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An efficient approximate solution of Riesz fractional advection-diffusion equation

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Abstract

The Riesz fractional advection-diffusion is a result of the mechanics of chaotic dynamics. It's of preponderant importance to solve this equation numerically. Moreover, the utilization of Chebyshev polynomials as a base in several mathematical equations shows the exponential rate of convergence. To this approach, we transform the interval of state space into the interval $[-1, 1] \times [-1, 1]$. Then, we use the operational matrix to discretize fractional operators. Applying the resulting discretization, we obtain a linear system of equations, which leads to the numerical solution. Examples show the effectiveness of the method.

Keywords. Operational matrices, Chebyshev polynomials, Fractional partial differential equations, Riesz fractional advection-diffusion.
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1. Introduction

Despite the long history of fractional calculus, over three centuries, this topic has been investigated explosively for the last four decades, and there exist comprehensive books about this subject. Some of them were introduced in [8, 14].

One of the most interesting equations in fractional calculus is the fractional advection-dispersion equation. Diffusion equations are an especial case of advection-dispersion equation. Advection translates the solute field by transferring the solutes at the flow rate. Dispersion expands the solute column and describes the diffusion propagation of particles through random movement from higher attentiveness regions to lower attentiveness regions. These equations are derived from kinetic dynamics [18, 25]. Also, another source of these equations can be found by extending a model of continuous-time random walk [19].

In this article, we study the advection-dispersion equation of the form

$$\frac{\partial u(z,t)}{\partial t} = p_1 \frac{\partial^{\mu} u(z,t)}{\partial |z|^{\mu}} + p_2 \frac{\partial^{\nu} u(z,t)}{\partial |z|^{\nu}} + f(z,t), \ \ [z,t] \in [t_0, T] \times [z_0, L], \tag{1.1}$$

with boundary conditions

$$\begin{aligned} &u(z,t_0) = g(z), \quad (g(z_0) = g(L) = 0, \text{(the consistence condition)} \\ &u(z_0,t) = u(L,t) = 0, \end{aligned} \tag{1.2}$$

where f is a continuous source function and g is a continuous function the indicates boundary conditions at the first moment. p_1 , p_2 are parameters of the problem, and $t_0, T, z_0, L \in \mathbb{R}$ determine the rectangular $[z_0, L] \times [t_0, T]$ on which the solution u is defined.

A large range of numerical solutions is developed for solving the equation (1.1), [2, 4, 13]. Here, we have a new approach to solve (1.1) by using the operational matrices supported by Chebyshev polynomials. We transform the

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coordinates of the equation (1.1) into $[-1,1]\times[-1,1]$, which has two important advantages. On the one hand, the resulting transform is still the same type; advection-dispersion equation. This surprising result may have an important implication on the realm of fractional calculus. The reason is that it makes the initializing of the fractional process invariant in such modeling.

On the other hand, this transformation numerically is important since we can directly use Chebyshev polynomials defined on the interval [-1,1]. It makes it more convenient and easy to implement.

We discretize all operators such as fractional integrals and derivatives by operational matrices based on Chebyshev polynomials. Then, we replace the corresponding operator to obtain a system of linear equations for the coefficient of the approximate solution. Solving the linear system, we can find the solution of the transformed equation. Inversely, transforming coordinates, we obtain the exact solution.

The memory terms in these varieties of equations build them fully totally different from integer-order partial differential equations (PDEs) and finding FPDEs numerically or analytically is more difficult than PDEs. However, the memory term within the integral type has its benefits and is beneficial within the modeling of a physical or a natural phenomenon during which the recent information depends fully on the information of the whole spare time activity [14]. Therefore, it's of overriding importance to seek out economical strategies for finding solutions for FPDEs.

Recently, new strategies for finding FPDEs are developed within the literature. These strategies embrace the variational iteration technique [12], Laplace transform method [6], wavelet operational method [23], Haar wavelet method [22], Adomian decomposition method [5], homotopy analysis method [7], Legendre base method [3], Bernstein polynomials [2] and Chebyshev collocation methods [1, 10, 15–17, 21].

The spectral strategies using Chebyshev polynomials well-known for differential and partial differential equations with speedy convergence property [9, 11, 20]. A crucial advantage of those strategies over finite-difference strategies is that computing the constant of the approximation, fully determines the answer at any purpose of the required region, whereas the finite-difference strategies area unit supported algorithmic formulas. Therefore, it is interesting to introduce an operational matrix spectral technique using Chebyshev polynomials for finding an approximate solution of FPDEs. This separates orthogonality attributes of the Chebyshev polynomials its benefits over different orthogonal polynomials like Legendre polynomials. Also, the zeros of the Chebyshev polynomials are known analytically. These attributes cause the Clenshaw-Curtis formula that makes numerical integration simple. we have a tendency to use this formula to get the operational matrix of the fractional integration.

The paper is organized as follows: In section 1, we provide preliminary definitions and theorems related to the fractional calculus. In section 2, we provide coordinate transformations for fractional operators. Sections 3-5 introduces chebyshev polynomials and discusses approximating functions by these polynomials. In section 5, the operational matrix for fractional operators is obtained. In section 6, the numerical method is introduced, and supporting examples are followed in section 7.

2. Preliminaries and notations

In this section, we use some of the well-known principles and theorems of the book [14] and [9] for the Chebyshev polynomial and fractional calculus topics, respectively.

2.1. Fractional calculus.

For description of the Caputo-fractional derivative, we first define the Riemann-Liouville fractional integral.

Definition 2.1. [8, 14] Suppose $\mu \geq 0$, $I=[a,b]\subset\mathbb{R}$, and f is integrable function on $[a,b]$. Then, the left and right-handed Riemann-Liouville fractional integral of order μ are described as

$$ {}_{a}^{RL}I_{z}^{\mu} f(z) = \frac{1}{\Gamma(\mu)}\int_a^z \frac{f(\xi)}{(z-\xi)^{1-\mu}}d\xi, \tag{2.1}$$

$$ {}_{z}^{RL}I_{b}^{\mu} f(z) = \frac{1}{\Gamma(\mu)}\int_z^b \frac{f(\xi)}{(\xi-z)^{1-\mu}}d\xi. \tag{2.2}$$

We denote by $AC[a,b]$ the space of absolutely continuous functions on $[a,b]$, and by AC^n $(n\in\mathbb{N})$ the space of differentiable functions f such that $f^n\in AC[a,b]$.

Definition 2.2. [8, 14] Suppose $m-1<\mu\leq m$, $m\in\mathbb{N}$, $I=[a,b]\subset\mathbb{R}$ and $f\in AC^{m-1}([a,b])$. Then, the left and right-handed Riemann-Liouville fractional derivatives of order μ are described as

$$\begin{aligned} {}_{a}^{RL}D_{z}^{\mu}f(z) &= \frac{1}{\Gamma(m-\mu)}\frac{d^m}{dz^m}\int_a^z \frac{f(\xi)}{(z-\xi)^{1+\mu-m}}d\xi &(2.3)\\ &= \frac{d^m}{dz^m}\,{}_{a}^{RL}I_{z}^{m-\mu}f(z), &(2.4)\\ {}_{z}^{RL}D_{b}^{\mu}f(z) &= \frac{(-1)^m}{\Gamma(m-\mu)}\frac{d^m}{dz^m}\int_z^b \frac{f(\xi)}{(\xi-z)^{1+\mu-m}}d\xi &(2.5)\\ &= (-1)^m\frac{d^m}{dz^m}\,{}_{z}^{RL}I_{b}^{m-\mu}f(z), \end{aligned}$$

and the left and right-handed Caputo fractional derivatives of order μ is described as

$$\begin{aligned} {}_{a}^{C}D_{z}^{\mu}f(z) &= \frac{1}{\Gamma(m-\mu)}\int_a^z \frac{f^{(m)}(\xi)}{(z-\xi)^{1+\mu-m}}d\xi &(2.6)\\ {}_{z}^{C}D_{b}^{\mu}f(z) &= \frac{(-1)^m}{\Gamma(m-\mu)}\int_z^b \frac{f^{(m)}(\xi)}{(\xi-z)^{1+\mu-m}}d\xi. &(2.7) \end{aligned}$$

Now, we can define the Riesz fractional derivative.

Definition 2.3. [8, 13] Suppose $\mu\geq 0$, and $I=[a,b]\subset\mathbb{R}$. Let f be a function and $f(a)=f(b)=0$. Then, Riesz fractional derivative is described as

$$\frac{d^\mu}{d|z|^\mu}f(z)=-c_\mu({}_{a}^{RL}D_{z}^{\mu}+{}_{z}^{RL}D_{b}^{\mu})f(z), \tag{2.8}$$

where

$$c_\mu=\frac{1}{2\cos(\frac{\pi\mu}{2})},\quad \mu\neq 1.$$

It can be easily verified that the Riemann-Liouville fractional integral of $(z-a)^\nu$ and $(b-z)^\nu$ produce power functions of the same form.

Theorem 2.4. *[8] Let $\alpha>0$ and $\beta>0$. Then,*

$${}_{a}^{RL}I_{z}^{\mu}(z-a)^\nu=\frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu+1)}(z-a)^{\nu+\mu}, \tag{2.9}$$

and

$${}_{z}^{RL}I_{b}^{\mu}(b-z)^\nu=\frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu+1)}(b-z)^{\nu+\mu}. \tag{2.10}$$

3. Transforming Riesz fractional advection-diffusion equation

The Chebyshev polynomials are delineated on the interval $[-1,1]$, and the problem is that we want to solve the equation (1.1) on the region $[z_0,L]\times[w_0,T]$. Two methods can be proposed. We can use shifted Chebyshev polynomials or we can use a linear transformation of the Riesz fractional advection-diffusion equation. We apply the second method. The following theorems are important for this purpose.

Theorem 3.1. *Let $(\bar{z},\bar{t})=(a_1z+b_1,a_2t+b_2)$ be a linear transformation which translate the $[a,b]\times[c,d]$ onto $[\bar{a},\bar{b}]\times[\bar{c},\bar{d}]$ and suppose $a_1,a_2\neq 0$. Define, $u(z,t):=v(\bar{z},\bar{t})$. Then,*

$$\begin{aligned} {}_{a}^{RL}I_{z}^{\mu}u(z,t) &= \left(\frac{1}{a_1}\right)^{\mu}{}_{\bar{a}}^{RL}I_{\bar{z}}^{\mu}v(\bar{z},\bar{t}),\\ {}_{z}^{RL}I_{b}^{\mu}u(z,t) &= \left(\frac{1}{a_1}\right)^{\mu}{}_{\bar{z}}^{RL}I_{\bar{b}}^{\mu}v(\bar{z},\bar{t}). \end{aligned} \tag{3.1}$$

Proof.

$$
\begin{aligned}
{}_{a}^{RL}I_{z}^{\mu}u(z,t) &= \frac{1}{\Gamma(\mu\mu)}\int_{a}^{z}\frac{u(\xi,w)}{(z-\xi)^{1-\mu}}d\xi \\
&= \frac{1}{\Gamma(\mu)}\int_{a}^{z}\frac{v(a_1\xi+b_1,a_2t+b_2)}{(z-\xi)^{1-\mu}}d\xi \\
&= \frac{1}{\Gamma(\mu)}\int_{a_1a+b_1}^{a_1z+b_1}\frac{v(s,a_2t+b_2)}{(z-\frac{s}{a_1}+\frac{b_1}{a_1})^{1-\mu}}\frac{1}{a_1}ds.
\end{aligned}
\tag{3.2}
$$

Substituting $s=a_1\xi+b_1$ into the right hand side of the Equation (3.2), we obtain

$$
{}_{a}^{RL}I_{z}^{\mu}u(z,t) = \frac{1}{\Gamma(\mu)}\int_{\bar{a}}^{\bar{z}}\frac{v(s,a_2t+b_2)}{(\frac{\bar{z}}{a_1}-\frac{s}{a_1})^{1-\mu}}\frac{1}{a_1}ds.
\tag{3.3}
$$

Using the notation $\bar{z}=a_1z+b_1$ and $\bar{a}=a_1a+b_1$, thus we obtain

$$
\begin{aligned}
{}_{a}^{RL}I_{z}^{\mu}u(z,t) &= \frac{1}{\Gamma(\mu)a_1^{\mu}}\int_{\bar{a}}^{\bar{z}}\frac{v(s,a_2t+b_2)}{(\bar{z}-s)^{1-\mu}}ds \\
&= \left(\frac{1}{a_1}\right)^{\mu}{}_{\bar{a}}^{RL}I_{z}^{\mu}v(a_1z+b_1,a_2t+b_2).
\end{aligned}
\tag{3.4}
$$

Similarly,

$$
\begin{aligned}
{}_{z}^{RL}I_{b}^{\mu}u(z,t) &= \frac{1}{\Gamma(\mu)}\int_{z}^{b}\frac{u(\xi,t)}{(\xi-z)^{1-\mu}}d\xi \\
&= \frac{1}{\Gamma(\mu)}\int_{z}^{b}\frac{v(a_1\xi+b_1,a_2t+b_2)}{(\xi-z)^{1-\mu}}d\xi \\
&= \frac{1}{\Gamma(\mu)}\int_{a_1z+b_1}^{a_1b+b_1}\frac{v(s,a_2t+b_2)}{(\frac{s}{a_1}-\frac{b_1}{a_1}-z)}^{1-\mu}\frac{1}{a_1}ds.
\end{aligned}
\tag{3.5}
$$

Substituting $s=a_1\xi+b_1$ into the right hand side of the Equation (3.5), we obtain

$$
{}_{z}^{RL}I_{b}^{\mu}u(z,t) = \frac{1}{\Gamma(\mu)}\int_{\bar{z}}^{\bar{b}}\frac{v(s,a_2t+b_2)}{(\frac{s}{a_1}-\frac{\bar{z}}{a_1})^{1-\mu}}\frac{1}{a_1}ds.
\tag{3.6}
$$

Using the notation $\bar{z}=a_1z+b_1$, $\bar{b}=a_1b+b_1$ finally we obtain

$$
\begin{aligned}
{}_{z}^{RL}I_{b}^{\mu}u(z,t) &= \frac{1}{\Gamma(\mu)a_1^{\mu}}\int_{\bar{z}}^{\bar{b}}\frac{v(s,a_2t+b_2)}{(s-\bar{z})^{1-\mu}}ds \\
&= \left(\frac{1}{a_1}\right)^{\mu}{}_{z}^{RL}I_{\bar{b}}^{\mu}v(a_1z+b_1,a_2t+b_2).
\end{aligned}
\tag{3.7}
$$

This completes the proof. □

Theorem 3.2. *Let the suppositions of the Theorem 3.1 be held. Then,*

$$
\begin{aligned}
{}_{a}^{RL}D_{z}^{\mu}u(z,t) &= a_1^{\mu}\,{}_{\bar{a}}^{RL}D_{\bar{z}}^{\mu}v(\bar{z},\bar{t}), \\
{}_{z}^{RL}D_{b}^{\mu}u(z,t) &= a_1^{\mu}\,{}_{\bar{z}}^{RL}D_{\bar{b}}^{\mu}v(\bar{z},\bar{t}).
\end{aligned}
\tag{3.8}
$$

Proof.

$$
\begin{aligned}
{}^{RL}_{a}D^{\mu}_{z}u(z,t) &= \frac{\partial^m}{\partial z^m}\left({}^{RL}_{a}I^{m-\mu}_{z}u(z,t)\right)\\
&= a_1^m \frac{\partial^m}{\partial z^m}\left(\frac{1}{a_1}\right)^{m-\mu} {}^{RL}_{\bar{a}}I^{m-\mu}_{\bar{z}}v(a_1z+b_1, a_2t+b_2)\\
&= a_1^{\mu}\,{}^{RL}_{\bar{a}}D^{\mu}_{\bar{z}}v(a_1z+b_1, a_2t+b_2),
\end{aligned}
\tag{3.9}
$$

and similarly

$$
\begin{aligned}
{}^{RL}_{z}D^{\mu}_{b}u(z,t) &= (-1)^m\frac{\partial^m}{\partial z^m}\left({}^{RL}_{z}I^{m-\mu}_{b}u(z,t)\right)\\
&= (-1)^m a_1^m \frac{\partial^m}{\partial z^m}\left(\frac{1}{a_1}\right)^{m-\mu} {}^{RL}_{\bar{z}}I^{m-\mu}_{\bar{b}}v(a_1z+b_1, a_2t+b_2)\\
&= a_1^{\mu}\,{}^{RL}_{\bar{z}}D^{\mu}_{\bar{b}}v(a_1z+b_1, a_2t+b_2).
\end{aligned}
\tag{3.10}
$$

□

Corollary 3.3. *Under the suppositions of the Theorem 3.1 and Definition 2.1, we have*

$$
\frac{\partial^{\gamma}u(z,t)}{\partial|z|^{\gamma}} = a_1^{\gamma}\frac{\partial^{\gamma}v(\bar{z},\bar{t})}{\partial|z|^{\gamma}},
\tag{3.11}
$$

for $\gamma > 0$.

Now, setting $(\bar{z},\bar{t}) = (\frac{2}{L-z_0}(z-z_0)-1, \frac{2}{T-t_0}(t-t_0)-1)$, and $u(z,t)=v(\bar{z},\bar{t})$ we get

$$
\frac{2}{T-t_0}\frac{\partial v(\bar{z},\bar{t})}{\partial t} = p_1\left(\frac{2}{L-z_0}\right)^{\mu}\frac{\partial^{\mu}v(\bar{z},\bar{t})}{\partial|z|^{\mu}} + p_2\left(\frac{2}{L-z_0}\right)^{\nu}\frac{\partial^{\nu}v(\bar{z},\bar{t})}{\partial|z|^{\nu}} + f(\bar{z},\bar{t}),
\tag{3.12}
$$

for $-1<\bar{z}<1$, and $-1<\bar{t}<1$, with boundary conditions

$$
\begin{aligned}
&v(\bar{z},-1) = g\left(\frac{(\bar{z}+1)(L-z_0)}{2}+z_0\right),\\
&v(-1,\bar{t}) = v(1,\bar{t}) = 0,
\end{aligned}
\tag{3.13}
$$

which its domain is $[-1,1]\times[-1,1]$. Therefore, without loss of generality, we obtain the numerical method for the following equation

$$
\frac{\partial u(z,t)}{\partial t} = p_1\frac{\partial^{\mu}u(z,t)}{\partial|z|^{\mu}} + p_2\frac{\partial^{\nu}u(z,t)}{\partial|z|^{\nu}} + f(z,t),
\tag{3.14}
$$

for $-1<z<1$, and $-1<t<1$, with boundary conditions

$$
\begin{aligned}
&u(z,-1) = g(z),\\
&u(-1,t) = u(1,t) = 0.
\end{aligned}
\tag{3.15}
$$

3.1. Chebyshev polynomials.

Definition 3.4. [14] The Chebyshev polynomial $T_n(z)$ of the first kind is a polynomials in z of degree n, described by the relation

$$
T_n(z) = \cos(n\theta) \quad \text{when } z=\cos(\theta).
\tag{3.16}
$$

Some of them are plotted in Figure 1. The Chebyshev polynomials are orthogonal with respect to the weight function $\omega(z)=\frac{1}{\sqrt{1-z^2}}$ and the corresponding inner product is

$$
<f,g> = \int_{-1}^{1}\omega(z)g(z)f(z)dz, \quad \text{for} \quad f,g\in\mathcal{C}[-1,1].
\tag{3.17}
$$

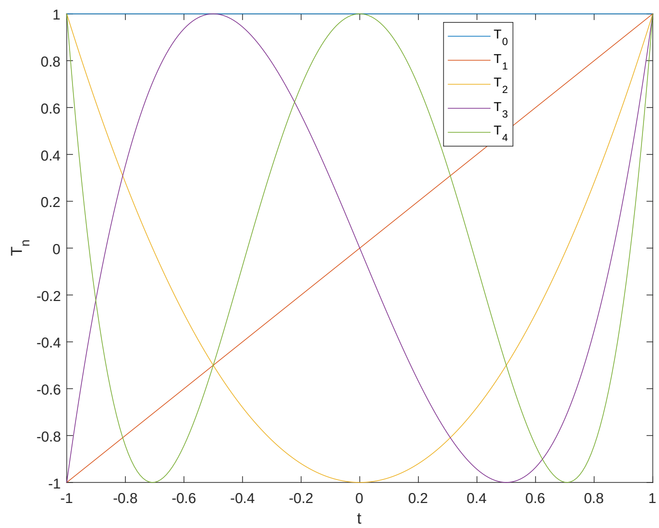


FIGURE 1. Chebyshev polynomials

The well-known recursive formula

$$T_0(z) = 1,\ T_1(z) = z,\ T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z),\ n \in \mathbb{N}, \tag{3.18}$$

is important to calculate these polynomials numerically, while we may use

$$T_n(z) = \sum_{\lambda=0}^{n} (-2)^{\lambda} \frac{n(n+\lambda-1)!}{(n-\lambda)!(2\lambda)!}(1-z)^{\lambda}, \qquad n > 0, \tag{3.19}$$

and

$$T_n(z) = (-1)^n T_n(-z) = \sum_{\lambda=0}^{n} (-1)^{n+\lambda} \frac{2^{\lambda} n(n+\lambda-1)!}{(n-\lambda)!(2\lambda)!}(1+z)^{\lambda}, \qquad n > 0, \tag{3.20}$$

To calculate the Chebyshev polynomial to obtain the operational matrix of the fraction operator. Chebyshev polynomials discrete orthogonality link to the Clenshaw-Curtis descripted by the formula [9]:

$$\int_{-1}^{1} w(z) f(z) dz \simeq \frac{\pi}{M+1} \sum_{\lambda=1}^{M+1} f(z_{\lambda}), \tag{3.21}$$

where z_{λ} for $\lambda = 1, \dots, M+1$ are zeros of $T_{M+1}(z)$. Also,

$$\gamma_n = \|T\|^2 = \begin{cases} \frac{\pi}{2}, & n > 0, \\ \pi, & n = 0. \end{cases}$$

will be of important use later. The integrals and the derivatives of the Chebyshev polynomials

$$\int T_0(z)dz = T_1(z), \quad \int T_1(z)dz = \frac{1}{4}T_2(z),$$

$$\int T_n(z)dz = \frac{1}{2}\left(\frac{T_{n+1}(z)}{n+1} - \frac{T_{n-1}(z)}{n-1}\right), \quad n > 1.$$

and

$$\frac{d}{dz}T_n(z) = 2n \sideset{}{'}\sum_{\substack{r=0 \\ n-r=odd}}^{n-1} T_r \tag{3.22}$$

where, $\sum'$ denotes that the last term in the sum is to be halved if n is even.

Since, the Equation (3.22) is fundamental for obtaining operational matrix, we write some of the sentences exactly

$$\begin{aligned}
T_1' &= 2(1)(T_0/2) = T_0, \\
T_2' &= 2(2)(T_1) = 4T_1, \\
T_3' &= 2(3)(T_0/2 + T_2) = 3T_0 + 6T_2, \\
T_4' &= 2(4)(T_1 + T_3) = 8T_1 + 8T_3, \\
T_5' &= 2(5)(T_0/2 + T_2 + T_4) = 5T_0 + 10T_2 + 10T_4, \\
T_6' &= 2(6)(T_1 + T_3 + T_5) = 12T_1 + 12T_3 + 12T_5.
\end{aligned} \tag{3.23}$$

Other derivatives have similarly pattern.

4. Function approximation

A function f on the interval $[0, 1]$, may be extended as follows:

$$f(z) \simeq \sum_{m=0}^{M} c_m T_m(z) = C^T \zeta(z), \ \ M \in \mathbb{N}, \tag{4.1}$$

where C and ζ are the matrices of size $(M + 1) \times 1$

$$\begin{aligned}
C^T &= [c_0, \cdots, c_M], \\
\zeta^T(z) &= [T_0, \ldots, T_M],
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
c_i &= \frac{1}{\gamma_i} \int_{-1}^{1} w(z) f(z) T_i(z) dz \\
&\simeq \frac{\pi}{\gamma_i(M+1)} \sum_{\lambda=1}^{M+1} f(z_\lambda) T_i(z_\lambda), \quad i = 0, \ldots, M.
\end{aligned}$$

The following error estimate for infinitely differentiable function f shows that the Chebyshev polynomials converge with exponential rate. We consider expansions of the form

$$f(z) = \sum_{m=0}^{\infty} c_m T_m(z), \qquad z \in [-1, 1].$$

with partial sum denoted by

$$S_n(z) = \sum_{m=0}^{n} c_m T_m(z).$$

Theorem 4.1. *[9] when a function f has $m + 1$ continuous derivatives on $[-1, 1]$, where m is natural number , then $|f(z) - S_n(z)| = O(n^{-m})$ for all $z \in [-1, 1]$.*

Remark 4.2. We note that if the equation (1.1) has a unique solution and the source function f and initial boundary condition g has $m+1$ continuous derivatives, then according to Fredholm alternative the solution of the equation (1.1) also have $m + 1$ continuous derivatives. Thus, the error estimate in the Theorem 4.1 can be used as an approximate estimation of errors.

Let u be a bivariate function defined on $[0,1]\times[0,1]$. can then be expanded utulize Chebyshev polynomials as follow:

$$u(z,t)\simeq\sum_{n=0}^{N}\sum_{m=0}^{N}u_{n,m}T_n(z)T_m(t)=\zeta(z)^TU\zeta(t),\ \ M\in\mathbb{N}. \tag{4.3}$$

where $U=(u_{i,j})$ is a matrix of size $(M+1)\times(M+1)$ with elements

$$\begin{aligned}u_{i,j}&=\frac{1}{\gamma_i\gamma_j}\int_{-1}^{1}\int_{-1}^{1}\omega(z)\omega(t)u(z,t)T_i(z)T_j(t)dzdw\\&\simeq\frac{\pi^2}{\gamma_i\gamma_j(M+1)^2}\sum_{r=1}^{M+1}\sum_{s=1}^{M+1}u(z_r,z_s)T_i(z_r)T_j(z_s).\end{aligned}$$

5. Construct operational matrices

Theorem 5.1. *Let $\zeta(z)$ be the vector of the Chebyshev polynomials described in (4.2). Then,*

$$\zeta'(z)=\mathrm{D}\zeta(z), \tag{5.1}$$

where the operational matrix D *can be described as*

$$\mathrm{D}=\begin{pmatrix}0&0&0&0&0&0&\dots\\1&0&0&0&0&0&\dots\\0&4&0&0&0&0&\dots\\3&0&6&0&0&0&\dots\\0&8&0&8&0&0&\dots\\5&0&10&0&10&0&\dots\\0&12&0&12&0&12&\dots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

Proof. The proof is simple application of the formula (3.23). □

Letting D be differential operator we have

$$\begin{aligned}{}^{RL}_{a}D^{\mu}_{z}f(z)&=D^m\left({}^{RL}_{a}I^{m-\mu}_{z}f(z)\right), &(5.2)\\ {}^{RL}_{z}D^{\mu}_{b}f(z)&=(-1)^mD^m\left({}^{RL}_{z}I^{m-\mu}_{b}f(z)\right). &(5.3)\end{aligned}$$

Theorem 5.2. *Let ${}_L\Gamma^{\gamma}$ and ${}_R\Gamma^{\gamma}$ be the operational matrices of the fractional differential operators ${}^{RL}_{-1}I^{\gamma}_{z}$ and ${}^{RL}_{z}I^{\gamma}_{1}$, respectively. Then,*

$$({}_L\Gamma^{\gamma})_{0m}=\frac{1}{\Gamma(\gamma+1)}\frac{\pi}{\gamma_m(N+1)}\sum_{j=1}^{N+1}(z_j+1)^{\gamma}T_m(z_j),$$

$$({}_R\Gamma^{\gamma})_{0m}=\frac{1}{\Gamma(\gamma+1)}\frac{\pi}{\gamma_m(N+1)}\sum_{j=1}^{N+1}(1-z_j)^{\gamma}T_m(z_j),$$

for $m=0,\cdots,N$, and

$$({}_L\Gamma^{\gamma})_{nm}=\sum_{\lambda=0}^{n}\frac{(-1)^{n+\lambda}2^{\lambda}n(n+\lambda-1)!}{(n-\lambda)!(2\lambda)!}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\gamma+1)}\frac{\pi}{\gamma_m(N+1)}\sum_{j=1}^{N+1}(z_j+1)^{\lambda+\gamma}T_m(z_j),$$

$$({}_R\Gamma^{\gamma})_{nm}=\sum_{\lambda=0}^{n}\frac{(-2)^{\lambda}n(n+\lambda-1)!}{(n-\lambda)!(2\lambda)!}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\gamma+1)}\frac{\pi}{\gamma_m(N+1)}\sum_{j=1}^{N+1}(1-z_j)^{\lambda+\gamma}T_m(z_j),$$

for $m=0,\cdots,N$ and $n=1,\cdots,N$.

$$
\begin{aligned}
{}^{RL}_{-1}I^{\gamma}_{z}T_n(z) &= {}^{RL}_{-1}I^{\gamma}_{z}\sum_{\lambda=0}^{n}(-1)^{n+\lambda}\frac{2^{\lambda}n(n+\lambda-1)!}{(n-\lambda)!(2\lambda)!}(1+z)^{\lambda}\\
&= \sum_{\lambda=0}^{n}\frac{(-1)^{n+\lambda}2^{\lambda}n(n+\lambda-1)!}{(n-\lambda)!(2\lambda)!}\,{}^{RL}_{-1}I^{\gamma}_{z}(1+z)^{\lambda}\\
&= \sum_{\lambda=0}^{n}\frac{(-1)^{n+\lambda}2^{\lambda}n(n+\lambda-1)!}{(n-\lambda)!(2\lambda)!}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\gamma+1)}(z+1)^{\lambda+\gamma},
\end{aligned}
$$

$$
(z+1)^{\lambda+\gamma} \simeq \sum_{m=0}^{N}\left(\frac{\pi}{\gamma_m(N+1)}\sum_{j=1}^{N+1}(z_j+1)^{\lambda+\gamma}T_m(z_j)\right)T_m(z),
$$

$$
\begin{aligned}
{}^{RL}_{z}I^{\gamma}_{1}T_n(z) &= {}^{RL}_{z}I^{\gamma}_{1}\sum_{\lambda=0}^{n}\frac{(-2)^{\lambda}n(n+\lambda-1)!}{(n-\lambda)!(2\lambda)!}(1-z)^{\lambda}\\
&= \sum_{\lambda=0}^{n}\frac{(-2)^{\lambda}n(n+\lambda-1)!}{(n-\lambda)!(2\lambda)!}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\gamma+1)}(1-z)^{\gamma+f},
\end{aligned}
$$

$$
(1-z)^{\lambda+\gamma} \simeq \sum_{m=0}^{N}\left(\frac{\pi}{\gamma_m(N+1)}\sum_{j=1}^{N+1}(1-z_j)^{\lambda+\gamma}T_m(z_j)\right)T_m(z),
$$

Theorem 5.3. *Let $m-1<\mu\leq m$, $m\in\mathbb{N}$ and ${}^{Ri}\mathrm{D}^{\mu}$ be an operational matrix of the Riesz fractional differential operator. Then,*

$$
{}^{Ri}\mathrm{D}^{\mu} = -c_{\mu}D^{m}\left({}_{L}\mathrm{I}^{m-\mu}+(-1)^{m}{}_{R}\mathrm{I}^{m-\mu}\right),
$$

where

$$
c_{\mu} = \frac{1}{2\cos(\frac{\pi\mu}{2})},\quad \mu\neq 1.
$$

6. Description of the proposed method

To obtain an operational Chebyshev method for Riesz fractional advection-diffusion we first recall it for advection-diffusion equation with ordinary derivatives:

$$
\frac{\partial u(z,t)}{\partial t} = p_1\frac{\partial u(z,t)}{\partial z}+p_2\frac{\partial^2 u(z,t)}{\partial z^2}+f(z,t), \tag{6.1}
$$

for $-1<z<1$, and $-1<w<1$, with boundary conditions

$$
\begin{aligned}
&u(z,-1)=g(z),\\
&u(-1,t)=u(1,t)=0.
\end{aligned} \tag{6.2}
$$

Now, supposing

$$
u(z,t)\simeq\zeta(z)^{T}U\zeta(t), \tag{6.3}
$$

using operational matrices we have

$$
\zeta(z)^{T}UD\zeta(t)\simeq p_1\zeta(z)^{T}D^{T}U\zeta(t)+p_2\zeta(z)^{T}(D^{T})^{2}U\zeta(t)+\zeta(z)^{T}K\zeta(t). \tag{6.4}
$$

Here, f was approximated by $\zeta(z)^{T}F\zeta(t)$. This leads to the following $(N+1)^2$ equations

$$
UD=p_1D^{T}U+p_2(D^{T})^{2}U+F, \tag{6.5}
$$

which the obtained system is singular when $F = 0$. Therefore, the first $(N-1)\times N$ equations will be used from (6.5). Other equations obtained using initial and boundary conditions:

$$\begin{aligned} F\zeta(-1) &= G^T,\\ \zeta(\pm 1)^T F &= [0,\dots,0], \end{aligned} \tag{6.6}$$

We note that $g = G\zeta(z) = \zeta^T(z)G^T$.

Similarly, the operational Chebyshev method for Riesz fractional advection-diffusion (3.14) is

$$UD = p_1\left({}^{Ri}\mathrm{D}^{\mu}\right)^T U + p_2\left({}^{Ri}\mathrm{D}^{\nu}\right)^T U + F. \tag{6.7}$$

Finally, operators can take to implement easy to get a system of linear algebraic equations in the standard form used. We show by vec, the vectorization of a matrix

$$vec(A) := (a_{1,1}\dots, a_{m,1},\dots, a_{1,n},\dots, a_{m,n})^T.$$

notice that

$$vec(ABC) = (C^T \otimes A)vec(B),$$

where $\otimes$ is the Kronecker product.

7. Examples

Example 7.1. Regard the Riesz fractional diffusion equation

$$\frac{\partial u(z,t)}{\partial t} = \frac{\partial^{\mu} u(z,t)}{\partial |z|^{\mu}}, \qquad 1<\mu<2, \tag{7.1}$$

on $[0,\pi]\times[0,1]$ subject to the boundary and initial conditions given by

$$\begin{aligned} u(z,0) &= z^2(\pi - z),\\ u(0,t) &= u(\pi,t) = 0. \end{aligned} \tag{7.2}$$

From [24], the analytical solution is

$$u(z,t) = \sum_{n=1}^{\infty}\left(\frac{8}{n^3}(-1)^{n+1} - \frac{4}{n^3}\right)e^{-\lambda_n^{\frac{\alpha}{2}} t}\sin(nz), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 = n^2. \tag{7.3}$$

Using the procedure of Section 3 the equation (7.1) can be converted to the standard form:

$$\frac{\partial v(z,t)}{\partial t} = \frac{\partial^{\mu} v(z,t)}{\partial |z|^{\mu}}, \qquad 1<\mu<2, \tag{7.4}$$

on $[-1,1]\times[-1,1]$ subject to the boundary and initial conditions given by

$$\begin{aligned} v(z,-1) &= \left(\frac{z+1}{2}\right)^2 \pi^2\left(\pi - \frac{z+1}{2}\pi\right),\\ v(-1,t) &= v(1,t) = 0. \end{aligned} \tag{7.5}$$

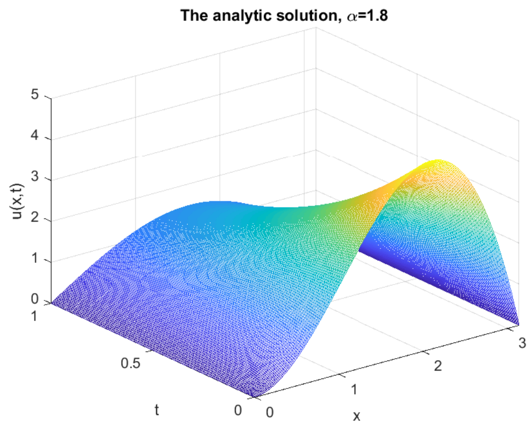
The solution of (7.1) can be computed using $u(z,w) = v(\frac{2z}{\pi} - 1, 2w-1)$, on the $[0,\pi]\times[0,1]$. Figures 2 and 3 show the analytical and approximate solution, respectively. Table 1 shows the exact solution and approximate solution on some uniform divided node points. The comparison confirms effectiveness of the proposed method.

Example 7.2. Regard the Riesz fractional advection-dispersion equation

$$\frac{\partial u(z,t)}{\partial t} = \frac{\partial^{\mu} u(z,t)}{\partial |z|^{\mu}} + \frac{\partial^{\nu} u(z,t)}{\partial |z|^{\nu}}, \quad 0<t<1,\ 0<z<L=\pi, \tag{7.6}$$

on $[0,\pi]\times[0,1]$ subject to the boundary and initial conditions given by

$$\begin{aligned} u(z,0) &= z^2(\pi - z),\\ u(0,t) &= u(\pi,t) = 0. \end{aligned} \tag{7.7}$$



(a) (b)

FIGURE 2. (a) The analytic solution of Example 7.1 (b) The approximate solution of Example 7.1.

TABLE 1. Comparison of the exact solution u and the approximate solution u_N, $N = 7$.

z	0.4712	0.9425	1.4137	1.8850	2.3562	2.8274
$u(z, 0.25)$	0.9325	1.9179	2.8315	3.2530	2.7270	1.2995
$u_N(z, 0.25)$	0.9510	1.9684	2.8804	3.2925	2.8066	1.3311
$u(z, 0.5)$	0.8728	1.6769	2.2424	2.3564	1.8731	0.8378
$u_N(z, 0.5)$	0.9161	1.7430	2.3223	2.4421	1.9495	0.8884
$u(z, 0.75)$	0.7237	1.3704	1.7467	1.7393	1.3567	0.5879
$u_N(z, 0.75)$	0.7843	1.4419	1.8377	1.8504	1.4276	0.6378
$u(z, 1)$	0.5728	1.0828	1.3503	1.3024	1.0105	0.4283
$u_N(z, 1)$	0.6390	1.1563	1.4423	1.4206	1.0765	0.4759

TABLE 2. The max error for Example 7.2

N	2	3	4	5	6	7	8
$E(N)$	0.9689	0.6409	0.4087	0.1344	0.0919	0.0549	0.0450

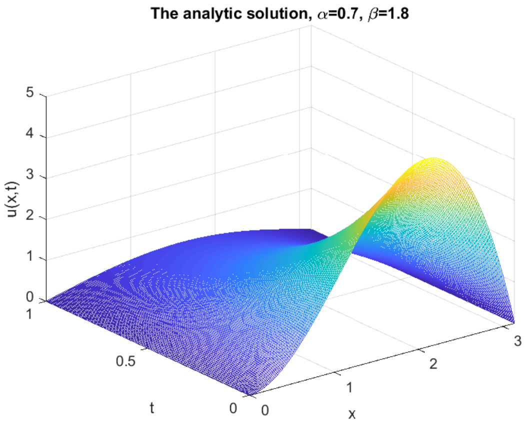
From [24], the analytical solution is

$$u(z,t) = \sum_{n=1}^{\infty} \left(\frac{8}{n^3}(-1)^{n+1} - \frac{4}{n^3}\right) e^{-(\lambda_n^{\frac{\alpha}{2}} + \lambda_n^{\frac{\beta}{2}})t} \sin(nz). \tag{7.8}$$

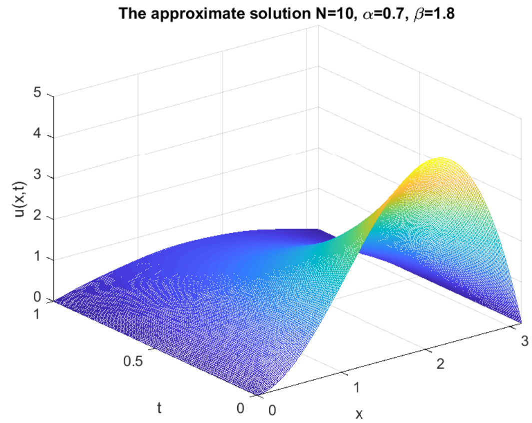
Figures 4 and 5 show the analytical and approximate solution, respectively. Table 2 show the max error defined by

$$E(N) = \max_{(z,y)\in D_{200}} |u(z,y) - u_N(z,y)|,$$

on $D_{200} = \{(z_i, y_j)|\ z_i = ih,\ y_j = jh,\ i,j = -100, \ldots,\ 99, 100,\ h = \frac{1}{100}\}$, where $u(z,y)$ is analytic solution and $u_N(z,y)$ is approximate solution.



(a)



(b)

FIGURE 3. (a) The analytic solution of Example 7.2 (b) The approximate solution of Example 7.2

8. Conclusion

In this article we have used Chebyshev polynomials to the solution of advection-diffusion equation. For this purpose, we converted the desired interval to $[-1, 1]$. Next, we transformed the differential equation into a system of linear algebraic equations using the operational matrix. By solving the system of algebraic equations, we obtain the unknowns, and then by placing them in the desired expansion, the approximation to the equation is obtained. We used this method for different examples. By reporting the error, we showed that this method is an efficient method with fast convergence.

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