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A meshless technique based on the radial basis functions for solving systems of partial differential equations

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Abstract

The radial basis functions (RBFs) methods were first developed by Kansa to approximate partial differential equations (PDEs). The RBFs method is being truly meshfree becomes quite appealing, owing to the presence of distance function, straight-forward implementation, and ease of programming in higher dimensions. Another considerable advantage is the presence of a tunable free shape parameter, contained in most of the RBFs that control the accuracy of the RBFs method. Here, the solution of the two-dimensional system of nonlinear partial differential equations is examined numerically by a Global Radial Basis Functions Collocation Method (GRBFCM). It can work on a set of random or uniform nodes with no need for element connectivity of input data. For the time-dependent partial differential equations, a system of ordinary differential equations (ODEs) is derived from this scheme. Like some other numerical methods, a comparison between numerical results with analytical solutions is implemented confirming the efficiency, accuracy, and simple performance of the suggested method.

Keywords. Global meshless method, Radial basis functions, Method of lines, Partial differential equations.
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1. Introduction

Meshfree methods do not need mesh generation, contrary to those of traditional numerical schemes used for solving partial differential equations [8]. The radial basis functions collocation methods are simple in programming. They are useful for a wide range of partial differential equations, especially for high-dimensional problems. It can be said that usage of kernels as trial functions, collocation in the symmetric and unsymmetric form first appeared in [22, 23]. It has been proved that this process is highly successful since the linear systems are generated easily along with proper accuracy and less computational expense. Furthermore, recently in [33] the optimality of the symmetric collocation [14, 20] which applies the kernel basis was proved in comparison with all existing linear PDE solvers which implement the same input data. The usage of RBFs for achieving the numerical solution of PDEs has been popular in science and engineering because it's a meshfree scheme and it can also be applied for multi-dimensional problems without difficulty. Many researchers have broadly used RBFs in various contexts including [7, 10, 14, 29] in recent years, and considered RBFs as a potential substitute in the area of numerical solution of PDEs. Hardy proposed RBFs interpolation in [16] for approximating two-dimensional geographical surfaces based on scattered data.

Kansa [24] applied a meshfree scheme based on multi-quadrics RBFs for approximating the solution of PDEs and then Golberg et. al extended this idea in [15]. The existence, uniqueness, and convergence of the approximation by RBFs was discussed in [20, 25, 27]. Huang et al. have discussed the significance of the role of shape parameter c in the MQ method in [21].

The solvability of the system of equations derived from this scheme for distinct points of interpolation has been presented in [27]. Very recently, in [1–4, 7, 28, 31, 35], the Local and Global Radial Basis Collocation Methods have

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been used to achieve the meshfree solution of the nonlinear coupled PDEs. Kernel-based collocation methods for time-dependent PDEs are based on a fixed spatial interpolation. However, as coefficients are time-dependent, a system of ordinary differential equations (ODEs) is obtained. This scheme is known as the method of lines, and it suggests acceptable agreement in terms of accuracy in a myriad of problems [13].

It is aimed to approximate function $f : \mathbb{R}^d \to \mathbb{R}$ by implementing RBF where d is the dimension of the problem and $f(X)$ is data given at points $X \in R^d$ (named the centers) by the following interpolant

$$I_N f(X) = \sum_{j=1}^{N} \lambda_j \varphi(r_j), \tag{1.1}$$

where N is the number of data points, $r_j = \|X - X_j\|$, φ is a standard RBF, $r \geq 0$ and $(\lambda_1, \lambda_2, ..., \lambda_N)$ are the unknown coefficients which must be determined such that $I_N f(X_j) = f(X_j)$ for $j = 1, ..., N$. The standard RBFs fall into two separate categories;

Class 1. Infinitely smooth RBFs. The basis functions of this category are infinitely differentiable and they are considerably dependent on the shape parameter c e.g. Hardy multiquadric (MQ, $\phi(r) = \sqrt{(r^2 + c^2)}$), Gaussian (GA, $\phi(r) = exp(-c^2 r^2)$), inverse multiquadric (IMQ, $(\phi(r) = \sqrt{r^2 + c^2})^{-1}$) and inverse quadric (IQ, $\phi(r) = (r^2 + c^2)^{-1}$) [10, 13, 15, 16, 24].

Class 2. Infinitely smooth (except at centers) RBFs. These basis functions are not infinitely differentiable. The accuracy of the aforementioned basis functions, which are free of the shape parameter, is relatively smaller in comparison with the basis functions of Class 1. For example, thin plate spline $(r^{2n} \log(r), n = 1, 2, 3, \cdots)$, cubic r^3 and linear r, etc.

In this work, we examine the numerical solution of the two-dimensional system of nonlinear (PDEs) by a Global Radial Basis Functions Collocation Method (GRBFCM). Such systems appear in many fields of mathematics, physical sciences, and engineering [9]. Obtaining the exact solution of these types of equations are usually too difficult and even if an exact solution is derived, the relevant calculations might very complex. Lately, some strong techniques have been proposed, such as the Adomian decomposition method [12, 34], the variational iteration method [17], the homotopy perturbation method [18, 19], and the differential transform method [11]. The GRBFCM belongs to a class of truly meshless methods which eliminates the need of underlying mesh. It works on a set of uniform or random nodes with no need for element connectivity of input data.

The rest of the paper is organized as follows. In section 2 we describe the governing equations and discuss the numerical method. Numerical experiments are given in section 3. A brief conclusion is presented in the last section.

2. Mathematical formulation

In this section, the focus is on the numerical solution of the following two-dimensional system of nonlinear partial differential equations as the governing equations which was previously considered in [6, 9, 26] as

$$\begin{cases} V_t(X,t) - W(X,t)V_x(X,t) - W_t(X,t)V_y(X,t) = F(X,t), \\ \\ W_t(X,t) - V(X,t)W_x(X,t) - V_t(X,t)W_y(X,t) = G(X,t), \end{cases} \quad X \in \Omega \subset \mathbb{R}^2, \qquad t \in (0,T], \tag{2.1}$$

subject to the initial conditions

$$V(X,0) = V_0(X), \qquad X \in \overline{\Omega}, \tag{2.2}$$

$$W(X,0) = W_0(X), \qquad X \in \overline{\Omega}, \tag{2.3}$$

and Dirichlet boundary conditions

$$\left(V\left(X,t\right), W\left(X,t\right)\right) = \left(f^D\left(X,t\right), g^D\left(X,t\right)\right), \qquad X \in \Gamma_D \subseteq \Gamma,\ t \in [0,T], \tag{2.4}$$

where $X = (x,y)$ and F, G, V_0, W_0, f^D and g^D are known functions, $\Omega \subset \mathbb{R}^2$ is the domain, Γ is the boundary of the domain Ω, V and W are unknown functions which must be determined. Nonlinear systems of partial differential equations with variable coefficients are often too complicated to be solved exactly and even if an exact solution is obtained, the required calculations may be too complicated. In the literature, many powerful methods have been proposed for (2.1). Arbabi et al. [6], extended the Haar wavelets method to obtain semi-analytical solutions for the nonlinear system of partial differential equations (2.1). Biazar and Eslami [9] proposed a new homotopy perturbation method (NHPM) for solving system (2.1). Matinfar et al. [26] introduced a new homotopy analysis method for obtaining solutions of systems of partial differential equations (2.1).

In the following, we choose discretization points X_i, $1 \leq i \leq n$, and then rearrange the points successively into points $\{X_I \cup X_D\}$ where $X_I = \{X_1, \dots, X_{ni}\}$ is introduced as the set of interior points, $X_D = \{X_{ni+1}, \dots, X_n\}$ is considered as the set of boundary points, and the notation ni is the number of interior points. Therefore, the unknown functions $V(X,t)$ and $W(X,t)$ can be approximated as

$$V(X,t) = \sum_{j=1}^{n} \alpha_j(t)\ \phi_j(X), \qquad X \in \overline{\Omega}, \tag{2.5}$$

$$W(X,t) = \sum_{j=1}^{n} \beta_j(t)\ \phi_j(X), \qquad X \in \overline{\Omega}, \tag{2.6}$$

where $\phi_j(X) = \phi(||X - X_j||)$, ϕ can be one of the RBFs listed in Table 1. The $c \in \mathbb{R}$ in Table 1 is the shape parameter. Then we have

TABLE 1. Global RBFs.

Name	$\phi(r)$	condition
Gaussian	$\exp(-c^2r^2)$	
Multiquadric	$(c^2+r^2)^{\beta/2}$	$\beta \in \mathbb{R}_{\neq 0} \setminus 2\mathbb{N}$
Matérn/Sobolev	$r^{\nu}K_{\nu}(r)$	$\nu > 0$
Powers	r^{β}	$0 < \beta \notin 2\mathbb{N}$
Thin-plate splines	$r^{2n}\ln(r)$	$n \in \mathbb{N}$

$$V = \Phi\alpha(t),$$
$$W = \Phi\beta(t),$$

which in turn gives

$$\alpha(t) = \Phi^{-1}V, \tag{2.7}$$
$$\beta(t) = \Phi^{-1}W, \tag{2.8}$$

where

$$V = \left(V\left(X_1,t\right), V\left(X_2,t\right), \dots, V\left(X_n,t\right)\right)^T,$$
$$W = \left(W\left(X_1,t\right), W\left(X_2,t\right), \dots, W\left(X_n,t\right)\right)^T,$$
$$\alpha = \left(\alpha_1\left(t\right), \alpha_2\left(t\right), \dots, \alpha_n\left(t\right)\right)^T,$$
$$\beta = \left(\beta_1\left(t\right), \beta_2\left(t\right), \dots, \beta_n\left(t\right)\right)^T,$$
$$\Phi = \left(\phi_j\left(X_i\right)\right)_{1 \leq i,j \leq n}.$$

Consequently, it is possible to estimate the partial derivatives of the approximate solution and from interpolants (2.5) and (2.6) we have

$$\frac{\partial}{\partial \xi}(V(X,t)) = \sum_{j=1}^{n} \alpha_j(t)\ \frac{\partial}{\partial \xi}(\phi_j(X)),$$
$$\frac{\partial}{\partial \xi}(W(X,t)) = \sum_{j=1}^{n} \beta_j(t)\ \frac{\partial}{\partial \xi}(\phi_j(X)),$$

where $X \in \overline{\Omega}$ and $\xi = x, y$. From the aforementioned relations we obtain

$$\begin{aligned} \frac{\partial}{\partial \xi}(V(X_i,t)) &= \frac{\partial \Phi_i}{\partial \xi}\Phi^{-1}V, \\ \frac{\partial}{\partial \xi}(W(X_i,t)) &= \frac{\partial \Phi_i}{\partial \xi}\Phi^{-1}W, \end{aligned} \tag{2.9}$$

where

$$\frac{\partial \Phi_i}{\partial \xi} = \left(\frac{\partial}{\partial \xi}\phi_1(X_i), \frac{\partial}{\partial \xi}\phi_2(X_i), \ldots, \frac{\partial}{\partial \xi}\phi_n(X_i) \right).$$

At this point, it is possible to write the PDE (2.1) at the interior points $X_i, i = 1, \ldots, ni$, as follows:

$$V_t(X_i,t) - W(X_i,t)\ \left(\sum_{j=1}^{n} \alpha_j(t)\ \frac{\partial}{\partial x}\phi_j(X_i)\right) - W_t(X_i,t)\ \left(\sum_{j=1}^{n} \alpha_j(t)\ \frac{\partial}{\partial y}\phi_j(X_i)\right) = F(X_i,t),$$
$$W_t(X_i,t) - V(X_i,t)\ \left(\sum_{j=1}^{n} \beta_j(t)\ \frac{\partial}{\partial x}\phi_j(X_i)\right) - V_t(X_i,t)\ \left(\sum_{j=1}^{n} \beta_j(t)\ \frac{\partial}{\partial y}\phi_j(X_i)\right) = G(X_i,t).$$

By taking (2.9), for $i = 1, \ldots, ni$, we have

$$\begin{aligned} &V_t(X_i,t) - W(X_i,t)\ ((\Phi_i)_x * \Phi^{-1} * V) - W_t(X_i,t)\ ((\Phi_i)_y * \Phi^{-1} * V) = F(X_i,t), \\ &W_t(X_i,t) - V(X_i,t)\ ((\Phi_i)_x * \Phi^{-1} * W) - V_t(X_i,t)\ ((\Phi_i)_y * \Phi^{-1} * W) = G(X_i,t). \end{aligned} \tag{2.10}$$

Matrix form of Eq. (2.10) is suggested in the following:

$$\begin{aligned} &\tilde{V}_t - \tilde{W}_t.*(\Phi_y\Phi^{-1}V) = \tilde{W}.*(\Phi_x\Phi^{-1}V) + F, \\ &\tilde{W}_t - \tilde{V}_t.*(\Phi_y\Phi^{-1}W) = \tilde{V}.*(\Phi_x\Phi^{-1}W) + G, \end{aligned} \tag{2.11}$$

where

$$V = (V(X_1,t), \ldots, V(X_n,t))^T,$$
$$W = (W(X_1,t), \ldots, W(X_n,t))^T,$$
$$\tilde{V} = (V(X_1,t), \ldots, V(X_{ni},t))^T,$$
$$\tilde{W} = (W(X_1,t), \ldots, W(X_{ni},t))^T,$$
$$\Phi_x = ((\Phi_1)_x, \ldots, (\Phi_{ni})_x)^T,$$
$$\Phi_y = \left((\Phi_1)_y, \ldots, (\Phi_{ni})_y\right)^T,$$
$$F = (F(X_1,t), \ldots, F(X_{ni},t))^T,$$
$$G = (G(X_1,t), \ldots, G(X_{ni},t))^T,$$

and $.*$ is a symbol referring to the pointwise product between two matrices or vectors.
The Dirichlet boundary condition which was appeared in (2.4), results in:

$$\begin{array}{ll} V(X_i,t) = f^D(X_i,t), & i = ni+1,\dots n, \\ W(X_i,t) = g^D(X_i,t), & i = ni+1,\dots n. \end{array} \tag{2.12}$$

By substituting (2.12) in (2.11) The system of ODEs is obtained with the following initial conditions:

$\tilde{V}(0) = (V_0(X_i), i = 1,\dots,ni)$,

$\tilde{W}(0) = (W_0(X_i), i = 1,\dots,ni)$.

It should be noted that the system of ODEs (2.11) is nonlinear, and an appropriate ODE solver in mathematical softwares will utilize a proper time-stepping automatically and recognize stiffness of the ODE system.
The following algorithm is considered as a summary for the above-mentioned section.

Algorithm

1: Select n discretization points in the domain $\overline{\Omega}$.
2: Rearrange discretization points into two sets of interior points and boundary points.
3: Form matrix Φ.
4: Form matrix Φ_x.
5: Form matrix Φ_y.
6: Put vector $F = (F(X_1,t),\dots,F(X_{ni},t))^T$.
7: Put vector $G = (G(X_1,t),\dots,G(X_{ni},t))^T$.
8: Put initial vectors $\tilde{V}(0) = (V_0(X_i))_{1\le i\le ni}$ and $\tilde{W}(0) = (W_0(X_i))_{1\le i\le ni}$.
9: Solve the system of ODEs
$\tilde{V}_t - \tilde{W}_t.*(\Phi_y * \Phi^{-1} * V) = \tilde{W}.*(\Phi_x * \Phi^{-1} * V) + F$,
$\tilde{W}_t - \tilde{V}_t.*(\Phi_y * \Phi^{-1} * W) = \tilde{V}.*(\Phi_x * \Phi^{-1} * W) + G$,
where $V = \left(\tilde{V}, \left(f^D(X_i,t)\right)_{ni+1\le i\le n}\right)^T$ and $W = \left(\tilde{W}, \left(g^D(X_i,t)\right)_{ni+1\le i\le n}\right)^T$.

3. Numerical results

In this section, the numerical solution of the equations (2.1) with the right-hand side functions

$F(X,t) = 1 - x + y + t$,

$G(X,t) = 1 - x - y - t$,

initial conditions

$V(X,0) = x + y - 1$,

$W(X,0) = x - y + 1$,

and Dirichlet boundary conditions

$V(0,y,t) = y + t - 1$,

$W(0,y,t) = -y - t + 1$,

$V(x,0,t) = x + t - 1$,

$W(x,0,t) = x - t + 1$.

are presented. The exact solution of this problem is

$V(X,t) = x + y + t - 1$,

$W(X,t) = x - y - t + 1$.

We consider $0 \le x \le 1$, $0 \le y \le 1$ and $t \ge 0$. Three types of discretization points $\{(x_i,x_j)\}_{1\le i,j\le m}$ have been applied in the implementation of the method, which are listed as follows.

- Uniform points:

$$x_i = \frac{(i-1)}{m-1}, \qquad i = 1, \ldots, m.$$

- Legendre-Gauss-Lobatto points:

  Let L_{m-1} be the Legendre polynomial of degree $m-1$. The x_i, $i = 1, \ldots, m$ are the zeros of $(1-x^2)L'_{m-1}(x)$. Then x_i, $i = 1, \ldots, m$ belong to the interval $[-1, 1]$ and can be easily transferred to the interval $[0, 1]$ by the transformation $y = \frac{1}{2}(x+1)$.
- Chebyshev-Gauss-Lobatto points:

$$x_i = -\cos\left(\frac{\pi(i-1)}{m-1}\right), \qquad i = 1, \ldots, m,$$

  which is in the interval $[-1, 1]$ and can be effortlessly transferred to the interval $[0, 1]$ by the transformation $y = \frac{1}{2}(1+x)$.

Here, the Powers ($\beta = 5$), Matérn ($\nu = 1$), Multiquadric ($\beta = 1$) and Thin-plate splines ($n = 3$) RBFs are applied. In order to have a well-conditioned system matrix, the shape parameter c should not be very large. Although in order to get proper accuracy by the RBF method large shape parameters are needed, this leads to the ill-conditioned system matrix. Thus, it is obvious that having the best accuracy and conditioning can not occur simultaneously. This fact is considered as the Uncertainty Principle, which shows the more desired value of one quantity is the less desired value of the other is [32].

The shape parameter c can be selected so that the collocation matrix Φ has a condition number, $\kappa(\Phi)$, in the range $10^{13} \leq \kappa(\Phi) \leq 10^{15}$ [31]. These limits for the condition number are reliable until computers that perform double-precision floating-point arithmetic are utilized, but they will differ when applying other floating-point number systems [30]. The selection of the shape parameter at each center is explained in the following pseudocode [31]:

```
kappa = 0
while kappa < kappaMin and kappa > kappaMax do
  form Φ
  [V, S, W] = svd(Φ)
  kappa = max(S)/min(S)
  if kappa < kappaMin then
    shape = shape − shapeIncrement
  else if kappa>kappaMax then
    shape = shape + shapeIncrement
  end if
end while
```

The ODE solver ode23t of MATLAB is applied to the last ODE system (2.11). The maximum absolute and root mean squared error norms are considered as follows.

$$L_2 = \sqrt{\frac{1}{n}\sum_{j=1}^{n} | V_j - \bar{V}_j |^2},$$

$$L_\infty = \max_{1 \leq j \leq n} | V_j - \bar{V}_j |,$$

where V and $\bar{V}$ are the exact and approximate solutions, respectively. The L_2 error norms using the proposed method with various numbers and kinds of points including Chebyshev, Legendre, and uniform points and with different RBFs including Powers ($\beta = 5$), Matérn ($\nu = 1$), Multiquadric ($\beta = 1$) and Thin-plate splines ($n = 3$) at time $T = 0.01$ are reported in Tables 2-5. Tables 2-5 show that by the present method, a bit more accurate solutions are obtained for Chebyshev points. Also for all the tested RBFs, the Powers RBF yield a little more accurate results. In Tables 6 and 7, implementation of the method with $n = 81$ uniform points and different RBFs are shown against some other methods [6, 9, 26] at selected points. Tables 6 and 7 show that the results of the presented method are quite accurate and stable. Distributions of absolute error with their contours at $T = 0.01$ for $n = 100$ uniform points with various RBFs including Powers ($\beta = 5$), Matérn ($\nu = 1$), Multiquadric ($\beta = 1$) and Thin-plate splines ($n = 3$) are shown in Figures 1-4.

TABLE 2. The L_2 error norms of V and W at time $T = 0.01$ with various numbers and kinds of points and with Powers RBF.

	V			W		
n	Uniform points	Legendre points	Chebyshev points	Uniform points	Legendre points	Chebyshev points
36	$3.49E-04$	$1.41E-04$	$1.07E-04$	$3.58E-04$	$1.39E-04$	$1.45E-04$
64	$1.02E-04$	$2.90E-05$	$2.14E-05$	$1.02E-04$	$2.83E-05$	$2.08E-05$
100	$4.48E-05$	$9.97E-06$	$6.79E-06$	$4.45E-05$	$9.87E-06$	$6.75E-06$
144	$2.37E-05$	$8.91E-06$	$2.75E-06$	$2.35E-05$	$8.84E-06$	$2.84E-06$

TABLE 3. The L_2 error norms of V and W at time $T = 0.01$ with various numbers and kindss of points and with Matérn RBF.

	V			W		
n	Uniform points	Legendre points	Chebyshev points	Uniform points	Legendre points	Chebyshev points
36	$2.33E-04$	$1.70E-04$	$1.51E-04$	$8.93E-05$	$7.23E-05$	$6.64E-05$
64	$1.55E-04$	$9.43E-05$	$8.13E-05$	$6.28E-05$	$4.44E-05$	$3.99E-05$
100	$1.12E-04$	$5.96E-05$	$5.07E-05$	$4.75E-05$	$3.06E-05$	$2.76E-05$
144	$8.68E-05$	$4.11E-05$	$3.48E-05$	$3.78E-05$	$2.30E-05$	$2.10E-05$

TABLE 4. The L_2 error norms of V and W at time $T = 0.01$ with various numbers and kinds of points and with Multiquadric RBF.

	V			W		
n	Uniform points	Legendre points	Chebyshev points	Uniform points	Legendre points	Chebyshev points
36	$1.15E-03$	$8.48E-04$	$7.42E-04$	$8.58E-04$	$6.30E-04$	$5.55E-04$
64	$7.70E-04$	$4.13E-04$	$3.34E-04$	$5.48E-04$	$3.10E-04$	$2.58E-04$
100	$5.36E-04$	$2.07E-04$	$1.64E-04$	$3.76E-04$	$1.63E-04$	$1.32E-04$
144	$3.84E-04$	$1.08E-04$	$8.72E-05$	$2.69E-04$	$8.81E-05$	$7.04E-05$

TABLE 5. The L_2 error norms of V and W at time $T = 0.01$ with various numbers and kinds of points and with Thin-plate splines RBF.

	V			W		
n	Uniform points	Legendre points	Chebyshev points	Uniform points	Legendre points	Chebyshev points
36	$1.09E-04$	$1.19E-04$	$8.88E-05$	$1.04E-04$	$1.15E-04$	$8.36E-05$
64	$9.62E-05$	$7.01E-05$	$6.36E-05$	$9.24E-05$	$6.77E-05$	$6.11E-05$
100	$7.60E-05$	$4.42E-05$	$3.78E-05$	$7.37E-05$	$4.33E-05$	$3.71E-05$
144	$6.19E-05$	$3.06E-05$	$2.56E-05$	$6.04E-05$	$3.03E-05$	$2.56E-05$

TABLE 6. Results for V at time $T = 0.01$, with $n = 81$ uniform points.

(x, y)	$(0.125, 0.125)$	$(0.125, 0.625)$	$(0.375, 0.375)$	$(0.375, 0.875)$	$(0.625, 0.125)$	$(0.875, 0.875)$
Exact	-0.74	-0.24	-0.24	0.26	-0.24	0.76
Matérn RBF	-0.740054	-0.239999	-0.240001	0.259932	-0.240069	0.759989
Power RBF	-0.740014	-0.240002	-0.240001	0.259996	-0.240008	0.759992
Multiquadric RBF	-0.740178	-0.240072	-0.240012	0.259603	-0.240183	0.759673
Thin-plate splines RBF	-0.740007	-0.239985	-0.240004	0.260042	-0.240082	0.759977
Ref. [6]	-0.739849	-0.239854	-0.239528	0.260446	-0.239391	0.760789
Ref. [9]	-0.730000	-0.230000	-0.230000	0.269999	-0.230000	0.769999
Ref. [26]	-0.730000	-0.230000	-0.230000	0.269999	-0.230000	0.769999

TABLE 7. Results for W at $T = 0.01$, with $n = 81$ uniform points.

(x, y)	$(0.125, 0.125)$	$(0.125, 0.625)$	$(0.375, 0.375)$	$(0.375, 0.875)$	$(0.625, 0.125)$	$(0.875, 0.875)$
Exact	-0.749	-0.249	-0.249	0.251	-0.249	0.751
Matérn RBF	-0.749005	-0.248999	-0.249000	0.250993	-0.249006	0.750999
Power RBF	-0.749001	-0.249000	-0.249000	0.250999	-0.249000	0.750999
Multiquadric RBF	-0.749016	-0.249007	-0.249001	0.250996	-0.249017	0.750969
Thin-plate splines RBF	-0.749007	-0.248998	-0.249000	0.251004	-0.249008	0.750997
Ref. [6]	-0.748998	-0.248998	-0.248995	0.251003	-0.248993	0.751007
Ref. [9]	-0.748000	-0.248000	-0.248000	0.251999	-0.248000	0.751999
Ref. [26]	-0.748000	-0.248000	-0.248000	0.251999	-0.248000	0.751999

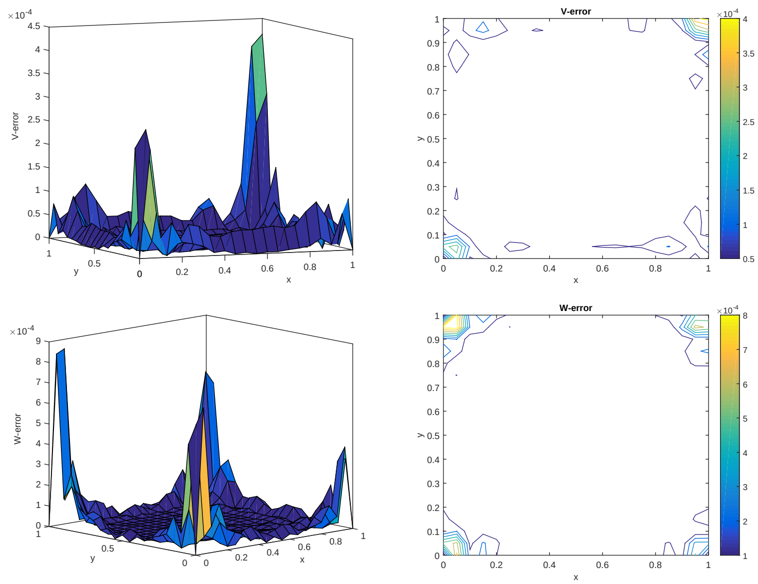

FIGURE 1. Graphs of absolute error with their contours at $T = 0.01$ by Powers RBF and $n = 100$ uniform points.

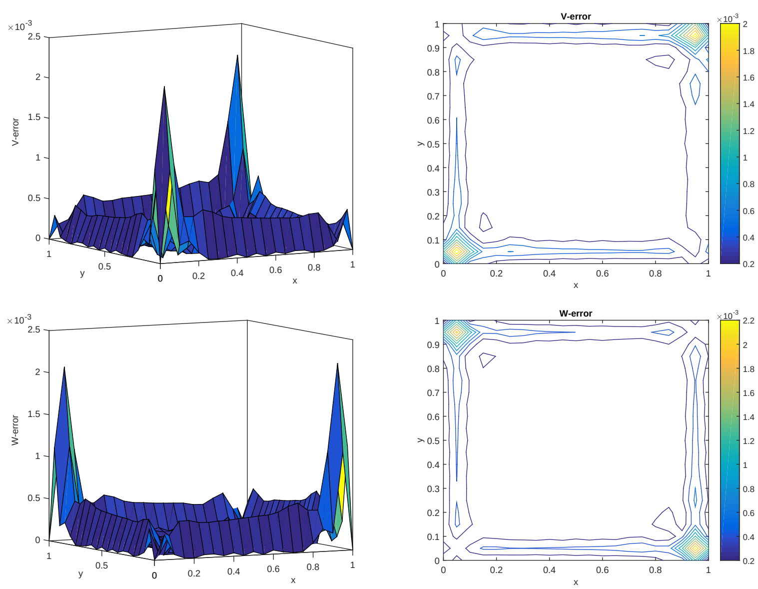

FIGURE 2. Graphs of absolute error with their contours at $T = 0.01$ by Matérn RBF and $n = 100$ uniform points.

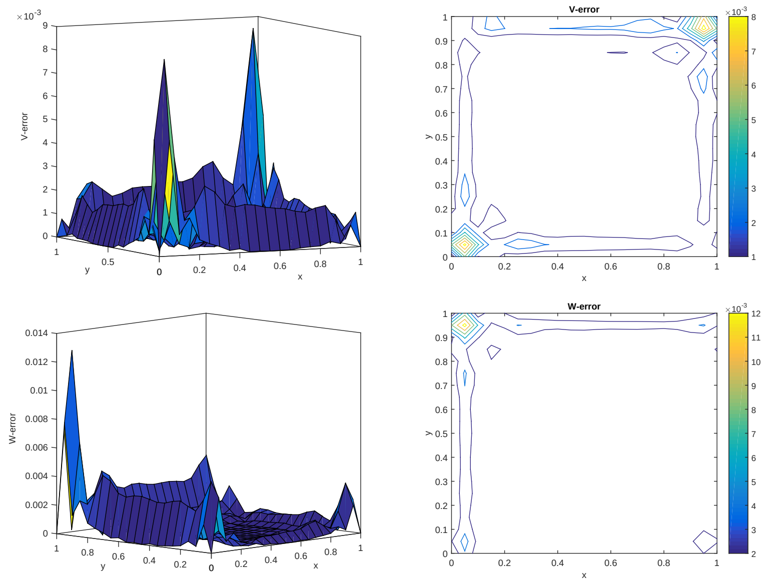

FIGURE 3. Graphs of absolute error and their contours at time $T = 0.01$ by Multiquadric RBF and $n = 100$ uniform points.

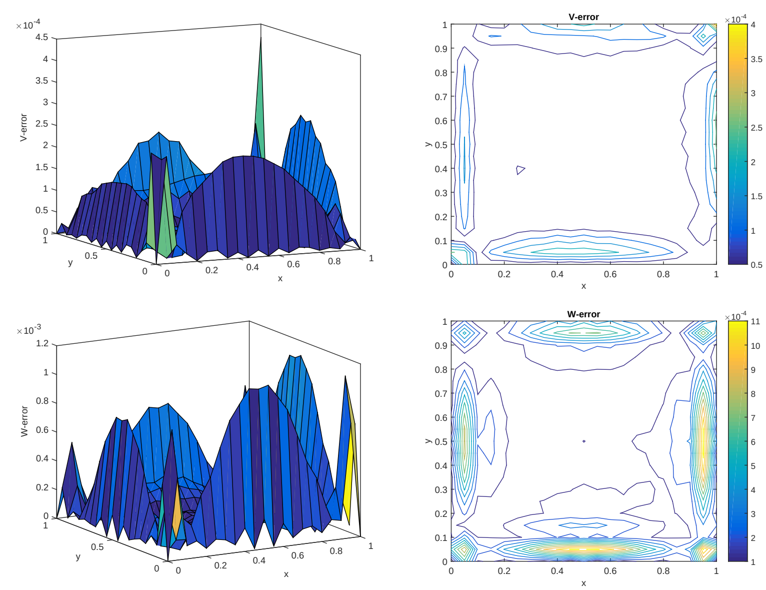

FIGURE 4. Graphs of absolute error and their contours at time $T = 0.01$ by Thin-plate splines RBF and $n = 100$ uniform points.

4. Conclusion

RBFs play an important role in solving PDEs as a meshfree scheme. The most important advantage of the RBFs method is the meshless property compared with the traditional mesh dependent techniques like finite element and finite difference schemes.

In this work, we extended the concept of efficient Global Radial Basis Functions Collocation Method which was presented in [6, 9, 26] for solving the two-dimensional system of nonlinear partial differential equations numerically which is applied to a set of uniform or random nodes with no need for element connectivity of input data. Obtained numerical results by the proposed method offer a very high accuracy in the computations. Also, using the method the cost of computations and discretization of variables reduced strongly.

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