f_{06}\Big)\frac{\Delta^3}{6} + \Big(f_{21} + b_{00}^2 f_{13} + b_{00}^3 b_{01} f_{04} \\
& + \frac{1}{4}b_{00}^4 f_{05}\Big)\frac{\Delta^2\Delta X}{2} + \Big(f_{12} + \frac{1}{2}b_{00}^2 f_{04}\Big)\frac{\Delta(\Delta X)^2}{2} + f_{03}\frac{(\Delta X)^3}{6}.
\end{aligned} \tag{3.3}$$

Remark 3.2. It is sufficient to suppose the existence and continuity of the partial derivatives of a, b and f which appear in the expressions $(L^{(i,j,k)}f)(t_0, X_{t_0})$ with $i,j,k = 0,1$.

Numerical solutions for weak approximation do not need information about the driving Wiener process and their random variables are simulated on different probability spaces. Thus, we can make use of random variables with distributions which are easy to simulate. In this section we modify the SRK3 schemes for the non-constant diffusion coefficient $b(t,x)$, and we will obtain now four families of SRK3 schemes for scalar SDE's ($d = m = 1$). In this case a three-stage stochastic Runge–Kutta scheme of the form (3.1) can be written as follows:

$$\begin{aligned}
\bar{X}_{n+1} = & \bar{X}_n + \Big(\alpha_1 a(t_n, \bar{X}_n) + \alpha_2 a(t_n + \mu_2\Delta, \eta_2) + \alpha_3 a(t_n + \mu_3\Delta, \eta_3)\Big)\Delta \\
& + \Big(\beta_1 b(t_n, \bar{X}_n) + \beta_2 b(t_n + \mu_2\Delta, \eta_2) + \beta_3 b(t_n + \mu_3\Delta, \eta_3)\Big)\Delta\hat{W}_n + R,
\end{aligned} \tag{3.4}$$

where

$$\eta_2 = \bar{X}_n + \lambda_{21} a(t_n + \mu_1\Delta, \eta_1)\Delta + \gamma_{21} b(t_n + \mu_1\Delta, \eta_1)\Delta\hat{W}_n, \tag{3.5}$$

and

$$\begin{aligned}
\eta_3 = & \bar{X}_n + \Big(\lambda_{31} a(t_n + \mu_1\Delta, \eta_1) + \lambda_{32} a(t_n + \mu_2\Delta, \eta_2)\Big)\Delta \\
& + \Big(\gamma_{31} b(t_n + \mu_1\Delta, \eta_1) + \gamma_{32} b(t_n + \mu_2\Delta, \eta_2)\Big)\Delta\hat{W}_n.
\end{aligned} \tag{3.6}$$

Here we have to find out the parameters and R in such a way that (3.4) is 3-equivalent to the third-order weak Taylor scheme (2.10). We begin by evaluating the third-order truncated expansion of $a\big(t_n + \mu_2\Delta, \eta_2\big)\Delta$ and $b\big(t_n + \mu_2\Delta, \eta_2\big)\Delta\hat{W}_n$.

By using relations (3.3) and (3.5) we get:

$$\begin{aligned} a\big(t_n+\mu_2\Delta,\eta_2\big)\Delta \overset{(3)}{\simeq} & a\Delta + a_{01}b\gamma_{21}\Delta\Delta\hat{W}_n + a_{10}\mu_2\Delta^2 + aa_{01}\lambda_{21}\Delta^2 \\ & + \frac{1}{2}a_{02}b^2\gamma_{21}^2\Delta(\Delta\hat{W}_n)^2 + a_{11}b\mu_2\gamma_{21}\Delta^2\Delta\hat{W}_n + aa_{02}b\lambda_{21}\gamma_{21}\Delta^2\Delta\hat{W}_n \\ & + \frac{1}{2}a_{03}b^3\gamma_{21}^3\Delta^2\Delta\hat{W}_n + aa_{11}\mu_2\lambda_{21}\Delta^3 + \frac{1}{2}a^2a_{02}\lambda_{21}^2\Delta^3 \\ & + \big(a_{12}+\frac{1}{2}a_{04}b_{00}^2\big)\frac{b^2}{2}\mu_2\gamma_{21}^2\Delta^3 + \frac{1}{2}aa_{03}b^2\gamma_{21}^2\lambda_{21}\Delta^3 \\ & + \frac{1}{2}\big(a_{20}-\frac{1}{4}b_{00}^4a_{04}\big)\mu_2^2\Delta^3, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} b\big(t_n+\mu_2\Delta,\eta_2\big)\Delta\hat{W}_n \overset{(3)}{\simeq} & b\Delta\hat{W}_n + b_{01}b\gamma_{21}(\Delta\hat{W}_n)^2 + b_{10}\mu_2\Delta\Delta\hat{W}_n + ab_{01}\lambda_{21}\Delta\Delta\hat{W}_n \\ & + \frac{1}{2}b_{02}b^2\gamma_{21}^2(\Delta\hat{W}_n)^3 + b_{11}b\mu_2\gamma_{21}\Delta(\Delta\hat{W}_n)^2 + ab_{02}b\lambda_{21}\gamma_{21}\Delta(\Delta\hat{W}_n)^2 \\ & + \frac{1}{6}b_{03}b^3\gamma_{21}^3(\Delta\hat{W}_n)^4 + ab_{11}\mu_2\lambda_{21}\Delta^2\Delta\hat{W}_n + \frac{1}{2}a^2b_{02}\lambda_{21}^2\Delta^2\Delta\hat{W}_n \\ & + \frac{1}{2}(b_{20}-\frac{1}{4}b_{00}^4b_{04})\mu_2^2\Delta^2\Delta\hat{W}_n + \frac{1}{2}\big(b_{12}+\frac{1}{2}b_{00}^2b_{04}\big)b^2\mu_2\gamma_{21}^2\Delta(\Delta\hat{W}_n)^3 \\ & + \frac{1}{2}ab_{03}b^2\lambda_{21}\gamma_{21}^2\Delta(\Delta\hat{W}_n)^3 + \frac{1}{2}\big(b_{21}+b_{00}^2b_{13}+b_{00}^3b_{01}b_{04}+\frac{1}{4}b_{00}^4b_{05}\big)b\mu_2^2\gamma_{21}\Delta^2(\Delta\hat{W}_n)^2 \\ & + \frac{1}{2}a^2bb_{03}\lambda_{21}^2\gamma_{21}\Delta^2(\Delta\hat{W}_n)^2 + \big(b_{12}+\frac{1}{2}b_{00}^2b_{04}\big)ab\mu_2\lambda_{21}\gamma_{21}\Delta^2(\Delta\hat{W}_n)^2. \end{aligned} \tag{3.8}$$

Now let's calculate the third-order truncated expansion of $a\big(t_n+\mu_3\Delta,\eta_3\big)\Delta$ and $b\big(t_n+\mu_3\Delta,\eta_3\big)\Delta\hat{W}_n$. For simplicity, we define:

$$\begin{aligned} N &:= \eta_3 - \bar{X}_n \\ &= \lambda_{31}a\Delta + \lambda_{32}a(t_n+\mu_2\Delta,\eta_2)\Delta + \gamma_{31}b\Delta\hat{W}_n + \gamma_{32}b(t_n+\mu_2\Delta,\eta_2)\Delta\hat{W}_n. \end{aligned} \tag{3.9}$$

Noting that $\Delta^4 \overset{(3)}{\simeq} 0 \overset{(3)}{\simeq} \Delta^3\cdot N$ and then, by using relations (3.3) and (3.6) we can obtaion:

$$\begin{aligned} a\big(t_n+\mu_3\Delta,\bar{X}_n+N\big)\Delta \overset{(3)}{\simeq} & a\Delta + a_{10}\mu_3\Delta^2 + a_{01}\Delta\cdot N + \big(a_{20}-\frac{1}{4}b_{00}^4a_{04}\big)\mu_3^2\frac{\Delta^3}{2} \\ & + a_{11}\mu_3\Delta^2\cdot N + a_{02}\frac{\Delta\cdot N^2}{2} + \big(a_{12}+\frac{1}{2}b_{00}^2a_{04}\big)\mu_3\frac{\Delta^2\cdot N^2}{2} + a_{03}\frac{\Delta\cdot N^3}{6}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} b\big(t_n+\mu_3\Delta,\bar{X}_n+N\big)\Delta\hat{W}_n \overset{(3)}{\simeq} & b\Delta\hat{W}_n + b_{10}\mu_3\Delta\Delta\hat{W}_n + b_{01}\Delta\hat{W}_n\cdot N \\ & + \big(b_{20}-\frac{1}{4}b_{00}^4b_{04}\big)\mu_3^2\frac{\Delta^2\Delta\hat{W}_n}{2} + b_{11}\mu_3\Delta\Delta\hat{W}_n\cdot N + b_{02}\frac{\Delta\hat{W}_n\cdot N^2}{2} \\ & + \big(b_{12}+\frac{1}{2}b_{00}^2b_{04}\big)\mu_3\frac{\Delta\Delta\hat{W}_n\cdot N^2}{2} + b_{03}\frac{\Delta\hat{W}_n\cdot N^3}{6} \\ & + \big(b_{21}+b_{00}^2b_{13}+b_{00}^3b_{01}b_{04}+\frac{1}{4}b_{00}^4b_{05}\big)\mu_3^2\frac{\Delta^2\Delta\hat{W}_n\cdot N}{2}. \end{aligned} \tag{3.11}$$

The third-order truncated expansion of N can be obtained from (3.7) and (3.8), it's easy to show the equivalences:

$$\begin{aligned}\varDelta\cdot N \overset{(3)}{\simeq}& \big(\gamma_{31}+\gamma_{32}\big)b\varDelta\Delta\hat{W}_n+\big(\lambda_{31}+\lambda_{32}\big)a\varDelta^2+b_{01}b\gamma_{21}\gamma_{32}\varDelta(\Delta\hat{W}_n)^2\\&+a_{01}b\lambda_{32}\gamma_{21}\varDelta^2\Delta\hat{W}_n+b_{10}\mu_2\gamma_{32}\varDelta^2\Delta\hat{W}_n+ab_{01}\lambda_{21}\gamma_{32}\varDelta^2\Delta\hat{W}_n\\&+\frac{3}{2}b_{02}b^2\gamma_{21}^2\gamma_{32}\varDelta^2\Delta\hat{W}_n+a_{10}\mu_2\lambda_{32}\varDelta^3+aa_{01}\lambda_{32}\lambda_{21}\varDelta^3+\frac{1}{2}a_{02}b^2\lambda_{32}\gamma_{21}^2\varDelta^3\\&+b_{11}b\mu_2\gamma_{21}\gamma_{32}\varDelta^3+ab_{02}b\lambda_{21}\gamma_{21}\gamma_{32}\varDelta^3+\frac{1}{2}b_{03}b^3\gamma_{21}^3\gamma_{32}\varDelta^3,\end{aligned}\tag{3.12}$$

$$\varDelta^2\cdot N \overset{(3)}{\simeq} \big(\gamma_{31}+\gamma_{32}\big)b\varDelta^2\Delta\hat{W}_n+\big(\lambda_{31}+\lambda_{32}\big)a\varDelta^3+b_{01}b\gamma_{21}\gamma_{32}\varDelta^3,\tag{3.13}$$

$$\begin{aligned}\varDelta\cdot N^2 \overset{(3)}{\simeq}& (\gamma_{31}+\gamma_{32})^2b^2\varDelta(\Delta\hat{W}_n)^2+2ab(\gamma_{31}+\gamma_{32})(\lambda_{31}+\lambda_{32})\varDelta^2\Delta\hat{W}_n\\&+2b^2b_{01}\gamma_{21}\gamma_{31}\gamma_{32}\varDelta(\Delta\hat{W}_n)^3+2b^2b_{01}\gamma_{21}\gamma_{32}^2\varDelta(\Delta\hat{W}_n)^3\\&+2bb_{10}\mu_2\gamma_{31}\gamma_{32}\varDelta^2(\Delta\hat{W}_n)^2+2bb_{10}\mu_2\gamma_{32}^2\varDelta^2(\Delta\hat{W}_n)^2\\&+2abb_{01}\lambda_{21}\gamma_{31}\gamma_{32}\varDelta^2(\Delta\hat{W}_n)^2+2abb_{01}(\lambda_{31}+\lambda_{32})\gamma_{21}\gamma_{32}\varDelta^2(\Delta\hat{W}_n)^2\\&+2abb_{01}\lambda_{21}\gamma_{32}^2\varDelta^2(\Delta\hat{W}_n)^2+2b^2a_{01}\lambda_{32}\gamma_{21}(\gamma_{31}+\gamma_{32})\varDelta^2(\Delta\hat{W}_n)^2\\&+b^2b_{01}^2\gamma_{21}^2\gamma_{32}^2\varDelta(\Delta\hat{W}_n)^4+\big(\lambda_{31}+\lambda_{32}\big)^2a^2\varDelta^3,\end{aligned}\tag{3.14}$$

$$\varDelta^2\cdot N^2 \overset{(3)}{\simeq} \big(\gamma_{31}+\gamma_{32}\big)^2b^2\varDelta^3,\tag{3.15}$$

$$\begin{aligned}\varDelta\cdot N^3 \overset{(3)}{\simeq}& \big(\gamma_{31}+\gamma_{32}\big)^3b^3\varDelta(\Delta\hat{W}_n)^3+3\big(\gamma_{32}^3\gamma_{21}+\gamma_{31}^2\gamma_{21}\gamma_{32}\big)b^3b_{01}\varDelta(\Delta\hat{W}_n)^4\\&+3ab^2\big(\gamma_{31}+\gamma_{32}\big)^2\big(\lambda_{31}+\lambda_{32}\big)\varDelta^3,\end{aligned}\tag{3.16}$$

$$\begin{aligned}\Delta\hat{W}_n\cdot N \overset{(3)}{\simeq}& \big(\gamma_{31}+\gamma_{32}\big)b(\Delta\hat{W}_n)^2+\big(\lambda_{31}+\lambda_{32}\big)a\varDelta\Delta\hat{W}_n+3bb_{01}\gamma_{21}\gamma_{32}\varDelta\Delta\hat{W}_n\\&+ba_{01}\lambda_{32}\gamma_{21}\varDelta(\Delta\hat{W}_n)^2+b_{10}\mu_2\gamma_{32}\varDelta(\Delta\hat{W}_n)^2+ab_{01}\lambda_{21}\gamma_{32}\varDelta(\Delta\hat{W}_n)^2\\&+\frac{3}{2}b^2b_{02}\gamma_{21}^2\gamma_{32}\varDelta^2+a_{10}\mu_2\lambda_{32}\varDelta^2\Delta\hat{W}_n+aa_{01}\lambda_{32}\lambda_{21}\varDelta^2\Delta\hat{W}_n\\&+\frac{3}{2}b^2a_{02}\lambda_{32}\gamma_{21}^2\varDelta^2\Delta\hat{W}_n+3bb_{11}\mu_2\gamma_{21}\gamma_{32}\varDelta^2\Delta\hat{W}_n+3abb_{02}\lambda_{21}\gamma_{21}\gamma_{32}\varDelta^2\Delta\hat{W}_n\\&+\frac{1}{2}b^3b_{03}\gamma_{21}^3\gamma_{32}\varDelta^2\Delta\hat{W}_n+ba_{11}\mu_2\lambda_{32}\gamma_{21}\varDelta^3+\frac{1}{2}b^3a_{03}\lambda_{32}\gamma_{21}^3\varDelta^3\\&+aba_{02}\lambda_{21}\lambda_{32}\gamma_{21}\varDelta^3+ab_{11}\mu_2\lambda_{21}\gamma_{32}\varDelta^3+\frac{1}{2}\big(b_{20}-\frac{1}{4}b_{00}^4b_{04}\big)\mu_2^2\gamma_{32}\varDelta^3\\&+\frac{1}{2}a^2b_{02}\lambda_{21}^2\gamma_{32}\varDelta^3+\frac{3}{2}\big(b_{12}+\frac{1}{2}b_{00}^2b_{04}\big)b^2\mu_2\gamma_{21}^2\gamma_{32}\varDelta^3+\frac{3}{2}ab^2b_{03}\lambda_{21}\gamma_{21}^2\gamma_{32}\varDelta^3,\end{aligned}\tag{3.17}$$

$$\begin{aligned}\varDelta\Delta\hat{W}_n\cdot N \overset{(3)}{\simeq}& \big(\gamma_{31}+\gamma_{32}\big)b\varDelta(\Delta\hat{W}_n)^2+\big(\lambda_{31}+\lambda_{32}\big)a\varDelta^2\Delta\hat{W}_n\\&+3bb_{01}\gamma_{21}\gamma_{32}\varDelta^2\Delta\hat{W}_n+ba_{01}\lambda_{32}\gamma_{21}\varDelta^3+b_{10}\mu_2\gamma_{32}\varDelta^3\\&+ab_{01}\lambda_{21}\gamma_{32}\varDelta^3+\frac{3}{2}b^2b_{02}\gamma_{21}^2\gamma_{32}\varDelta^3,\end{aligned}\tag{3.18}$$

$$
\begin{aligned}
\Delta\hat{W}_n\cdot N^2 \overset{(3)}{\simeq} & \big(\lambda_{31}+\lambda_{32}\big)^2 a^2 \varDelta^2 \Delta\hat{W}_n + \big(\gamma_{31}+\gamma_{32}\big)^2 b^2 (\Delta\hat{W}_n)^3 \\
& + 2ab\big(\lambda_{31}+\lambda_{32}\big)\big(\gamma_{31}+\gamma_{32}\big)\varDelta(\Delta\hat{W}_n)^2 + 2b^2 b_{01}\gamma_{21}\gamma_{31}\gamma_{32}(\Delta\hat{W}_n)^4 \\
& + 2b^2 b_{01}\gamma_{21}\gamma_{32}^2(\Delta\hat{W}_n)^4 + 2bb_{10}\mu_2\gamma_{31}\gamma_{32}\varDelta(\Delta\hat{W}_n)^3 + 2bb_{10}\mu_2\gamma_{32}^2\varDelta(\Delta\hat{W}_n)^3 \\
& + 2abb_{01}\lambda_{21}\gamma_{31}\gamma_{32}\varDelta(\Delta\hat{W}_n)^3 + 2abb_{01}\big(\lambda_{31}+\lambda_{32}\big)\gamma_{21}\gamma_{32}\varDelta(\Delta\hat{W}_n)^3 \\
& + 2abb_{01}\lambda_{21}\gamma_{32}^2\varDelta(\Delta\hat{W}_n)^3 + 2a_{01}b^2\lambda_{32}\gamma_{21}\big(\gamma_{31}+\gamma_{32}\big)\varDelta(\Delta\hat{W}_n)^3 \\
& + b^2 b_{01}^2\gamma_{21}^2\gamma_{32}^2(\Delta\hat{W}_n)^5 + 2a_{01}ab\lambda_{31}\lambda_{32}\gamma_{21}\varDelta^2(\Delta\hat{W}_n)^2 \\
& + 2b_{10}a\mu_2\big(\lambda_{31}+\lambda_{32}\big)\gamma_{32}\varDelta^2(\Delta\hat{W}_n)^2 + 2aba_{01}\lambda_{32}^2\gamma_{21}\varDelta^2(\Delta\hat{W}_n)^2 \\
& + 2b_{01}a^2\lambda_{21}\big(\lambda_{31}+\lambda_{32}\big)\gamma_{32}\varDelta^2(\Delta\hat{W}_n)^2 + 2a_{10}b\lambda_{32}\big(\gamma_{31}+\gamma_{32}\big)\varDelta^2(\Delta\hat{W}_n)^2 \\
& + 2a_{01}b_{01}b^2\lambda_{32}\gamma_{21}^2\varDelta(\Delta\hat{W}_n)^4 + 2a_{01}ab\lambda_{21}\lambda_{32}\big(\gamma_{31}+\gamma_{32}\big)\varDelta^2(\Delta\hat{W}_n)^2 \\
& + a_{02}b^3\lambda_{32}\gamma_{21}^2\big(\gamma_{31}+\gamma_{32}\big)\varDelta(\Delta\hat{W}_n)^4 + 2b_{11}b^2\mu_2\gamma_{21}\gamma_{31}\gamma_{32}\varDelta(\Delta\hat{W}_n)^4 \\
& + 2b_{02}ab^2\lambda_{21}\gamma_{21}\gamma_{31}\gamma_{32}\varDelta(\Delta\hat{W}_n)^4 + b_{03}b^4\gamma_{21}^3\gamma_{31}\gamma_{32}\varDelta^2(\Delta\hat{W}_n)^2 \\
& + 2b_{11}b^2\mu_2\gamma_{21}\gamma_{32}^2\varDelta(\Delta\hat{W}_n)^4 + 2b_{02}ab^2\lambda_{21}\gamma_{21}\gamma_{32}^2\varDelta(\Delta\hat{W}_n)^4 \\
& + b_{03}b^4\gamma_{21}^3\gamma_{32}^2\varDelta^2(\Delta\hat{W}_n)^2 + 2b_{10}b_{01}b\mu_2\gamma_{21}\gamma_{32}^2\varDelta(\Delta\hat{W}_n)^4 \\
& + 2b_{01}^2ab\lambda_{21}\gamma_{21}\gamma_{32}^2\varDelta(\Delta\hat{W}_n)^4 + b_{02}b^3\gamma_{21}^2\gamma_{31}\gamma_{32}(\Delta\hat{W}_n)^5 \\
& + b_{02}b^3\gamma_{21}^2\gamma_{32}^2(\Delta\hat{W}_n)^5,
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
\varDelta\Delta\hat{W}_n\cdot N^2 \overset{(3)}{\simeq} & \big(\gamma_{31}+\gamma_{32}\big)^2 b^2\varDelta(\Delta\hat{W}_n)^3 + 2b^2b_{01}\gamma_{21}\gamma_{31}\varDelta(\Delta\hat{W}_n)^4 \\
& + 2b^2b_{01}\gamma_{21}\gamma_{31}\gamma_{32}\varDelta(\Delta\hat{W}_n)^4 + 2b^2b_{01}\gamma_{21}\gamma_{32}^2\varDelta(\Delta\hat{W}_n)^4 \\
& + 2ab\big(\lambda_{31}+\lambda_{32}\big)\big(\gamma_{31}+\gamma_{32}\big)\varDelta^2(\Delta\hat{W}_n)^2,
\end{aligned}
\tag{3.20}
$$

$$
\varDelta^2\Delta\hat{W}_n\cdot N \overset{(3)}{\simeq} \big(\gamma_{31}+\gamma_{32}\big)b\varDelta^3,
\tag{3.21}
$$

$$
\begin{aligned}
\Delta\hat{W}_n\cdot N^3 \overset{(3)}{\simeq} & \big(\gamma_{31}+\gamma_{32}\big)^3 b^3(\Delta\hat{W}_n)^4 + 3ab^2\big(\lambda_{31}+\lambda_{32}\big)\big(\gamma_{31}+\gamma_{32}\big)^2\varDelta(\Delta\hat{W}_n)^3 \\
& + 3a^2b\big(\lambda_{31}+\lambda_{32}\big)^2\big(\gamma_{31}+\gamma_{32}\big)\varDelta^2(\Delta\hat{W}_n)^2 + 3b^2b_{10}\mu_2\big(\gamma_{31}^2\gamma_{32}+\gamma_{32}^3\big)\varDelta(\Delta\hat{W}_n)^4 \\
& + 3ab^2b_{01}\lambda_{21}\big(\gamma_{31}^2\gamma_{32}+\gamma_{32}^3\big)\varDelta(\Delta\hat{W}_n)^4 + 3b^3a_{01}\gamma_{21}\big(\gamma_{31}^2+\gamma_{32}^2\big)\varDelta(\Delta\hat{W}_n)^4 \\
& + 3b^3b_{01}\gamma_{21}\big(\gamma_{31}^2\gamma_{32}+\gamma_{32}^3\big)(\Delta\hat{W}_n)^5 + 3b^3b_{01}^2\gamma_{21}^2\gamma_{31}\gamma_{32}^2(\Delta\hat{W}_n)^6 \\
& + 3b^3b_{01}^2\gamma_{21}^2\gamma_{32}^3(\Delta\hat{W}_n)^6.
\end{aligned}
\tag{3.22}
$$

Now substituting (3.12)-(3.22) into (3.10), (3.11) and using the obtained expression together with (3.7) and (3.8) we have that the scheme (3.4) shall be 3-equivalent to:

$$
\begin{aligned}
\bar{X}_{n+1} =& \bar{X}_n + \big(\beta_1+\beta_2+\beta_3\big) b\Delta\hat{W}_n + \big(\alpha_1+\alpha_2+\alpha_3\big) a\varDelta \\
&+ a_{01}b\big(\alpha_2\gamma_{21}+\alpha_3(\gamma_{31}+\gamma_{32})\big)\varDelta\Delta\hat{W}_n + b_{10}\beta_2\mu_2\varDelta\Delta\hat{W}_n + ab_{01}\lambda_{21}\beta_2\varDelta\Delta\hat{W}_n \\
&+ \frac{3}{2}b_{02}b^2\gamma_{21}^2\beta_2\varDelta\Delta\hat{W}_n + b_{10}\mu_3\beta_3\varDelta\Delta\hat{W}_n + ab_{01}\big(\lambda_{31}+\lambda_{32}\big)\beta_3\varDelta\Delta\hat{W}_n \\
&+ 3b_{01}^2b\gamma_{21}\gamma_{32}\beta_3\varDelta\Delta\hat{W}_n + \frac{3}{2}b_{02}b^2\big(\gamma_{31}+\gamma_{32}\big)^2\beta_3\varDelta\Delta\hat{W}_n \\
&+ b_{01}b\big(\gamma_{21}\beta_2+(\gamma_{31}+\gamma_{32})\beta_3\big)(\Delta\hat{W}_n)^2 + a_{10}\big(\alpha_2\mu_2+\alpha_3\mu_3\big)\varDelta^2 \\
&+ aa_{01}\big(\alpha_2\lambda_{21}+\alpha_3(\lambda_{31}+\lambda_{32})\big)\varDelta^2 + \frac{1}{2}b_{03}b^3\gamma_{21}^3\beta_2\varDelta^2 + \frac{3}{2}b_{01}b_{02}b^2\gamma_{21}^2\gamma_{32}\beta_3\varDelta^2 \\
&+ 3b_{01}b_{02}b^2\gamma_{21}\gamma_{31}\gamma_{32}\beta_3\varDelta^2 + 3b_{01}b_{02}b^2\gamma_{21}\gamma_{32}^2\beta_3\varDelta^2 + \frac{1}{2}b_{03}b^3\big(\gamma_{31}+\gamma_{32}\big)^3\beta_3\varDelta^2 \\
&+ \frac{1}{2}a_{02}b^2\big(\alpha_2\gamma_{21}^2+\alpha_3(\gamma_{31}+\gamma_{32})^2\big)\varDelta(\Delta\hat{W}_n)^2 + a_{01}b_{01}b\alpha_3\gamma_{21}\gamma_{32}\varDelta(\Delta\hat{W}_n)^2 \\
&+ b_{11}b\mu_2\gamma_{21}\beta_2\varDelta(\Delta\hat{W}_n)^2 + abb_{02}\lambda_{21}\gamma_{21}\beta_2\varDelta(\Delta\hat{W}_n)^2 + a_{01}b_{01}b\lambda_{32}\gamma_{21}\beta_3\varDelta(\Delta\hat{W}_n)^2 \\
&+ b_{01}b_{10}\mu_2\gamma_{32}\beta_3\varDelta(\Delta\hat{W}_n)^2 + ab_{01}^2\lambda_{21}\gamma_{32}\beta_3\varDelta(\Delta\hat{W}_n)^2 + bb_{11}\mu_3\big(\gamma_{31}+\gamma_{32}\big)\beta_3\varDelta(\Delta\hat{W}_n)^2 \\
&+ abb_{02}\big(\gamma_{31}+\gamma_{32}\big)\big(\lambda_{31}+\lambda_{32}\big)\beta_3\varDelta(\Delta\hat{W}_n)^2 + ba_{01}^2\alpha_3\lambda_{32}\gamma_{21}\varDelta^2\Delta\hat{W}_n \\
&+ a_{01}b_{10}\mu_2\gamma_{32}\alpha_3\varDelta^2\Delta\hat{W}_n + ba_{11}\big(\alpha_2\mu_2\gamma_{21}+\alpha_3\mu_3(\gamma_{31}+\gamma_{32})\big)\varDelta^2\Delta\hat{W}_n \\
&+ aa_{02}b\big(\alpha_2\lambda_{21}\gamma_{21}+\alpha_3(\lambda_{31}+\lambda_{32})(\gamma_{31}+\gamma_{32})\big)\varDelta^2\Delta\hat{W}_n + aa_{01}b_{01}\lambda_{21}\gamma_{32}\alpha_3\varDelta^2\Delta\hat{W}_n \\
&+ \frac{1}{2}a_{03}b^3\big(\alpha_2\gamma_{21}^3+\alpha_3(\gamma_{31}+\gamma_{32})^3\big)\varDelta^2\Delta\hat{W}_n + \frac{3}{2}a_{01}b_{02}b^2\gamma_{21}^2\gamma_{32}\alpha_3\varDelta^2\Delta\hat{W}_n \\
&+ 3a_{02}b_{01}b^2\gamma_{21}\big(\gamma_{31}\gamma_{32}+\gamma_{32}^2\big)\alpha_3\varDelta^2\Delta\hat{W}_n + \frac{1}{2}\big(b_{20}-\frac{1}{2}b_{00}^4b_{04}\big)\mu_2^2\beta_2\varDelta^2\Delta\hat{W}_n \\
&+ ab_{11}\mu_2\lambda_{21}\beta_2\varDelta^2\Delta\hat{W}_n + \frac{1}{2}a^2b_{02}\lambda_{21}^2\beta_2\varDelta^2\Delta\hat{W}_n + \frac{3}{2}\big(b_{12}+\frac{1}{4}b_{00}^2b_{04}\big)b^2\mu_2\gamma_{21}^2\beta_2\varDelta^2\Delta\hat{W}_n \\
&+ \frac{3}{2}ab^2b_{03}\lambda_{21}\gamma_{21}^2\beta_2\varDelta^2\Delta\hat{W}_n + b_{01}a_{10}\mu_2\lambda_{32}\beta_3\varDelta^2\Delta\hat{W}_n + aa_{01}b_{01}\lambda_{21}\lambda_{32}\beta_3\varDelta^2\Delta\hat{W}_n \\
&+ \frac{3}{2}a_{02}b_{01}b^2\lambda_{32}\gamma_{21}^2\beta_3\varDelta^2\Delta\hat{W}_n + 3abb_{01}b_{02}\lambda_{21}\gamma_{21}\gamma_{32}\beta_3\varDelta^2\Delta\hat{W}_n \\
&+ \frac{1}{2}b^3b_{01}b_{03}\gamma_{21}^3\gamma_{32}\beta_3\varDelta^2\Delta\hat{W}_n + \frac{1}{2}\big(b_{20}-\frac{1}{2}b_{00}^4b_{04}\big)\mu_3^2\beta_3\varDelta^2\Delta\hat{W}_n \\
&+ ab_{11}\big(\lambda_{31}+\lambda_{32}\big)\mu_3\beta_3\varDelta^2\Delta\hat{W}_n + \frac{1}{2}a^2b_{02}\big(\lambda_{31}+\lambda_{32}\big)^2\beta_3\varDelta^2\Delta\hat{W}_n \\
&+ \frac{3}{2}b^2b_{01}^2b_{02}\gamma_{21}^2\gamma_{32}^2\beta_3\varDelta^2\Delta\hat{W}_n + 3bb_{10}b_{02}\mu_2\big(\gamma_{31}\gamma_{32}+\gamma_{32}^2\big)\beta_3\varDelta^2\Delta\hat{W}_n \\
&+ 3abb_{01}b_{02}\big(\lambda_{31}+\lambda_{32}\big)\gamma_{21}\gamma_{32}\beta_3\varDelta^2\Delta\hat{W}_n + 3abb_{01}b_{02}\lambda_{21}(\gamma_{31}\gamma_{32}+\gamma_{32}^2)\beta_3\varDelta^2\Delta\hat{W}_n \\
&+ \frac{3}{2}\big(b_{12}+\frac{1}{4}b_{00}^2b_{04}\big)b^2\mu_3(\gamma_{31}+\gamma_{32})^2\beta_3\varDelta^2\Delta\hat{W}_n + \frac{3}{2}b^3b_{01}b_{03}\gamma_{21}\big(\gamma_{31}^2+\gamma_{32}^2\big)\beta_3\varDelta^2\Delta\hat{W}_n \\
&+ \frac{3}{2}ab^2b_{03}\big(\lambda_{31}+\lambda_{32}\big)\big(\gamma_{31}^2+\gamma_{32}^2\big)\beta_3\varDelta^2\Delta\hat{W}_n + \frac{1}{2}\big(a_{20}-\frac{1}{2}b_{00}^4a_{04}\big)\big(\alpha_2\mu_2^2+\alpha_3\mu_3^2\big)\varDelta^3 \\
&+ \frac{1}{2}\big(a_{12}+\frac{1}{2}a_{04}b_{00}^2\big)b^2\big(\alpha_2\mu_2\gamma_{21}^2+\alpha_3\mu_3(\gamma_{31}+\gamma_{32})^2\big)\varDelta^3 \\
&+ aa_{11}\big(\alpha_2\mu_2\lambda_{21}+\alpha_3\mu_3(\lambda_{31}+\lambda_{32})\big)\varDelta^3 + \frac{1}{2}a^2a_{02}\big(\alpha_2\lambda_{21}^2+\alpha_3(\lambda_{31}+\lambda_{32})^2\big)\varDelta^3 \\
&+ \frac{1}{2}aa_{03}b^2\big(\alpha_2\lambda_{21}\gamma_{21}^2+\alpha_3(\lambda_{31}+\lambda_{32})(\gamma_{31}+\gamma_{32})^2\big)\varDelta^3
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}a_{01}a_{02}b^2\alpha_3\lambda_{32}\gamma_{21}\big(\gamma_{21}+2(\gamma_{31}+\gamma_{32})\big)\varDelta^3+a_{01}a_{10}\alpha_3\mu_2\lambda_{32}\varDelta^3+aa_{01}^2\alpha_3\lambda_{21}\lambda_{32}\varDelta^3\\
&+a_{02}b_{10}b\alpha_3\mu_2(\gamma_{31}\gamma_{32}+\gamma_{32}^2)\varDelta^3+\frac{3}{2}a_{02}b^2b_{01}^2\alpha_3\gamma_{21}^2\gamma_{32}^2\varDelta^3+aa_{02}bb_{01}\alpha_3\lambda_{21}(\gamma_{31}\gamma_{32}+\gamma_{32}^2)\varDelta^3\\
&+\frac{3}{2}a_{03}b_{01}b^3\alpha_3\gamma_{21}(\gamma_{31}^2\gamma_{32}+\gamma_{32}^3)\varDelta^3+aa_{02}bb_{01}\alpha_3(\lambda_{31}+\lambda_{32})\gamma_{21}\gamma_{32}\varDelta^3\\
&+ab\big(b_{12}+\frac{1}{4}b_{00}^2b_{04}\big)\mu_2\lambda_{21}\gamma_{21}\beta_2\varDelta^3+\frac{1}{2}a^2bb_{03}\lambda_{21}^2\gamma_{21}\beta_2\varDelta^3+\frac{1}{2}a_{03}b^3b_{01}\lambda_{32}\gamma_{21}^3\beta_3\varDelta^3\\
&+aba_{02}b_{01}\lambda_{21}\lambda_{32}\gamma_{21}\beta_3\varDelta^3+\frac{1}{2}\big(b_{20}-\frac{1}{2}b_{00}^4b_{04}\big)\mu_2^2\gamma_{32}\beta_3\varDelta^3+ab_{11}\mu_2\lambda_{21}\gamma_{32}\beta_3\varDelta^3\\
&+\frac{1}{2}a^2b_{02}\lambda_{21}^2\gamma_{32}\beta_3\varDelta^3+\frac{3}{2}ab^2b_{03}\lambda_{21}\gamma_{21}^2\gamma_{32}\beta_3\varDelta^3+a_{01}bb_{11}\mu_3\lambda_{32}\gamma_{21}\beta_3\varDelta^3\\
&+ab_{01}b_{11}\mu_3\lambda_{21}\gamma_{32}\beta_3\varDelta^3+\frac{3}{2}b^2b_{02}b_{11}\mu_3\gamma_{21}^2\gamma_{32}\beta_3\varDelta^3+aa_{01}bb_{02}\lambda_{31}\lambda_{32}\gamma_{21}\beta_3\varDelta^3\\
&+aa_{01}bb_{02}\lambda_{32}^2\gamma_{21}\beta_3\varDelta^3+ab_{10}b_{02}\mu_2\lambda_{31}\gamma_{32}\beta_3\varDelta^3+a^2b_{01}b_{02}\lambda_{21}\lambda_{31}\gamma_{32}\beta_3\varDelta^3\\
&+ab_{10}b_{02}\mu_2\lambda_{32}\gamma_{32}\beta_3\varDelta^3+a^2b_{01}b_{02}\lambda_{21}\lambda_{32}\gamma_{32}\beta_3\varDelta^3\\
&+3\big(b_{12}+\frac{1}{4}b_{00}^2b_{04}\big)b^2b_{01}\mu_3\gamma_{21}\gamma_{31}\beta_3\varDelta^3+\frac{3}{2}b^2b_{10}b_{03}\mu_2\gamma_{31}^2\gamma_{32}\beta_3\varDelta^3\\
&+\frac{3}{2}ab^2b_{01}b_{03}\lambda_{21}\gamma_{31}\gamma_{32}^2\beta_3\varDelta^3+\frac{3}{2}b^2b_{10}b_{03}\mu_2\gamma_{32}^2\beta_3\varDelta^3\\
&+\frac{3}{2}ab^2b_{01}b_{03}\lambda_{21}\gamma_{32}^2\beta_3\varDelta^3+R, \qquad (3.23)
\end{aligned}
$$

where

$$R=ba_{01}\Big(\Delta\hat{Z}_n-\frac{1}{2}\Delta\Delta\hat{W}_n\Big)+\frac{1}{12}b^2a_{02}\varDelta^2+\frac{1}{6}\Big(a_{10}b_{01}+bb_{10}b_{02}+abb_{01}b_{02}+\frac{1}{2}b^2b_{01}^2b_{02}+\frac{1}{2}b^3b_{02}^2+b^3b_{01}b_{03}\Big)\varDelta^2\Delta\hat{W}_n. \qquad (3.24)$$

Schemes (2.10) and (3.23) coincide if the constants satisfy the following system:

$$
\begin{aligned}
&\alpha_1+\alpha_2+\alpha_3=1, && \alpha_3\mu_2(\gamma_{31}\gamma_{32}+\gamma_{32}^2)=\frac{1}{6},\\
&\alpha_2\mu_2+\alpha_3\mu_3=\frac{1}{2}, && \alpha_3\lambda_{21}(\gamma_{31}\gamma_{32}+\gamma_{32}^2)=\frac{1}{6},\\
&\alpha_2\lambda_{21}+\alpha_3(\lambda_{31}+\lambda_{32})=\frac{1}{2}, && \alpha_2\mu_2\gamma_{21}^2+\alpha_3\mu_3(\gamma_{31}+\gamma_{32})^2=\frac{1}{3},\\
&\alpha_2\gamma_{21}+\alpha_3(\gamma_{31}+\gamma_{32})=\frac{1}{2}, && \alpha_2\lambda_{21}\gamma_{21}^2+\alpha_3(\lambda_{31}+\lambda_{32})(\gamma_{31}+\gamma_{32})^2=\frac{1}{3},\\
&\alpha_2\mu_2^2+\alpha_3\mu_3^2=\frac{1}{3}, && 3\alpha_3\gamma_{21}\gamma_{32}(\gamma_{31}+\gamma_{32})=\frac{4}{3},
\end{aligned}
$$

$$\begin{aligned}
&\alpha_2\gamma_{21}^2+\alpha_3(\gamma_{31}+\gamma_{32})^2=\frac{1}{3}, && \frac{3}{2}\alpha_3\gamma_{21}^2\gamma_{32}^2=\frac{4}{3},\\
&\alpha_2\lambda_{21}^2+\alpha_3(\lambda_{31}+\lambda_{32})^2=\frac{1}{3}, && \alpha_3\lambda_{32}\gamma_{21}\big(\gamma_{21}+2(\gamma_{31}+\gamma_{32})\big)=\frac{1}{2},\\
&\alpha_2\mu_2\lambda_{21}+\alpha_3\mu_3(\lambda_{31}+\lambda_{32})=\frac{1}{3}, && \alpha_2\gamma_{21}^3+\alpha_3(\gamma_{31}+\gamma_{32})^3=\frac{1}{3},\\
&\alpha_2\mu_2\gamma_{21}+\alpha_3\mu_3(\gamma_{31}+\gamma_{32})=\frac{1}{3}, && \frac{3}{2}\alpha_3\gamma_{21}^2\gamma_{32}=\frac{1}{6},\\
&\alpha_2\lambda_{21}\gamma_{21}+\alpha_3(\lambda_{31}+\lambda_{32})(\gamma_{31}+\gamma_{32})=\frac{1}{3}, && \\
&\alpha_3\mu_2\lambda_{32}=\frac{1}{6}, && \alpha_3\gamma_{21}\gamma_{32}=\frac{1}{3},\\
&\alpha_3\lambda_{21}\lambda_{32}=\frac{1}{6}, && \alpha_3\mu_2\gamma_{32}=\frac{1}{3},\\
&\alpha_3\lambda_{32}\gamma_{21}=\frac{1}{6}, && \alpha_3\lambda_{21}\gamma_{32}=\frac{1}{3},\\
&\beta_1+\beta_2+\beta_3=1, && \gamma_{21}\beta_2+(\gamma_{31}+\gamma_{32})\beta_3=\frac{1}{3},\\
&\gamma_{32}\gamma_{21}\beta_3=\frac{1}{6}, && \mu_2\beta_2+\mu_3\beta_3=\frac{1}{3},\\
&\gamma_{32}\mu_2\beta_3=\frac{1}{6}, && \lambda_{21}\beta_2+(\lambda_{31}+\lambda_{32})\beta_3=\frac{1}{3},\\
&\gamma_{32}\lambda_{21}\beta_3=\frac{1}{6}, && \gamma_{21}^2\beta_2+(\gamma_{31}+\gamma_{32})^2\beta_3=\frac{1}{3},\\
&\lambda_{32}\gamma_{21}\beta_3=\frac{1}{12}, && \mu_2^2\beta_2+\mu_3^2\beta_3=\frac{1}{3},\\
&\lambda_{32}\lambda_{21}\beta_3=\frac{1}{12}, && \lambda_{21}^2\beta_2+(\lambda_{31}+\lambda_{32})^2\beta_3=\frac{1}{3},
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
&\lambda_{32}\mu_2\beta_3=\frac{1}{12}, && \gamma_{21}^3\beta_2+(\gamma_{31}+\gamma_{32})^3\beta_3=\frac{1}{2},\\
&\lambda_{32}\lambda_{21}\gamma_{21}\beta_3=\frac{1}{6}, && \lambda_{21}^3\beta_2+(\lambda_{31}+\lambda_{32})^3\beta_3=\frac{1}{2},\\
&\mu_2\gamma_{21}\beta_2+\mu_3(\gamma_{31}+\gamma_{32})\beta_3=\frac{1}{3}, && \gamma_{21}^2\mu_2\beta_2+(\gamma_{31}+\gamma_{32})^2\mu_3\beta_3=\frac{1}{2},\\
&\mu_2\lambda_{21}\beta_2+\mu_3(\lambda_{31}+\lambda_{32})\beta_3=\frac{1}{3}, && 3\gamma_{32}\mu_2\beta_3\big(\gamma_{31}+\gamma_{32}\big)=\frac{1}{4},\\
&\gamma_{21}\lambda_{21}\beta_2+(\gamma_{31}+\gamma_{32})(\lambda_{31}+\lambda_{32})\beta_3=\frac{1}{3}, && \lambda_{32}\gamma_{21}\beta_3\big(\gamma_{21}+2(\gamma_{31}+\gamma_{32})\big)=\frac{1}{4}.
\end{aligned} \tag{3.26}$$

The whole equations of relation (3.25) can be simplified as follows:

$$\alpha_1+\alpha_2+\alpha_3=1, \tag{3.27}$$

$$\alpha_2\mu_2+\alpha_3\mu_3=\frac{1}{2}, \tag{3.28}$$

$$\alpha_2\mu_2^2+\alpha_3\mu_3^2=\frac{1}{3}, \tag{3.29}$$

$$\alpha_3\mu_2\lambda_{32}=\frac{1}{6}, \tag{3.30}$$

$$\alpha_2\mu_2\gamma_{21}^2 + \alpha_3\mu_3(\gamma_{31}+\gamma_{32})^2 = \frac{1}{3}, \tag{3.31}$$

$$3\alpha_3\gamma_{21}\gamma_{32}(\gamma_{31}+\gamma_{32}) = \frac{4}{3}, \tag{3.32}$$

$$\alpha_3\lambda_{32}\gamma_{21}\big(\gamma_{21}+2(\gamma_{31}+\gamma_{32})\big) = \frac{1}{2}, \tag{3.33}$$

$$\alpha_2\gamma_{21}^3 + \alpha_3(\gamma_{31}+\gamma_{32})^3 = \frac{1}{3}, \tag{3.34}$$

$$\mu_2 = \lambda_{21} = \gamma_{21}, \tag{3.35}$$

$$\mu_3 = \lambda_{31} + \lambda_{32} = \gamma_{31} + \gamma_{32}. \tag{3.36}$$

Now by using equations (3.30),(3.35), and (3.36), equations (3.33) and (3.34) can be rewritten as follows:

$$\begin{aligned} &\alpha_2\mu_2^3 + \alpha_3\mu_3^3 = \frac{1}{3},\\ &\mu_2 + 2\mu_3 = 3. \end{aligned} \tag{3.37}$$

It's easy to see that none of the solutions of the one-parameter families $\big($one of the families corresponds to the case $\mu_3 = 0, \mu_2 = \frac{2}{3}$, the other one to the case $\mu_2 = \mu_3 = \frac{2}{3}\big)$ can be a solution of (3.37) . On the other hand, since we have in the two-parameter family that:

$$\begin{aligned} \alpha_2 &= \frac{\frac{1}{2}\mu_3 - \frac{1}{3}}{\mu_2(\mu_3-\mu_2)}, \qquad \mu_2 \neq \mu_3, \quad \mu_2 \neq 0,\\ \alpha_3 &= \frac{\frac{1}{3} - \frac{1}{2}\mu_2}{\mu_3(\mu_3-\mu_2)}, \qquad \mu_3 \neq 0, \quad \mu_2 \neq \frac{2}{3}. \end{aligned} \tag{3.38}$$

By imposing on a solution of this family to verify (3.37), we get:

$$6\mu_3^3 - 17\mu_3^2 + 15\mu_3 - 4 = 0, \tag{3.39}$$

the roots of this equation are $\mu_3 = 1$, $\mu_3 = \frac{1}{2}$, and $\mu_3 = \frac{4}{3}$. If $\mu_3 = 1$, then $\mu_2 = \mu_3$ and, as we have said, these solutions do not verify (3.37). Each of the other roots leads to a solution of system (3.39). If $\mu_3 = \frac{1}{2}$, we have the solution:

Family A:

$$\begin{aligned} &\mu_2 = 2, \quad \gamma_{21} = 2, \quad \gamma_{31} = \frac{5}{16}, \quad \gamma_{32} = \frac{3}{16}, \quad \alpha_1 = \frac{1}{12}, \quad \alpha_2 = \frac{1}{36}, \quad \alpha_3 = \frac{8}{9},\\ &\lambda_{21} = 2, \quad \lambda_{31} = \frac{13}{32}, \quad \lambda_{32} = \frac{3}{32}, \quad \beta_1 = \frac{1}{2}, \quad \beta_2 = \frac{1}{18}, \quad \beta_3 = \frac{4}{9}, \end{aligned}$$

and if $\mu_3 = \frac{4}{3}$, we have:
Family B:

$$\begin{aligned} &\mu_2 = \frac{1}{3}, \quad \gamma_{21} = \frac{1}{3}, \quad \gamma_{31} = -\frac{20}{3}, \quad \gamma_{32} = 8, \quad \alpha_1 = -\frac{1}{8}, \quad \alpha_2 = 1, \quad \alpha_3 = \frac{1}{8},\\ &\lambda_{21} = \frac{1}{3}, \quad \lambda_{31} = -\frac{8}{3}, \quad \lambda_{32} = 4, \quad \beta_1 = \frac{3}{16}, \quad \beta_2 = \frac{3}{4}, \quad \beta_3 = \frac{1}{16}. \end{aligned}$$

Now to find third family by using equations (3.32) and (3.33), we can obtain:

$$\mu_2 - \frac{1}{4}\mu_3 = 0, \tag{3.40}$$

if $\mu_3 = 1$, we have the solution:
Family C:

$$\mu_2 = \frac{1}{4}, \quad \gamma_{21} = \frac{1}{4}, \quad \gamma_{31} = -\frac{19}{5}, \quad \gamma_{32} = \frac{24}{5}, \quad \alpha_1 = -\frac{1}{6}, \quad \alpha_2 = \frac{8}{9}, \quad \alpha_3 = \frac{5}{18},$$
$$\lambda_{21} = \frac{1}{4}, \quad \lambda_{31} = -\frac{14}{10}, \quad \lambda_{32} = \frac{24}{10}, \quad \beta_1 = \frac{1}{12}, \quad \beta_2 = \frac{7}{9}, \quad \beta_3 = \frac{5}{36}.$$

All the coefficients in the families A-C are satisfying in the whole equations of relations (3.26). Moreover the equations of relation (3.26) are reduced as follows:

$$\beta_1 + \beta_2 + \beta_3 = 1, \tag{3.41}$$

$$\gamma_{32}\mu_2\beta_3 = \frac{1}{6}, \tag{3.42}$$

$$\lambda_{32}\mu_2\beta_3 = \frac{1}{12}, \tag{3.43}$$

$$\mu_2\beta_2 + \mu_3\beta_3 = \frac{1}{3}, \tag{3.44}$$

$$\mu_2^2\beta_2 + \mu_3^2\beta_3 = \frac{1}{3}, \tag{3.45}$$

$$\gamma_{21}^3\beta_2 + (\gamma_{31} + \gamma_{32})^3\beta_3 = \frac{1}{2}, \tag{3.46}$$

$$3\gamma_{32}\mu_2\beta_3\big(\gamma_{31} + \gamma_{32}\big) = \frac{1}{4}, \tag{3.47}$$

$$\lambda_{32}\gamma_{21}\beta_3\big(\gamma_{21} + 2(\gamma_{31} + \gamma_{32})\big) = \frac{1}{4}. \tag{3.48}$$

By using equations (3.35), (3.36), and (3.43) equations (3.46) and (3.48) can be rewritten as follows:

$$\beta_2\mu_2^3 + \beta_3\mu_3^3 = \frac{1}{2},$$
$$\mu_2 + 2\mu_3 = 3. \tag{3.49}$$

Therefore we have in the two-parameter family that:

$$\beta_2 = \frac{\frac{1}{3} - \frac{1}{3}\mu_3}{\mu_2(\mu_3 - \mu_2)}, \qquad \mu_2 \neq \mu_3, \quad \mu_2 \neq 0,$$
$$\beta_3 = \frac{\frac{1}{3} - \frac{1}{3}\mu_2}{\mu_3(\mu_3 - \mu_2)}, \qquad \mu_3 \neq 0, \quad \mu_2 \neq \frac{2}{3}. \tag{3.50}$$

Also to find fourth family by using equation (3.40), if $\mu_3 = \frac{1}{2}$, we have the solution:
Family D:

$$\mu_2 = \frac{1}{8}, \quad \gamma_{21} = \frac{1}{8}, \quad \gamma_{31} = -\frac{35}{26}, \quad \gamma_{32} = \frac{24}{13}, \quad \alpha_1 = \frac{12}{9}, \quad \alpha_2 = -\frac{16}{9}, \quad \alpha_3 = \frac{13}{9},$$
$$\lambda_{21} = \frac{1}{8}, \quad \lambda_{31} = -\frac{11}{26}, \quad \lambda_{32} = \frac{12}{13}, \quad \beta_1 = \frac{19}{36}, \quad \beta_2 = -\frac{1}{4}, \quad \beta_3 = \frac{13}{18}.$$

Also the coefficients of family D are satisfying in the whole equations of relations (3.25). Each solution defines a scheme 3-equivalent to the simplified order 3 Taylor scheme. The family A is given by:

$$\begin{aligned}\bar{X}_{n+1} =&\bar{X}_n + \frac{1}{12}a(t_n,\bar{X}_n)\mathit{\Delta} + \frac{1}{36}a\big(t_n+2\mathit{\Delta},\bar{X}_n+2a\mathit{\Delta}+2b\Delta\hat{W}_n\big)\mathit{\Delta}\\&+\frac{8}{9}a\Big(t_n+\frac{1}{2}\mathit{\Delta},\bar{X}_n+\frac{13}{32}a\mathit{\Delta}+\frac{3}{32}a\big(t_n+2\mathit{\Delta},\bar{X}_n+2a\mathit{\Delta}+2b\Delta\hat{W}_n\big)\mathit{\Delta}\\&+\frac{5}{16}b\Delta\hat{W}_n+\frac{3}{16}b\big(t_n+2\mathit{\Delta},\bar{X}_n+2a\mathit{\Delta}+2b\Delta\hat{W}_n\big)\Delta\hat{W}_n\Big)\mathit{\Delta}\\&+\frac{1}{2}b(t_n,\bar{X}_n)\Delta\hat{W}_n+\frac{1}{18}b\big(t_n+2\mathit{\Delta},\bar{X}_n+2a\mathit{\Delta}+2b\Delta\hat{W}_n\big)\Delta\hat{W}_n\\&+\frac{4}{9}b\Big(t_n+\frac{1}{2}\mathit{\Delta},\bar{X}_n+\frac{13}{32}a\mathit{\Delta}+\frac{3}{32}a\big(t_n+2\mathit{\Delta},\bar{X}_n+2a\mathit{\Delta}+2b\Delta\hat{W}_n\big)\mathit{\Delta}\\&+\frac{5}{16}b\Delta\hat{W}_n+\frac{3}{16}b\big(t_n+2\mathit{\Delta},\bar{X}_n+2a\mathit{\Delta}+2b\Delta\hat{W}_n\big)\Delta\hat{W}_n\Big)\Delta\hat{W}_n+R,\end{aligned} \tag{3.51}$$

the family B is written as follows:

$$\begin{aligned}\bar{X}_{n+1} =&\bar{X}_n - \frac{1}{8}a(t_n,\bar{X}_n)\mathit{\Delta} + a\big(t_n+\frac{1}{3}\mathit{\Delta},\bar{X}_n+\frac{1}{3}a\mathit{\Delta}+\frac{1}{3}b\Delta\hat{W}_n\big)\mathit{\Delta}\\&+\frac{1}{8}a\Big(t_n+\frac{4}{3}\mathit{\Delta},\bar{X}_n-\frac{8}{3}a\mathit{\Delta}+4a\big(t_n+\frac{1}{3}\mathit{\Delta},\bar{X}_n+\frac{1}{3}a\mathit{\Delta}+\frac{1}{3}b\Delta\hat{W}_n\big)\mathit{\Delta}\\&-\frac{20}{3}b\Delta\hat{W}_n+8b\big(t_n+\frac{1}{3}\mathit{\Delta},\bar{X}_n+\frac{1}{3}a\mathit{\Delta}+\frac{1}{3}b\Delta\hat{W}_n\big)\Delta\hat{W}_n\Big)\mathit{\Delta}\\&+\frac{3}{16}b(t_n,\bar{X}_n)\Delta\hat{W}_n+\frac{3}{4}b\big(t_n+\frac{1}{3}\mathit{\Delta},\bar{X}_n+\frac{1}{3}a\mathit{\Delta}+\frac{1}{3}b\Delta\hat{W}_n\big)\Delta\hat{W}_n\\&+\frac{1}{16}b\Big(t_n+\frac{4}{3}\mathit{\Delta},\bar{X}_n-\frac{8}{3}a\mathit{\Delta}+4a\big(t_n+\frac{1}{3}\mathit{\Delta},\bar{X}_n+\frac{1}{3}a\mathit{\Delta}+\frac{1}{3}b\Delta\hat{W}_n\big)\mathit{\Delta}\\&-\frac{20}{3}b\Delta\hat{W}_n+8b\big(t_n+\frac{1}{3}\mathit{\Delta},\bar{X}_n+\frac{1}{3}a\mathit{\Delta}+\frac{1}{3}b\Delta\hat{W}_n\big)\Delta\hat{W}_n\Big)\Delta\hat{W}_n+R,\end{aligned} \tag{3.52}$$

the family C is written as follows:

$$\begin{aligned}\bar{X}_{n+1} =&\bar{X}_n - \frac{1}{6}a(t_n,\bar{X}_n)\mathit{\Delta} + \frac{8}{9}a\big(t_n+\frac{1}{4}\mathit{\Delta},\bar{X}_n+\frac{1}{4}a\mathit{\Delta}+\frac{1}{4}b\Delta\hat{W}_n\big)\mathit{\Delta}\\&+\frac{5}{18}a\Big(t_n+\mathit{\Delta},\bar{X}_n-\frac{11}{26}a\mathit{\Delta}+\frac{12}{13}a\big(t_n+\frac{1}{4}\mathit{\Delta},\bar{X}_n+\frac{1}{4}a\mathit{\Delta}+\frac{1}{4}b\Delta\hat{W}_n\big)\mathit{\Delta}\\&-\frac{19}{5}b\Delta\hat{W}_n+\frac{24}{5}b\big(t_n+\frac{1}{4}\mathit{\Delta},\bar{X}_n+\frac{1}{4}a\mathit{\Delta}+\frac{1}{4}b\Delta\hat{W}_n\big)\Delta\hat{W}_n\Big)\mathit{\Delta}\\&+\frac{1}{12}b(t_n,\bar{X}_n)\Delta\hat{W}_n+\frac{7}{9}b\big(t_n+\frac{1}{4}\mathit{\Delta},\bar{X}_n+\frac{1}{4}a\mathit{\Delta}+\frac{1}{4}b\Delta\hat{W}_n\big)\Delta\hat{W}_n\\&+\frac{5}{36}b\Big(t_n+\mathit{\Delta},\bar{X}_n-\frac{11}{26}a\mathit{\Delta}+\frac{12}{13}a\big(t_n+\frac{1}{4}\mathit{\Delta},\bar{X}_n+\frac{1}{4}a\mathit{\Delta}+\frac{1}{4}b\Delta\hat{W}_n\big)\mathit{\Delta}\\&-\frac{19}{5}b\Delta\hat{W}_n+\frac{24}{5}b\big(t_n+\frac{1}{4}\mathit{\Delta},\bar{X}_n+\frac{1}{4}a\mathit{\Delta}+\frac{1}{4}b\Delta\hat{W}_n\big)\Delta\hat{W}_n\Big)\Delta\hat{W}_n+R,\end{aligned} \tag{3.53}$$

and the family D is written as follows:

$$\begin{aligned}
\bar{X}_{n+1} =&\bar{X}_n + \frac{12}{9}a(t_n,\bar{X}_n)\varDelta - \frac{16}{9}a\big(t_n+\frac{1}{8}\varDelta,\bar{X}_n+\frac{1}{8}a\varDelta+\frac{1}{8}b\Delta\hat{W}_n\big)\varDelta\\
&+\frac{13}{9}a\Big(t_n+\varDelta,\bar{X}_n-\frac{11}{26}a\varDelta+\frac{12}{13}a\big(t_n+\frac{1}{8}\varDelta,\bar{X}_n+\frac{1}{8}a\varDelta+\frac{1}{8}b\Delta\hat{W}_n\big)\varDelta\\
&-\frac{35}{26}b\Delta\hat{W}_n+\frac{24}{13}b\big(t_n+\frac{1}{8}\varDelta,\bar{X}_n+\frac{1}{8}a\varDelta+\frac{1}{8}b\Delta\hat{W}_n\big)\Delta\hat{W}_n\Big)\varDelta\\
&+\frac{19}{36}b(t_n,\bar{X}_n)\Delta\hat{W}_n-\frac{1}{4}b\big(t_n+\frac{1}{8}\varDelta,\bar{X}_n+\frac{1}{8}a\varDelta+\frac{1}{8}b\Delta\hat{W}_n\big)\Delta\hat{W}_n\\
&+\frac{5}{36}b\Big(t_n+\varDelta,\bar{X}_n-\frac{11}{26}a\varDelta+\frac{12}{13}a\big(t_n+\frac{1}{8}\varDelta,\bar{X}_n+\frac{1}{8}a\varDelta+\frac{1}{8}b\Delta\hat{W}_n\big)\varDelta\\
&-\frac{35}{26}b\Delta\hat{W}_n+\frac{24}{13}b\big(t_n+\frac{1}{8}\varDelta,\bar{X}_n+\frac{1}{8}a\varDelta+\frac{1}{8}b\Delta\hat{W}_n\big)\Delta\hat{W}_n\Big)\Delta\hat{W}_n+R. \qquad (3.54)
\end{aligned}$$

The Butcher arrays for families A-D are illustrated as follows:

TABLE 1. Butcher arrays for family A.

2	2			2		
$\frac{1}{2}$	$\frac{13}{32}$	$\frac{3}{32}$		$\frac{5}{16}$	$\frac{3}{16}$	
R	$\frac{1}{12}$	$\frac{1}{36}$	$\frac{8}{9}$	$\frac{1}{2}$	$\frac{1}{18}$	$\frac{4}{9}$

TABLE 2. Butcher arrays for family B.

$\frac{1}{3}$	$\frac{1}{3}$			$\frac{1}{3}$		
$\frac{4}{3}$	$-\frac{8}{3}$	4		$-\frac{20}{3}$	8	
R	$-\frac{1}{8}$	1	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{3}{4}$	$\frac{1}{16}$

TABLE 3. Butcher arrays for family C.

$\frac{1}{4}$	$\frac{1}{4}$			$\frac{1}{4}$		
1	$-\frac{14}{10}$	$\frac{24}{10}$		$-\frac{19}{5}$	$\frac{24}{5}$	
R	$-\frac{1}{6}$	$\frac{8}{9}$	$\frac{5}{18}$	$\frac{1}{12}$	$\frac{7}{9}$	$\frac{5}{36}$

Table 4. Butcher arrays for family D.

$\frac{1}{8}$	$\frac{1}{8}$			$\frac{1}{8}$		
$\frac{1}{2}$	$-\frac{11}{26}$	$\frac{12}{13}$		$-\frac{35}{26}$	$\frac{24}{13}$	
R	$\frac{12}{9}$	$-\frac{16}{9}$	$\frac{13}{9}$	$\frac{19}{36}$	$-\frac{1}{4}$	$\frac{13}{18}$

4. MS-stability analysis of SRK3 schemes

In this section, we consider the following scalar linear test equation of Itô type:

$$dX_t = \lambda X_t dt + \mu X_t dW(t), \qquad t_0 < t < T, \quad \lambda, \mu \in \mathbb{R}, \tag{4.1}$$

with nonrandom initial condition $X_{t_0} = x_0 \in \mathbb{R}$, and $x_0 \neq 0$. To study stochastic stability, this test equation has been widely used (see [15, 16, 23]). The exact solution of equation (4.1) is given by [17]:

$$X_t = x_0 \exp\Big((\lambda - \frac{1}{2}\mu^2)t + \mu W(t) \Big). \tag{4.2}$$

Definition 4.1. The zero solution of SDE (4.1) is said to be:

(i) MS-stable, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$\sup_{t_0 \leq t \leq T} \mathbb{E}\big(|X_t|^2\big) \leq \epsilon, \quad \textit{for} \quad \mathbb{E}\big(|x_0|^2\big) \leq \delta. \tag{4.3}$$

(ii) Asymptotically MS-stable, if it is stable in mean square and, when $\mathbb{E}\big(|x_0|^2\big) \leq \delta$:

$$\mathbb{E}\big(|X_t|^2\big) \to 0, \quad \textit{as} \quad t \to \infty. \tag{4.4}$$

According to Definition 4.1 one can conclude that the exact solution X_t of equation (4.1) is asymptotically MS-stable if (see [16]):

$$\lim_{t\to\infty} \mathbb{E}\big(|X_t|^2\big) = 0, \quad \Leftrightarrow \quad \Re(\lambda) + \frac{1}{2}|\mu|^2 < 0. \tag{4.5}$$

Applying SRK3 scheme (3.51)-(3.54) to the linear test equation (4.1) yields the unit relation as follows:

$$\begin{aligned}\bar{X}_{n+1} = \bar{X}_n \Big(& 1 + \lambda\Delta + \frac{1}{2}\lambda^2\Delta^2 + \frac{5}{6}\lambda\mu\Delta\Delta\hat{W}_n + \frac{1}{6}\lambda^3\Delta^3 + \frac{7}{12}\lambda^2\mu\Delta^2\Delta\hat{W}_n \\ & + \frac{7}{12}\lambda\mu^2\Delta(\Delta\hat{W}_n)^2 + \mu\Delta\hat{W}_n + \frac{1}{3}\mu^2(\Delta\hat{W}_n)^2 + \frac{1}{6}\mu^3(\Delta\hat{W}_n)^3 \Big) + R.\end{aligned} \tag{4.6}$$

For numerical solution, we apply a one-step numerical scheme to the linear test equation (4.1) and represent a recurrence formula of the form:

$$\bar{X}_{n+1} = R\big(\Delta, \lambda, \mu\big)\bar{X}_n. \tag{4.7}$$

Saito and Mitsui [23] introduce the following definition of MS-stability for a numerical scheme.

Definition 4.2. A numerical scheme is said to be MS-stable for Δ, λ, μ, if:

$$\overline{R}\big(\Delta, \lambda, \mu\big) = \mathbb{E}\Big(|R\big(\Delta, \lambda, \mu\big)|^2\Big) < 1. \tag{4.8}$$

$\overline{R}\big(\Delta, \lambda, \mu\big)$ is called the MS-stability function of the numerical scheme.

By Definition 4.2, SRK3 scheme (4.6) is MS-stable if:

$$\mathbb{E}\Big(\Big|1+\lambda\Delta+\frac{1}{2}\lambda^2\Delta^2+\frac{5}{6}\lambda\mu\Delta\Delta\hat{W}_n+\frac{1}{6}\lambda^3\Delta^3+\frac{7}{12}\lambda^2\mu\Delta^2\Delta\hat{W}_n+\frac{7}{12}\lambda\mu^2\Delta(\Delta\hat{W}_n)^2+\mu\Delta\hat{W}_n+\frac{1}{3}\mu^2(\Delta\hat{W}_n)^2+\frac{1}{6}\mu^3(\Delta\hat{W}_n)^3\Big|^2\Big)<1. \tag{4.9}$$

Then, we reach the following relation:

$$\begin{aligned}
&1+\Big(2\lambda+\frac{5}{3}\mu^2\Big)\Delta+\Big(2\lambda^2+\frac{4}{3}\mu^4+\frac{7}{2}\lambda\mu^2\Big)\Delta^2\\
&+\Big(\frac{4}{3}\lambda^3+\frac{121}{36}\lambda^2\mu^2+2\lambda\mu^4+\frac{5}{12}\mu^6\Big)\Delta^3+\Big(\frac{7}{12}\lambda^4+\frac{77}{48}\lambda^2\mu^4+\frac{5}{3}\lambda^3\mu^2\Big)\Delta^4\\
&+\Big(\frac{77}{144}\lambda^4\mu^2+\frac{1}{6}\lambda^5\Big)\Delta^5+\frac{1}{36}\lambda^6\Delta^6<1,
\end{aligned} \tag{4.10}$$

where we have used the following properties in relation (4.10):

$$\begin{aligned}
&\mathbb{E}\big(\Delta\hat{W}_n\big)=0, && \mathbb{E}\big((\Delta\hat{W}_n)^2\big)=\Delta, && \mathbb{E}\big((\Delta\hat{W}_n)^3\big)=0,\\
&\mathbb{E}\big((\Delta\hat{W}_n)^4\big)=3\Delta^2, && \mathbb{E}\big((\Delta\hat{W}_n)^5\big)=0, && \mathbb{E}\big((\Delta\hat{W}_n)^6\big)=15\Delta^3.
\end{aligned} \tag{4.11}$$

To compare the stability condition (4.8) of the numerical method with the stability condition (4.5) of the test problem, we prefer to introduce stability region as follows:

$$p:=\lambda\Delta, \qquad q:=\mu\sqrt{\Delta}. \tag{4.12}$$

Therefore the MS-stability function of SRK3 schemes is obtained as follows:

$$\begin{aligned}
\overline{R}_{SRK3}(p,q):=\;&1+2p+\frac{5}{3}q^2+2p^2+\frac{7}{2}pq^2+\frac{4}{3}q^4+\frac{4}{3}p^3+\frac{121}{36}p^2q^2+2pq^4\\
&+\frac{5}{12}q^6+\frac{7}{12}p^4+\frac{77}{48}p^2q^4+\frac{5}{3}p^3q^2+\frac{77}{144}p^4q^2+\frac{1}{6}p^5+\frac{1}{36}p^6,
\end{aligned} \tag{4.13}$$

In this setting, MS-stability conditions for linear test problem and the proposed scheme are, respectively, equivalent to:

$$p+\frac{1}{2}q^2<0, \tag{4.14}$$

and

$$\overline{R}_{SRK3}(p,q)<1. \tag{4.15}$$

The MS-stability regions of (4.1) and (4.6) can thus be, respectively, defined as:

$$S_{SDE}:=\Big\{p,q\in\mathbb{R};\ (4.14)\ \textit{holds}\Big\},$$

$$S_{SRK3}:=\Big\{p,q\in\mathbb{R};\ (4.15)\ \textit{holds}\Big\}.$$

In Figure 1 the regions of MS-stability of both the approximate solution SRK3 and RK3-T schemes and the SDE solution of (4.1) are illustrated.

5. Numerical experiments

In this section, numerical results from the implementation of the SRK3 schemes proposed is compared to those from the implementation of well-known schemes of the same order (see [27]).

The simplied order 3 weak Taylor scheme (2.10), denoted by Taylor3, the order 3 Runge–Kutta method proposed by Tocino [27], denoted by RK3-T. Also the third-order SRK Schemes proposed in (3.51)-(3.54) will be denoted by

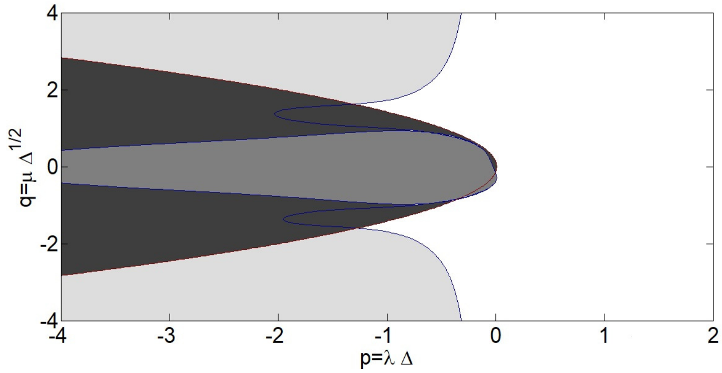

FIGURE 1. MS-stability regions. SRK3 scheme (light gray), RK3-T scheme (dark gray) and SDE (4.1) (dark).

SRK3. Denoting $\bar{X}_T$ and $X(T)$ as the numerical and the exact solutions at time T in ith simulation, respectively. We use mean of absolute errors, "$MeanError$", defined by:

$$MeanError := \frac{1}{2000} \sum_{i=1}^{2000} \parallel X(T) - \bar{X}_T \parallel, \tag{5.1}$$

to measure accuracy of the proposed schemes. For each example we have used $N = 2000$ simulations for stepsizes $\Delta = 2^{-1}, 2^{-2}, \ldots, 2^{-5}$ to compute the approximated value of the known expectation. The mean of absolute errors for each considered scheme are summarized in Tables 5-8.

Example 5.1. At first we consider the linear SDE (4.1). In order to analyze the numerical MS-stability, we consider $x_0 = 1$ and different values for λ and μ. We consider $T = 1$, and the proposed methods with various stepsizes Δ. The results in Tables 5-6 confirm a better convergence properties of the SRK3 schemes in comparison with other methods.

Case (i): To test problem (4.1) we consider $T = 1$, $\lambda = -4$ and $\mu = 0.2$. The computed results are shown in Table 5. From these table we can see that the schemes Taylor3, RK3-T and SRK3 are MS-stable for all stepsizes Δ.

TABLE 5. Means of absolute errors for proposed schemes at $T = 1$ for test problem (4.1) with $\lambda = -4$, $\mu = 0.2$.

Δ	$Taylor3$	$RK3-T$	$SRK3$
2^{-1}	4.97×10^{-1}	3.45×10^{-1}	1.85×10^{-1}
2^{-2}	1.37×10^{-2}	8.87×10^{-3}	5.38×10^{-4}
2^{-3}	8.30×10^{-3}	9.41×10^{-4}	1.76×10^{-4}
2^{-4}	6.59×10^{-3}	5.16×10^{-4}	5.50×10^{-5}
2^{-5}	2.14×10^{-3}	3.94×10^{-4}	3.04×10^{-5}

Case (ii): Eventually to test problem (4.1) we consider $T = 1$, $\lambda = -10$ and $\mu = 0.2$. The computed results are shown in Table 6. From these table we can see that the Taylor3 scheme is MS-stable for stepsizes $\Delta < 2^{-1}$, and the RK3-T scheme is stable for stepsizes $\Delta < 2^{-3}$, where as, the SRK3 scheme is stable for stepsizes $\Delta < 2^{-2}$.

TABLE 6. Means of absolute errors for proposed schemes at $T = 1$ for test problem (4.1) with $\lambda = -10$, $\mu = 0.2$.

Δ	$Taylor3$	$RK3-T$	$SRK3$
2^{-1}	$unstable$	$unstable$	$unstable$
2^{-2}	6.26×10^{-6}	$unstable$	$unstable$
2^{-3}	1.08×10^{-6}	$unstable$	1.09×10^{-9}
2^{-4}	5.20×10^{-7}	5.86×10^{-7}	1.07×10^{-9}
2^{-5}	1.64×10^{-7}	9.01×10^{-8}	4.66×10^{-10}

Example 5.2. The second example is a linear SDE in Itô sense with two standard Brownian motions as below:

$$dX_t = \lambda X_t dt + \mu_1 X_t dW_1(t) + \mu_2 X_t dW_2(t), \qquad t_0 < t < T, \quad \lambda, \mu_1, \mu_2 \in \mathbb{R}, \tag{5.2}$$

with nonrandom initial condition $x_0 = 1$. The exact solution of equation (5.2) is given by [17]:

$$X_t = x_0 \exp\left(\left(\lambda - \frac{1}{2}(\mu_1^2 + \mu_2^2)\right)t + \mu_1 W_1(t) + \mu_2 W_2(t)\right). \tag{5.3}$$

In order to analyze the numerical MS-stability, we consider different values for λ and μ. We consider $T = 1$, and the proposed methods with various stepsizes Δ. The results in Tables 7,8 confirm a better convergence properties of the SRK3 schemes in comparison with other methods.

Case (i): To test problem (5.2) we consider $T = 1$, $\lambda = -10$, $\mu_1 = 0.2$ and $\mu_2 = 0.5$. The computed results are shown in Table 7. From these table we can see that the Taylor3 scheme is MS-stable for all stepsizes Δ, and the RK3-T scheme is stable for stepsizes $\Delta < 2^{-1}$, where as, the SRK3 scheme is stable for stepsizes $\Delta < 2^{-2}$.

TABLE 7. Means of absolute errors for proposed schemes at $T = 1$ for test problem (5.2) with $\lambda = -10$, $\mu_1 = 0.2$ and $\mu_2 = 0.5$.

Δ	$Taylor3$	$RK3-T$	$SRK3$
2^{-1}	3.01×10^{-6}	$unstable$	$unstable$
2^{-2}	2.78×10^{-6}	4.62×10^{-7}	$unstable$
2^{-3}	9.86×10^{-7}	9.43×10^{-8}	7.10×10^{-9}
2^{-4}	5.31×10^{-7}	3.95×10^{-8}	5.69×10^{-9}
2^{-5}	4.10×10^{-7}	2.99×10^{-8}	2.75×10^{-9}

Case (ii): Also, to test problem (5.2) we consider $T = 1$, $\lambda = -15$, $\mu_1 = 0.1$ and $\mu_2 = 0.3$. The computed results are shown in Table 8. From these table we can see that the Taylor3 scheme is MS-stable for all stepsizes Δ, where as the RK3-T and the SRK3 schemes are stable for stepsizes $\Delta < 2^{-2}$.

Example 5.3. The third example is scalar nonlinear SDE in Itô sense as below:

$$dX_t = -\big(\lambda + \mu^2 X_t\big)\big(1 - X_t^2\big)dt + \mu\big(1 - X_t^2\big)dW(t), \qquad t_0 < t < T, \quad \lambda, \mu \in \mathbb{R}, \tag{5.4}$$

with nonrandom initial condition $x_0 = 0$. The exact solution of equation (5.4) is given by [17]:

$$X_t = \frac{(1 + x_0)\exp\Big(-2\lambda t + 2\mu W(t)\Big) + x_0 - 1}{(1 + x_0)\exp\Big(-2\lambda t + 2\mu W(t)\Big) - x_0 + 1}. \tag{5.5}$$

In order to analyze the numerical MS-stability, we consider different values for λ and μ. We consider $T = 1$, and the proposed methods with various stepsizes Δ. The results in Tables 9,10 confirm a better convergence properties of the SRK3 schemes in comparison with other methods.

TABLE 8. Means of absolute errors for proposed schemes at $T = 1$ for test problem (5.2) with $\lambda = -15$, $\mu_1 = 0.1$ and $\mu_2 = 0.3$.

Δ	$Taylor3$	$RK3-T$	$SRK3$
2^{-1}	6.45×10^{-7}	*unstable*	*unstable*
2^{-2}	2.59×10^{-7}	*unstable*	*unstable*
2^{-3}	7.32×10^{-8}	4.91×10^{-9}	4.13×10^{-12}
2^{-4}	4.96×10^{-8}	3.53×10^{-12}	5.18×10^{-14}
2^{-5}	4.12×10^{-8}	7.13×10^{-13}	4.49×10^{-14}

Case (i): To test problem (5.4) we consider $T = 1$, $\lambda = -2$ and $\mu = 0.2$. The computed results are shown in Table 9. From these table we can see that the schemes Taylor3, RK3-T and SRK3 are MS-stable for all stepsizes Δ.

TABLE 9. Means of absolute errors for proposed schemes at $T = 1$ for test problem (5.4) with $\lambda = -2$ and $\mu = 0.2$.

Δ	$Taylor3$	$RK3-T$	$SRK3$
2^{-1}	9.21×10^{-1}	8.24×10^{-1}	2.70×10^{-1}
2^{-2}	7.33×10^{-2}	6.58×10^{-3}	6.50×10^{-4}
2^{-3}	5.02×10^{-3}	4.63×10^{-3}	3.07×10^{-4}
2^{-4}	1.97×10^{-3}	3.27×10^{-3}	2.25×10^{-4}
2^{-5}	1.25×10^{-3}	9.67×10^{-4}	2.02×10^{-4}

Case (ii): Also, to test problem (5.4) we consider $T = 1$, $\lambda = -5$, and $\mu = 0.1$. The computed results are shown in Table 10. From these table we can see that the Taylor3 scheme is MS-stable for all stepsizes Δ, where as the RK3-T and the SRK3 schemes are stable for stepsizes $\Delta < 2^{-2}$.

TABLE 10. Means of absolute errors for proposed schemes at $T = 1$ for test problem (5.4) with $\lambda = -5$ and $\mu = 0.1$.

Δ	$Taylor3$	$RK3-T$	$SRK3$
2^{-1}	7.49×10^{-2}	*unstable*	*unstable*
2^{-2}	1.50×10^{-2}	*unstable*	*unstable*
2^{-3}	8.34×10^{-3}	7.64×10^{-3}	1.11×10^{-3}
2^{-4}	3.94×10^{-3}	5.81×10^{-3}	4.46×10^{-5}
2^{-5}	8.84×10^{-4}	9.12×10^{-4}	3.84×10^{-6}

6. Conclusion

Based on the papers which are introducing some families of third–order stochastic Runge–Kutta schemes by using Itô-Taylor expansion, namely for constant diffusion part, we consider the SDEs with general diffusion term and succeed to deal with the obtained equations and introduce more extra SRK3 families. Moreover, the stability region is illustrated and numerical results are shown to reveal the accuracy of the proposed SRK3 families.

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