Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 10, No. 2, 2022, pp. 419-430
DOI:110.22034/cmde.2021.38468.1692

Stochastic analysis and invariant subspace method for handling option pricing with numerical simulation

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Abstract

In this paper, option pricing is given via stochastic analysis and invariant subspace method. Finally numerical solutions is driven and shown via diagram. The considered model is one of the most well known non-linear time series model in which the switching mechanism is controlled by an unobservable state variable that follows a first-order Markov chain. Some analytical solutions for option pricing are given under our considered model. Then numerical solutions are presented via finite difference method.

Keywords. Option pricing; Markov chain; Geometric Brownian motion; Finite difference method.

2010 Mathematics Subject Classification. 58J65, 65M06.

1. Introduction

Stochastic analysis specially stochastic process and stochastic differential equation theory have a key role in option pricing theory. A stochastic process is a phenomenon that can be thought to evolve over time in a random way. Common examples are the location of a particle in a physical system, the price of the stock in a financial market, interest rates, mobile phone networks, Internet traffic, etc. A basic example is the erratic movement of pollen grains suspended in water, called Brownian motion [15].

Stochastic differential equations are not only used to price options but are also applied in a more general way to describe optimal consumption and investment decisions in a continuous-time setting [17].

The pricing of contingent claims has been widely used and studied in a lot of literatures [4–6]. For this reason, it's one of the most important topic in financial mathematics. The methodology for option pricing is developed by Black and Scholes. A number of papers raised to their model. The defection of the Black-Scholes model is that interest and volatility rate are supposed to be non random which are not consistent with reality of market.

To get more realistic models, many extensions to the Black-Scholes model have been introduced. Among them our considered model that is one of the most well known non-linear time series model provide more realistic description for asset price dynamics. In this model the switching mechanism is controlled by an unobservable state variable that follows a first-order Markov chain.This model involves multiple structures (equations) that can characterize the time series behaviors in different regimes. By permitting switching between these structures, this model is able to capture more complex dynamic patterns [13]. In this paper, invariant subspace method, has been applied to solve the system of coupled partial differential equations for option pricing. In fact the pricing of power options has been studied when the price dynamics of the underlying risky asset are assumed to follow a Markov-modulated geometric Brownian motion and find an exact and explicit formula for power options pricing with regime switching via invariant subspace method. This methodology is discussed in [10–12, 14, 16] comprehensively.

Received: 10 February 2020 ; Accepted: 07 April 2021.
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The payoff of power option depend on the price of the risky underlying asset raised to power $\alpha \geq 0$. In this paper our financial market satisfies by the regime switching Black-Scholes formula which has been widely used by traders and investors.

This paper is organized as follows. Section 2 describes the notations, definitions and the asset price dynamics under the Markov-modulated geometric Brownian motion. Section 3 formulates the partial differential equation system for power options pricing. Section 4 and 5 derive exact and approximated solutions for power options pricing and these solutions compares via some plotted graphs.

2. Mathematical Formulation

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space where $\{\mathcal{F}_t : t \geq 0\}$ is the filtration generated by Brownian motion and Q is a risk-neutral probability. Suppose the states of an economy are modeled by a finite state continuous-time Markov chain $\{X_t : t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$. Without loss of generality, we can identify the state space of $\{X_t : t \geq 0\}$ with a finite set of unit vectors $\chi := \{e_1, e_2, \ldots, e_M\}$, where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^M$. We suppose that $\{X_t : t \geq 0\}$ and $\{W_t : t \geq 0\}$ are independent.

Let H be the generator $[q_{ij}]_{i,j=1,2,\ldots,M}$, of the Markov chain process. We have the following semi-martingale representation theorem for $\{X_t : t \geq 0\}$,

$$X_t = X_0 + \int_0^t H X_s \,\mathrm{d}s + M_t,$$

where $\{M_t : t \geq 0\}$ is an $\mathbb{R}^M$-valued martingale increment process with respect to the filtration generated by $\{X_t : t \geq 0\}$ [8]. The underling asset price S_t at time t is as follows:

$$\mathrm{d}S_t = \mu_{X_t} S_t \mathrm{d}t + \sigma_{X_t} S_t \mathrm{d}W_t,$$

where the stock appreciation rate $\{\mu_t : t \geq 0\}$ and the volatility $\{\sigma_t : t \geq 0\}$ of S depend on $\{X_t : t \geq 0\}$ are described by $\mu_{X_t} := \mu(t, X_t) = \langle \mu, X_t \rangle$, $\sigma_{X_t} := \sigma(t, X_t) = \langle \sigma, X_t \rangle$, where $\mu := (\mu_1, \mu_2, \ldots, \mu_M)$, $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_M)$ with $\sigma_i > 0$, $i = 1, 2, \ldots, M$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^M$.

3. Option pricing

In this section, we derive the pricing of power call options in a regime switching model. The payoff of power call option depends on the price of the risky underlying asset raised to the power $\alpha > 0$. For the standard power call option, the payoff is equal to $\max\{{S_t}^\alpha - K^\alpha\}$, where K is strike price and T is maturity time. The price at time t prior to the expiration time T of this call option under Q is given by

$$\begin{aligned} V(S,t,T,X) &= E^Q\left[\exp\left(-\int_0^t r_u \,\mathrm{d}u\right) V(T) \mid S_t = s, X_t = x\right], \\ &= E^Q\left[\exp\left(-\int_0^t r_u \,\mathrm{d}u\right) ({S_t}^\alpha - K^\alpha)^+ \mid S_t = s, X_t = x\right]. \end{aligned}$$

Let $\tilde{V}(S,t,X) = \exp\left(-\int_0^t r_u \,\mathrm{d}u\right) V(S,t,T,X)$. So

$$\begin{aligned} \tilde{V}(S,t,X) &= E^Q\left[\exp\left(-\int_0^t r_u \,\mathrm{d}u\right) ({S_t}^\alpha - K^\alpha)^+ \mid S_t = s, X_t = x\right], \\ &= E^Q\left[\exp\left(-\int_0^t r_u \,\mathrm{d}u\right) ({S_t}^\alpha - K^\alpha)^+ \mid G_t\right]. \end{aligned}$$

We assume a filtration $\{G_t : t \geq 0\}$, where G_t is the smallest σ-algebra generated by $\{W_u, X_u : u \leq t\}$ and $\tilde{V}$ is a martingale under Q.

Let $\tilde{V}(S,t) = (\tilde{V}(S,t,e_1), \ldots, \tilde{V}(S,t,e_M))$. So $\tilde{V}(S_t,t,x_t) = \langle \tilde{V}(S_t,t), X_t \rangle$.

In the sequel, we apply Ito formula for $\tilde{V}(S,t,X)$ and find its dynamics:

$$\begin{aligned} d\tilde{V}(S_t,t,X_t) &= \frac{\partial \tilde{V}}{\partial t}\,dt + \frac{\partial \tilde{V}}{\partial S}dS + \frac{1}{2}\frac{\partial^2 \tilde{V}}{\partial S^2}d\langle S,S\rangle + \tilde{V}\,dX_t \\ &= -r_t \exp\left(-\int_0^t r_u\,du\right) V(S_t,t,X_t)dt + \exp\left(-\int_0^t r_u\,du\right)\frac{\partial V}{\partial t}dt \\ &+ \exp\left(-\int_0^t r_u\,du\right)\frac{\partial V}{\partial S}\left(r_t S_t dt + \sigma_t S_t dW_t\right) \\ &+ \frac{1}{2}\exp\left(-\int_0^t r_u\,du\right)\frac{\partial^2 V}{\partial S^2}(\sigma_t S_t)^2\,dt + \exp\left(-\int_0^t r_u\,du\right) V\,dX_t. \end{aligned}$$

Since $\tilde{V}(S,t,X)$ is a Q martingale, the drift term must be identical to zero [1]. Hence, we have

$$-r_t V + \frac{\partial V}{\partial t} + r_t S\frac{\partial V}{\partial s} + \frac{1}{2}{\sigma_t}^2 S^2\frac{\partial^2 V}{\partial S^2} + \langle V,H\rangle = 0. \tag{3.1}$$

Let $V_i := V(t,s,e_i)$, $i=1,2,\ldots,M$. Therefor V satisfies the below regime switching equation

$$-r_i V_i + \frac{\partial V_i}{\partial t} + r_i S\frac{\partial V_i}{\partial s} + \frac{1}{2}{\sigma_i}^2 S^2\frac{\partial^2 V_i}{\partial S^2} + \langle V,He_i\rangle = 0, \tag{3.2}$$

which is a parabolic differential equation that is equal to

$$-r_i V_i + \frac{\partial V_i}{\partial t} + r_i S\frac{\partial V_i}{\partial S} + \frac{1}{2}{\sigma_i}^2 S^2\frac{\partial^2 V_i}{\partial S^2} + \sum_{i\neq j} q_{ij}(V_j - V_i) = 0, \tag{3.3}$$

with the terminal condition

$$V(T,s,e_i) = V(s), \qquad i=1,2,\ldots,M. \tag{3.4}$$

4. Exact solutions via invariant subspace method

To be self-contained, we first present a brief details on the invariant subspace method applicable to time PDEs involving two independent variables t,S in the form of

$$\frac{\partial V_i}{\partial t} = \widehat{F}_i[V_i], \tag{4.1}$$

where $\frac{\partial}{\partial t}$ is a time derivative, and $\widehat{F}_i[V_i]$ are nonlinear differential operators of order k. This method was introduced by Galaktionov [9].

Theorem 4.1. *Let W_n be the linear space spanned by n-linearly independent functions $f_i(S), i=1,\ldots,n$ and suppose that W_n is invariant under the operator $\widehat{F}[V]$. Then there exist n functions $\Phi_1,\Phi_2,\ldots,\Phi_n$ such that*

$$\widehat{F}\left[\sum_{i=1}^n c_i f_i(S)\right] = \sum_{i=1}^n \Phi_i(c_1,c_2,\ldots,c_n) f_i(S), \qquad c_i\in\mathbb{R}, \tag{4.2}$$

where Φ_i are the expansion coefficients of $\widehat{F}[V]\in W_n$ in the basis f_i. It follows that the evolution equation (4.1) has solution of the form

$$V_i(t,S) = \sum_{i=1}^n c_i(t) f_i(S), \tag{4.3}$$

where the coefficients $c_1(t),c_2(t),\ldots,c_n(t)$ satisfy a system of ODEs

$$\frac{dc_i(t)}{dt} = \Phi_i(c_1(t),c_2(t),\ldots,c_n(t)), \quad i=1,2,\ldots,n. \tag{4.4}$$

According to these basics consider equation (3.3) in the following form

$$\frac{\partial V_i}{\partial t} = \widehat{F}_i[V_i] = r_i V_i - r_i S\frac{\partial V_i}{\partial S} - \frac{1}{2}{\sigma_i}^2 S^2 \frac{\partial^2 V_i}{\partial S^2} - \sum_{i\neq j} q_{ij}(V_j - V_i). \tag{4.5}$$

It is straight forward to check that the differential operator (4.5) admits an invariant subspace $W = \mathcal{L}\{1, x\} \times \mathcal{L}\{1, x\}$ because

$$\begin{aligned}
\widehat{F_1}[C_{11} + C_{21}S] &= r_1 C_{11} - \sum_{1\neq j} q_{1j}\Big(C_{1j} - C_{11} + S\Big(C_{2j} - C_{21}\Big)\Big) \in W,\\
\widehat{F_2}[C_{12} + C_{22}S] &= r_2 C_{12} - \sum_{2\neq j} q_{2j}\Big(C_{1j} - C_{12} + S\Big(C_{2j} - C_{22}\Big)\Big) \in W,\\
&\vdots\\
\widehat{F_n}[C_{1n} + C_{2n}S] &= r_n C_{1n} - \sum_{n\neq j} q_{nj}\Big(C_{1j} - C_{1n} + S\Big(C_{2j} - C_{2n}\Big)\Big) \in W.
\end{aligned} \tag{4.6}$$

It is clear that the dimension of W is two, which suggests that an exact solution of (4.5) in the form

$$\begin{cases}
V_1(S,t) = C_{11}(t) + C_{21}(t)S\\
V_2(S,t) = C_{12}(t) + C_{22}(t)S\\
\vdots\\
V_n(S,t) = C_{1n}(t) + C_{2n}(t)S
\end{cases} \tag{4.7}$$

where $C_{11}(t), C_{21}(t), \cdots, C_{2n}(t)$ are unknown functions to be computed. Inserting solution (4.7) in (4.5) and equating different powers of S to zero yield a simple system of ODEs:

$$\begin{cases}
\dfrac{\mathrm{d}C_{11}(t)}{\mathrm{d}t} = r_1 C_{11} - \displaystyle\sum_{1\neq j} q_{1j}\Big(C_{1j} - C_{11}\Big),\\
\dfrac{\mathrm{d}C_{21}(t)}{\mathrm{d}t} = -\displaystyle\sum_{1\neq j} q_{1j}\Big(C_{2j} - C_{21}\Big),\\
\dfrac{\mathrm{d}C_{12}(t)}{\mathrm{d}t} = r_2 C_{12} - \displaystyle\sum_{2\neq j} q_{2j}\Big(C_{1j} - C_{12}\Big),\\
\dfrac{\mathrm{d}C_{22}(t)}{\mathrm{d}t} = -\displaystyle\sum_{2\neq j} q_{2j}\Big(C_{2j} - C_{22}\Big),\\
\qquad\vdots\\
\dfrac{\mathrm{d}C_{1n}(t)}{\mathrm{d}t} = r_n C_{1n} - \displaystyle\sum_{n\neq j} q_{nj}\Big(C_{1j} - C_{1n}\Big),\\
\dfrac{\mathrm{d}C_{2n}(t)}{\mathrm{d}t} = -\displaystyle\sum_{n\neq j} q_{nj}\Big(C_{2j} - C_{2n}\Big).
\end{cases} \tag{4.8}$$

The solution $V_1(t,S), V_2(t,S), \cdots, V_n(t,S)$ will be found by solving above system. In order to summarizing the system (4.7) we solve this system only for $j = 1, 2$ or $V_1(t,S), V_2(t,S)$. Thus the system is reduced to

$$\begin{cases}
\dfrac{\partial V_1}{\partial t} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 V_1}{\partial s^2} + r_1 s\dfrac{\partial V_1}{\partial s} - (r_1 + q_{12})V_1 + q_{12}V_2 = 0,\\
\dfrac{\partial V_2}{\partial t} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 V_2}{\partial s^2} + r_2 s\dfrac{\partial V_2}{\partial s} - (r_2 + q_{21})V_1 + q_{21}V_1 = 0.
\end{cases} \tag{4.9}$$

Consequently, we have:

$$\begin{cases} \dfrac{\mathrm{d}C_{11}(t)}{\mathrm{d}t} = r_1 C_{11} - q_{12}\Big(C_{12} - C_{11}\Big), \\ \dfrac{\mathrm{d}C_{21}(t)}{\mathrm{d}t} = -q_{12}\Big(C_{22} - C_{21}\Big), \\ \dfrac{\mathrm{d}C_{12}(t)}{\mathrm{d}t} = r_2 C_{12} - q_{21}\Big(C_{11} - C_{12}\Big), \\ \dfrac{\mathrm{d}C_{22}(t)}{\mathrm{d}t} = -q_{21}\Big(C_{21} - C_{22}\Big). \end{cases} \tag{4.10}$$

The solution of the system (4.10) is

$$C_{11}(t) = A\left(\frac{-q_{12} + q_{21} - \sqrt{Q} - r_1 + r_2}{2q_{21}}\right)\exp\left[\frac{(r_1 + r_2 + q_{12} + q_{21} + \sqrt{Q})\,t}{2}\right]$$

$$+B\left(\frac{-q_{12} + q_{21} + \sqrt{Q} - r_1 + r_2}{2q_{21}}\right)\exp\left[-\frac{\Big(- r_1 - r_2 - q_{12} - q_{21} + \sqrt{Q}\Big)t}{2}\right],$$

$$C_{12}(t) = A\exp\left[\frac{(r_1 + r_2 + q_{12} + q_{21} + \sqrt{Q})\,t}{2}\right] - B\exp\left[-\frac{(-r_1 - r_2 - q_{12} - q_{21} + \sqrt{Q})\,t}{2}\right],$$

$$C_{21}(t) = L + E\exp\left[(q_{12} + q_{21})\,t\right],$$

$$C_{22}(t) = L - E\frac{q_{21}}{q_{12}}\exp\left[(q_{12} + q_{21})\,t\right],$$

where

$$Q = r_1^2 - 2r_1 r_2 + 2r_1 q_{12} - 2r_1 q_{21} + r_2^2 - 2r_2 q_{12} + 2r_2 q_{21} + q_{12}^2 + 2q_{21}q_{12} + q_{21}^2,$$

and A, B, L, E are arbitrary constants. The solution of the considered equation for $q_{12} = q_{21} = r_1 = r_2 = 1$, is obtained by:

$$\begin{cases} V_1(S,t) = \Big(- Ae^{3t} + Be^{t}\Big) + S\Big(L + Ee^{2t}\Big), \\ V_2(S,t) = \Big(Ae^{3t} + Be^{t}\Big) + S\Big(L - Ee^{2t}\Big), \end{cases} \tag{4.11}$$

and are plotted in Figure 4.

5. Numerical Simulation

In this section, a numerical solution for the system of Black-Scholes equations (4.9), equipped with a suitable final time and boundary conditions which will be defined exactly later is investigated.

The finite difference method is one of the most popular numerical schemes to solve differential equations especially for boundary value problems. It is based on replacing the presence derivatives in the differential equation with some suitable approximations in order to reduce the differential equation to an algebraic system of equations. In financial mathematics, an explicit finite difference method was implemented for the first time by Brennan and Schwartz [2]. The disadvantage of their work was the conditional stability of the explicit method. Later some implicit methods were used such as a work presented by Courtadon [3], with the Crank-Nicolson approach. To comparison the three explicit, implicit and Crank-Nicolson approaches see [7]. In this part, a numerical solution for the system of the Black-Scholes equation applying implicit finite difference method is presented. Coupling of the equations of this system is considered to obtain the structure of all the matrices and vectors.

For discretization the system (4.9), we follow the method of line approach, replacing any spatial derivative by a discrete formula obtained here by the finite difference method. Thus, the PDE (4.9) is transformed into a large set of ODEs. Finally, applying a time-stepping procedure, we obtain a full discretized equations.

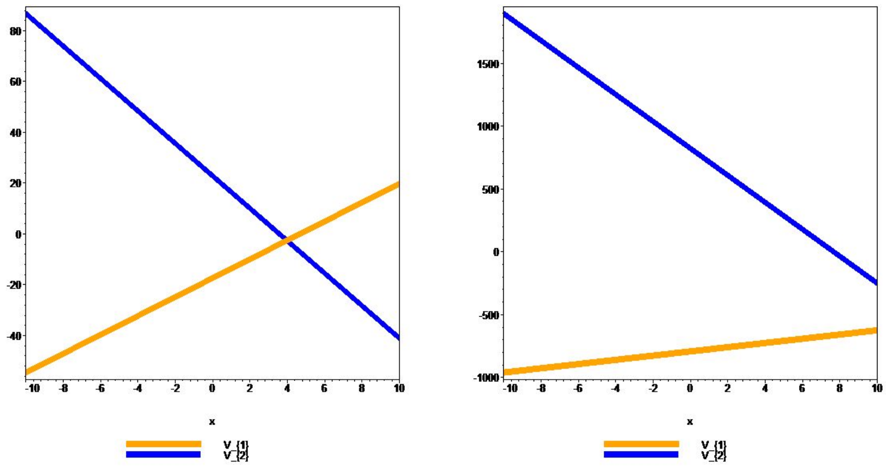

FIGURE 1. Effect of the solution $V(S,t)$ of Eq. (3.3) with $A=B=E=L=t=1$ in left and $A=B=E=L=T=2$ in right.

Applying a change of variable, $k(t)=T-t$, we can transform the backward system into a forward one. To be more precise, we have:

$$\frac{\partial V_i(s,t)}{\partial t}=\frac{\partial V_i(s,k(t))}{\partial k(t)}k'(t)=-\frac{\partial V_i(s,k)}{\partial k},\quad i\in\{1,2\}.$$

For the ease of notation, in the rest of this section, we replace again $k(t)$ with t, obtaining:

$$\begin{cases}\frac{\partial V_1}{\partial t}-\frac{1}{2}\sigma^2 s^2\frac{\partial^2 V_1}{\partial s^2}-r_1 s\frac{\partial V_1}{\partial s}+(r_1+q_{12})V_1-q_{12}V_2=0,\\ \frac{\partial V_2}{\partial t}-\frac{1}{2}\sigma^2 s^2\frac{\partial^2 V_2}{\partial s^2}-r_2 s\frac{\partial V_2}{\partial s}+(r_2+q_{21})V_2-q_{21}V_1=0.\end{cases}\tag{5.1}$$

Considering the interval $[0,T]$ where T is the final time, we divide this interval into M equally distance subintervals of length τ. In a similar way, we divide spatial interval $[0,S_{\max}]$ into N equally sized subintervals of length h. A grid point in this mesh is denoted by $(nh,m\tau)$ where $n=1,2,3,\dots,N$ and $m=1,2,3,\dots,M$. Using a fully implicit method, the discretization of the equation (5.1) will be as follows:

$$\frac{V_{i,n}^{m+1}-V_{i,n}^{m}}{\tau}-\frac{1}{2}\sigma^2n^2h^2\frac{V_{i,n+1}^{m+1}-2V_{i,n}^{m+1}+V_{i,n-1}^{m+1}}{h^2}-r_inh\frac{V_{i,n+1}^{m+1}-V_{i,n-1}^{m+1}}{2h}+(q_{ij}+r_i)V_{i,n}^{m+1}-q_{ij}V_{j,n}^{m+1}=0,$$

for $i,j\in\{1,2\}$, $i\neq j$. Simplifying it, we obtain

$$\frac{1}{2}(r_in\tau-\sigma^2n^2\tau)V_{i,n-1}^{m+1}+(1+\sigma^2n^2\tau+(q_{ij}+r_i)\tau)V_{i,n}^{m+1}-\frac{1}{2}(\sigma^2n^2\tau+r_in\tau)V_{i,n+1}^{m+1}-q_{ij}\tau V_{j,n}^{m+1}=V_{i,n}^{m},\tag{5.2}$$

for $i\in\{1,2\}$, $n=1,\dots,N$ and $m=0,\dots,M-1$.

To solve equation (5.2) collectively, we define $V_n^m=(V_{1,n}^m,V_{2,n}^m)^T$ indicating the variables $V_{1,n}^m$, $V_{2,n}^m$ at the grid point n,m. We denote an iteration of the collective linear system (5.2) at the time step m by

$$AV^{m+1}=b^m,\quad m=0,1,2,\dots,M-1,$$

where A is a block tridiagonal matrix defined as

$$A = \begin{pmatrix} D_1 & U_2 & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ L_1 & D_2 & U_3 & & \vdots \\ 0_{2\times 2} & \ddots & & & \\ & & & \ddots & 0_{2\times 2} \\ \vdots & & \ddots & \ddots & U_{N-1} \\ 0_{2\times 2} & & \cdots & L_{N-2} & D_{N-1} \end{pmatrix},$$

where,

$$D_n = \begin{pmatrix} d_{1,n} & -q_{12}\tau \\ -q_{21}\tau & d_{2,n} \end{pmatrix}, \quad U_n = \begin{pmatrix} u_{1,n} & 0 \\ 0 & u_{2,n} \end{pmatrix}, \quad L_n = \begin{pmatrix} l_{1,n} & 0 \\ 0 & l_{2,n} \end{pmatrix},$$

with,

$$\begin{array}{ll} d_{i,n} = 1 + (r_i + q_{ij})\tau + \sigma^2 n^2 \tau, & i \in \{1,2\},\ n = 1,\dots,N-1, \\ u_{i,n} = -\frac{1}{2}(r_i\tau(n-1) + \sigma^2\tau(n-1)^2), & i \in \{1,2\},\ n = 2,\dots,N, \\ l_{i,n} = \frac{1}{2}(r_i\tau(n+1) - \sigma^2\tau(n+1)^2), & i \in \{1,2\},\ n = 0,\dots,N-2, \end{array}$$

and,

$$\begin{aligned} V^{m+1} &= \left(V^{m+1}_{1,1}, V^{m+1}_{2,1}, V^{m+1}_{1,2}, V^{m+1}_{2,2}, \dots, V^{m+1}_{1,N-1}, V^{m+1}_{2,N-1}\right)^T, \\ b^m &= \big(V^m_{1,1} - l_{1,0}V^{m+1}_{1,0}, V^m_{2,1} - l_{2,0}V^{m+1}_{2,0}, V^m_{1,2}, V^m_{2,2}, \\ &\dots, V^m_{1,N-2}, V^m_{2,N-2}, V^m_{1,N-1} - u_{1,N}V^{m+1}_{1,N}, V^m_{2,N-1} - u_{2,N}V^{m+1}_{2,N}\big)^T. \end{aligned}$$

The values $V^0_{i,n}, V^m_{i,0}, V^m_{i,N}$, $i \in \{1,2\}$, $n = 0,\dots,N$ and $m = 0,\dots,M$ are known from the initial and boundary conditions.

5.1. **Numerical Examples.** In this section, we present some numerical examples to illustrate the efficiency and reliability of the proposed method. All examples are performed by using `Matlab 2017a`.

Example 5.1. *Consider the following coupled linear system as a test function*

$$\begin{cases} \dfrac{\partial V_1}{\partial t} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 V_1}{\partial s^2} + r_1 s \dfrac{\partial V_1}{\partial s} - (r_1 + q_{12})V_1 + q_{12}V_2 = 0, \\ \dfrac{\partial V_2}{\partial t} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 V_2}{\partial s^2} + r_2 s \dfrac{\partial V_2}{\partial s} - (r_2 + q_{21})V_2 + q_{21}V_1 = b(s,t), \end{cases} \tag{5.3}$$

where $0 \le s \le 2$, $0 \le t \le 0.25$ *and* $r_1 = r_2 = q_{ij} = 0.1$, *and* $\sigma = 0.4$ *equiped with the initial conditions*

$$V_1(s,0) = V_2(s,0) = 0,$$

and the boundary conditions

$$V_1(0,t) = V_1(2,t) = V_2(0,t) = 0, \quad V_2(2,t) = \frac{52}{5}t^2, \quad 0 \le t \le 0.25,$$

where

$$b(s,t) = s\left[20(2-s) + 8t(1+8s) - \frac{39}{250}st^2\right].$$

The exact solution of this problem is:

$$V_1(s,t) = t^2 s(2-s), \quad V_2(s,t) = st\left[20(2-s) + \frac{2}{5}t(5+8s)\right].$$

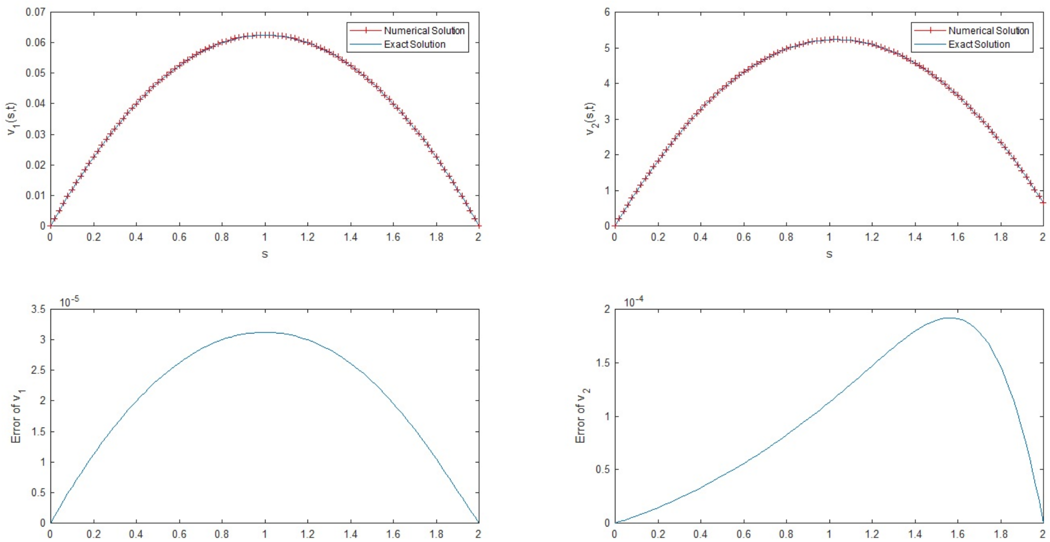

FIGURE 2. The exact and numerical solutions for both V_1 and V_2 functions illustrated in the final time step together with the maximum errors.

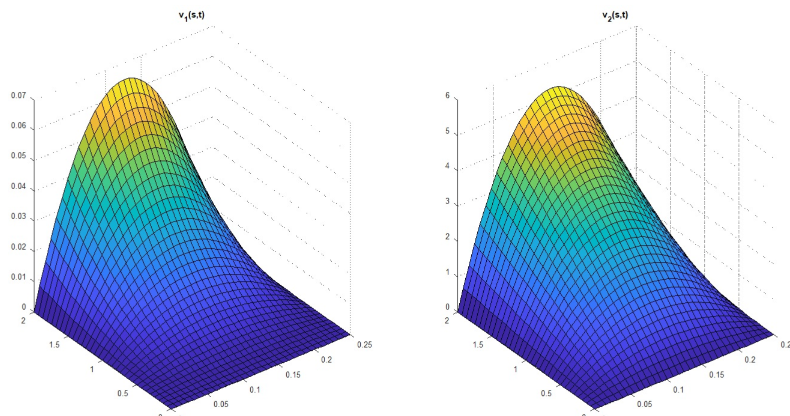

FIGURE 3. The exact and numerical solutions for both V_1 and V_2 functions illustrated in all the time steps. This figure illustrates that numerical and exact solutions of our option pricing coincide approximately. Notice: The axes show two parameters; the risky asset price and time parameter.

From Figures. 2 and 3, it can be seen that the presented method is very efficient and accurate in solving this problem because the graphics of the approximate solutions coincide with the graphics of the exact solutions and the maximum errors are around 10^{-5} and 10^{-4} correspond to the V_1 and V_2, respectively. Moreover, in Table 1 the numerical and exact solution for some spatial points at the final time step together with their errors are presented. As we expected, the presented numerical solutions are also very satisfactory.

Example 5.2. *Consider the system of equations (5.1), we define the following initial conditions*

$$V_i(s,0) = \max(s^\alpha - E^\alpha, 0), \text{ for } i = 1, 2, \quad 0 \le s \le s_{\max}$$

Table 1. The numerical and exact solution of the example 2 at the final time applying two different values of r_i considering $q_{ij} = r_i$, $E = 1$, $s_{\max} = 2E$, $T = 0.5$ and the number of spatial and temporal steps are set to be 200 and 2000, respectively. The maximum error at every presented spatial point is computed.

		$r_1 = r_2 = 0.1$			$r_1 = r_2 = 0.6$		
x		numeric	exact	Error	numeric	exact	Error
0.3	V_1	0.1274	0.1274	$6.3725E{-}5$	0.1274	0.1274	$6.3554E{-}5$
	V_2	5.2834	5.2833	$9.3404E{-}5$	1.0055	1.0054	$7.6339E{-}5$
0.6	V_1	0.2099	0.2098	$1.0498E{-}4$	0.2099	0.2098	$1.0429E{-}4$
	V_2	8.8396	8.8394	$2.2369E{-}4$	1.7231	1.7230	$1.5541E{-}4$
0.9	V_1	0.2474	0.2473	$1.2372E{-}4$	0.2474	0.2473	$1.2155E{-}4$
	V_2	10.6687	10.6683	$3.8981E{-}4$	2.1529	2.1527	$2.3415E{-}4$
1.2	V_1	0.2399	0.2398	$1.1928E{-}4$	0.2399	0.2398	$1.1136E{-}4$
	V_2	5.2834	5.2833	$9.3404E{-}4$	2.2948	2.2945	$2.8609E{-}4$
1.5	V_1	0.1874	0.1873	$9.0044E{-}5$	0.1874	0.1873	$7.3795E{-}5$
	V_2	9.1452	9.1446	$6.4656E{-}4$	2.1487	2.1485	$2.6238E{-}4$
1.8	V_1	0.0899	0.0899	$3.9031E{-}5$	0.0899	0.0899	$2.5375E0{-}5$
	V_2	5.7924	5.7920	$4.1867E{-}4$	1.7147	1.7146	$1.3774E{-}4$

$s_{max} = 3E$ *where* E *is the exercise price and* $\alpha > 0$. *The boundary conditions are set to be as follows*

$$\begin{cases} V_i(0,t) = 0, \\ \lim_{s\to\infty} \frac{V_i(s,t)}{s^\alpha} = 1, \end{cases}$$

where $0 \le t \le T$. *The rest of parameters are set to be as follows:*

$$T = 1,\ \alpha = 2,\ 0 \le r_1, r_2 \le 1,\ E = 50,\ \textit{and } \sigma = 0.4.$$

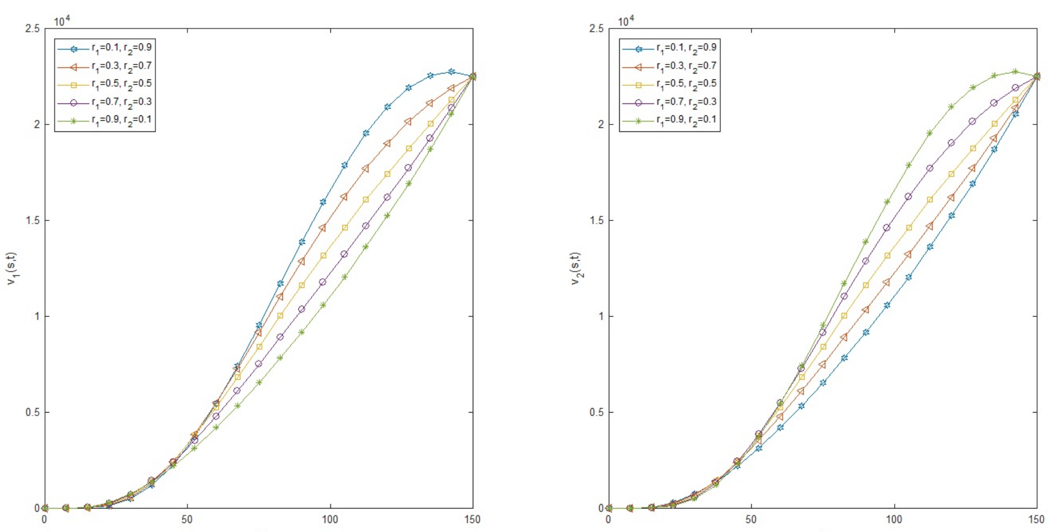

Figure 4. The numerical solutions for both V_1 and V_2 functions illustrated in the final time step with $q_{12} = r_1$ ranging from 0.1 to 0.9 and $q_{21} = r_2 = 1 - r_1$.

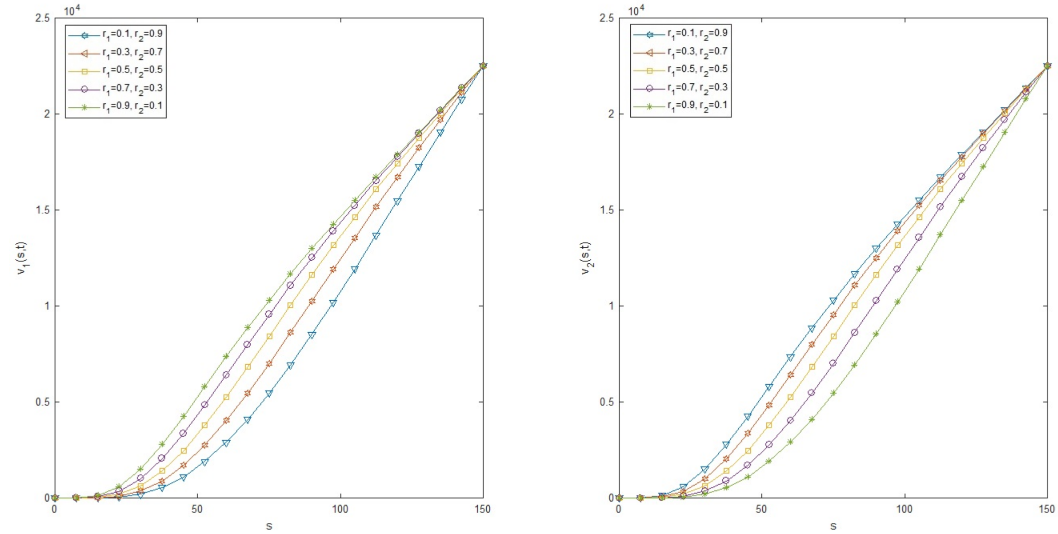

FIGURE 5. The numerical solutions for both V_1 and V_2 functions illustrated in the final time step where $q_{12} = q_{21} = 0.1$ and r_i is ranging from 0.1 to 0.9.

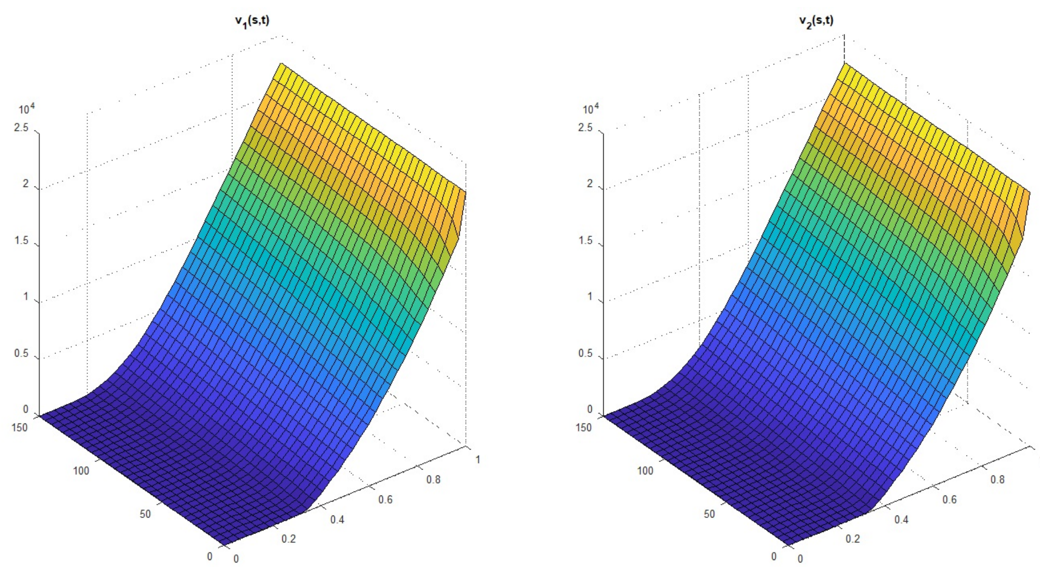

FIGURE 6. The numerical solutions for both V_1 and V_2 functions illustrated in all the time steps where $q_{ij} = r_i = 0.1$ for $j, i = 1, 2$.

In Figure. 6, we show the obtained results by using the finite difference method where the number of temporal and spatial steps are set to be 32 and $r_1 = r_2 = q_1 = q_2 = 0.1$*. In Figures 4 and 5,* V_1 *and* V_2 *functions at the final time step for different values of* r_i *and* q_{ij}*,* $i, j \in \{1, 2\}$ *are presented. Figure 4 is related to the case, where* r_1 *is varying from* 0.1 *to* 0.9 *and* $r_2 = 1 - r_1$*,* $q_{ij} = r_i$*. In the other case, we consider* $q_{12} = q_{21} = 1$ *fixed, and again* r_1 *is varying from* 0.1 *to* 0.9 *and* $r_2 = 1 - r_1$*. Its graphics is illustrated in the Figure 5. Moreover, in Table 2 the numerical solutions for some spatial points at the final time step, where the parameter* r_1 *and* r_2 *are varying from* 0.1 *to* 0.9*, are presented.*

Table 2. The numerical solution of the example 2 at the final time applying some different values of r_i considering $q_{ij} = r_i$, $E = 20$, $s_{\max} = 2E$, $T = 1$ and the number of spatial and temporal steps are set to be 200 and 2000, respectively.

x		$r_1 = 0.1$, $r_2 = 0.9$	$r_1 = 0.3$, $r_2 = 0.7$	$r_1 = 0.5$, $r_2 = 0.5$	$r_1 = 0.7$, $r_2 = 0.3$	$r_1 = 0.9$, $r_2 = 0.1$
5	V_1	0.7142	0.5075	0.6803	1.4021	2.9238
	V_2	2.9238	1.4021	0.6803	0.5075	0.7142
10	V_1	37.0718	39.6468	47.5287	59.0461	70.7608
	V_2	70.7608	59.0461	47.5287	39.6468	37.0718
15	V_1	202.3390	219.8212	234.1130	239.5424	234.2582
	V_2	234.2582	239.5424	234.1130	219.8212	202.3390
20	V_1	519.2549	533.0258	523.1868	491.9509	448.6284
	V_2	448.6284	491.9509	523.1868	533.0258	519.2549
25	V_1	915.0686	887.4569	832.0927	762.6361	692.5017
	V_2	692.5017	762.6361	832.0927	887.4569	915.0686
30	V_1	1283.2689	1204.5929	1117.2672	1034.8536	963.8028
	V_2	963.8028	1034.8536	1117.2672	1204.5929	1283.2689
35	V_1	1531.7565	1445.5279	1370.9592	1311.4411	1265.2823
	V_2	1265.2823	1311.4411	1370.9592	1445.5279	1531.7565

6. Conclusion

The motivation of the present study was to analysis on the exact and numerical solutions of the handling option pricing. An analytical method called invariant subspace method has been applied in order to find some exact solution of the system. The methodology of this method is based on a given set of fundamental solutions to construct some non-trivial solutions by solving a linear system of differential equations. In the second part of the paper a numerical simulation based on finite difference method is given. Approximated solutions and the corresponding errors are calculated and the advantages of this method are shown by some plotted graphs.

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