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New analytical methods for solving a class of conformable fractional differential equations by fractional Laplace transform

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Abstract

In this paper, new analytical solutions for a class of conformable fractional differential equations (CFDEs) and some more results about Laplace transform introduced by Abdeljawad are investigated. The Laplace transform method is developed to get the exact solution of CFDEs. The aim of this paper is to convert the CFDEs into ordinary differential equations (ODEs), this is done by using the fractional Laplace transform of $(\alpha + \beta)$ order.

Keywords. Conformable fractional differential equations, Fractional Laplace transform, Exact analytical solutions.

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1. Introduction

Fractional differential equations sometimes called as extraordinary differential equations because of their nature and easily find in various fields of applied sciences [11, 18]. For example, fractional-order differential equations have been established for modeling of real phenomena in various fields such as physics, engineering, mechanics, control theory, economics, medical science, finance and etc, [1, 3, 5–8, 11, 14, 18, 20]. Modeling of spring pendulum in fractional sense and it's numerical solution proposed in [5]. Study of the motion of a capacitor microphone in fractional calculus proposed in [6]. So the scientific and engineering problems which involve fractional calculus are very large and still very effective. In recent years, scientists have proposed many efficient and powerful methods to obtain exact or numerical solutions of fractional differential equations [8, 14]. In addition, many researchers have been trying to form a new definition of fractional derivative. Most of these definitions include integral form for fractional derivatives. There are many types of differential derivatives in fractional calculus e.g. Grunwald-Letnikov, Riemann-Liouville, Caputo [20], Caputo-Fabrizio [7], Atangana-Baleanu [3] and more recently one, the conformable fractional derivative (CFD) [14]. The chain rule which is an applicable and useful rule in the calculus, is hold only for conformable fractional derivatives. Recently, some authors introduced the concept of non-local derivative. In [14], Khalil presented a new definition of derivative prominently compatible with the classical derivative, this operator is called "conformable derivative". This derivative satisfied some conventional properties, for instance, the chain rule. This operator can be used to solve conformable differential equations. The Conformable fractional derivative has some advantages in properties. Thus now it is widely used in many research fields. However, Ortigueira [19], figured out that the CFD, is not a real fractional definition. Since,

1) The zero order derivative of a function does not return the function. In fact

$$(T_0^0 u)(t) = \lim_{\epsilon \to 0} \frac{u\,(t + \epsilon t) - u(t)}{\epsilon} = \lim_{\epsilon \to 0} \frac{u\,(t(1+\epsilon)) - u(t)}{1+\epsilon} . \lim_{\epsilon \to 0} \frac{1+\epsilon}{\epsilon}.$$

2) The index law does not hold, that is $T_\alpha(T_\beta u(t)) \neq T_{\alpha+\beta} u(t)$, for any α and β. See Theorem 4.2.

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3) In special case we have, $({}_tT^a_\alpha fg)(t) = g(t)({}_tT^a_\alpha f)(t) + f(t)({}_tT^a_\alpha g)(t)$, but the generalized Leibniz rule is not valid .

In [8], the author showed the exact solutions of time heat differential equations by using the conformable derivative. Atangana in [4], investigated some properties of this derivative, related theorems and new definitions were introduced. Interesting works related with operator are given by [9, 13].

The rest of this study is organized as follows. In section 2, we give some important theorems based on conformable fractional derivative. In section 3, we present some conformable fractional Laplace transform theorems. In section 4, new methods for solving a class of CFDEs by fractional Laplace transforms of $(\alpha+\beta)$ order are investigated.

2. Basic definitions and tools about CFD

The CFD with a limit operator which was first introduced Khalil et al. [14]. After then Abdeljawad [1], has also presented fractional versions of the chain rule, exponential functions, Gronwalls inequality, Taylor power series expansions and Laplace transform for the CFD. Khalil et al. [14], introduced a new kind of fractional derivatives as follows:

Definition 2.1. *The left conformable fractional derivative of order $0 < \alpha \leq 1$ starting from $a \in \mathbb{R}$ of function $u : [a, +\infty) \to \mathbb{R}$, is defined by*

$$(T^a_\alpha u)(t) = \lim_{\epsilon \to 0} \frac{u\left(t + \epsilon(t-a)^{1-\alpha}\right) - u(t)}{\epsilon}, \tag{2.1}$$

When $a = 0$, we have:

$$(T^0_\alpha u)(t) = \lim_{\epsilon \to 0} \frac{u\left(t + \epsilon t^{1-\alpha}\right) - u(t)}{\epsilon}.$$

If $(T^a_\alpha u)(t)$ exists on $(a, +\infty)$, then $(T^a_\alpha u)(a) = \lim_{t \to a^+}(T^a_\alpha u)(t)$. If $(T^a_\alpha u)(t_0)$ exists and is finite, then we say that u is left $\alpha-$differentiable at t_0.

The right conformable fractional derivative of order $0 < \alpha \leq 1$ terminating at $b \in \mathbb{R}$ of function $u : (-\infty, b] \to \mathbb{R}$, is defined by

$$({}^bT_\alpha u)(t) = -\lim_{\epsilon \to 0} \frac{u\left(t + \epsilon(b-t)^{1-\alpha}\right) - u(t)}{\epsilon}, \tag{2.2}$$

T_α. If $({}^bT_\alpha u)(t)$ exists on $(-\infty, b)$, then $({}^bT_\alpha u)(a) = \lim_{t \to b^-}({}^bT_\alpha u)(t)$. If $({}^bT_\alpha u)(t_0)$ exists and is finite, then we say that u is right $\alpha-$differentiable at t_0. See [16].

Definition 2.2. *Let $\alpha \in (n, n+1]$, and u be an n-differentiable at t, where $t > 0$, then the conformable derivative of u of order α is defined by*

$$({}_tT^0_\alpha u)(t) = \lim_{\epsilon \to 0} \frac{u^{([\alpha]-1)}(t + \epsilon t^{([\alpha]-\alpha)}) - u^{([\alpha]-1)}(t)}{\epsilon}, \tag{2.3}$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Remark 2.3. *Let $\alpha \in (n, n+1]$, and u is (n+1)-differentiable at $t > 0$. Then*

$$({}_tT^0_\alpha u)(t) = t^{([\alpha]-\alpha)} u^{([\alpha])}(t), \tag{2.4}$$

Theorem 2.4. *Let $0 < \alpha \leq 1$, and f, g be left(right) $\alpha-$differentiable functions. Then,*

1) $\forall c_1, c_2 \in \mathbb{R}$, $\left(T^a_\alpha(c_1 f + c_2 g)\right)(t) = c_1\left(T^a_\alpha f\right)(t) + c_2\left(T^a_\alpha g\right)(t)$,
2) $\forall c_1, c_2 \in \mathbb{R}$, $\left({}^bT_\alpha(c_1 f + c_2 g)\right)(t) = c_1\left({}^bT_\alpha f\right)(t) + c_2\left({}^bT_\alpha g\right)(t)$,
3) $\forall \lambda \in \mathbb{R}$, $T^a_\alpha\left((t-a)^\lambda\right) = \lambda(t-a)^{\lambda-\alpha}$, ${}^bT_\alpha\left((b-t)^\lambda\right) = \lambda(b-t)^{\lambda-\alpha}$,
4) $T^a_\alpha(C) = 0$, ${}^bT_\alpha(C) = 0$, *where C is a constant.*
5) *(Product rule for left and right CF derivative)*
$\left(T^a_\alpha(fg)\right)(t) = f(t)\left(T^a_\alpha g\right)(t) + g(t)\left(T^a_\alpha f\right)(t)$,
$\left({}^bT_\alpha(fg)\right)(t) = f(t)\left({}^bT_\alpha g\right)(t) + g(t)\left({}^bT_\alpha f\right)(t)$,

6) *(Quotient rule for left and right CF derivative)*
$\left(T_\alpha^a(\frac{f}{g})\right)(t) = \frac{g(t)(T_\alpha^a f)(t) - f(t)(T_\alpha^a g)(t)}{g(t)^2}, \quad g(t) \neq 0,$
$\left({}^bT_\alpha(\frac{f}{g})\right)(t) = \frac{g(t)\left({}^bT_\alpha f\right)(t) - f(t)\left({}^bT_\alpha g\right)(t)}{g(t)^2}, \quad g(t) \neq 0,$
7) $(T_\alpha^a f)(t) = (t-a)^{1-\alpha} f'(t), \quad \left({}^bT_\alpha f\right)(t) = -(b-t)^{1-\alpha} f'(t),$
where $f'(t) = \lim_{\epsilon \to 0}\left(\frac{f(t+\epsilon)-f(t)}{\epsilon}\right)$.

Proof. See [16]. □

Theorem 2.5. *If a function* $u : [0, +\infty) \to \mathbb{R}$*, is* $\alpha-$*differentiable at* x_0*,* $\alpha \in (0, 1]$*. Then,* u *is continuous at* x_0*.*

The chain rule is valid for conformable fractional derivatives.

Theorem 2.6. *(CF-chain rull). Let* $\alpha \in (0, 1]$ *and* $f, g : [0, \infty) \to \mathbb{R}$ *be* $\alpha-$*differentiable functions and* $h(t) = (fog)(t)$*. Then,* $h(t)$ *is left* $\alpha-$*differentiable and for all* t *(*$t \neq a$ *and* $g(t) \neq a$*), we have*

$$1)\ If\ g(t) \geq a,\ then \quad (T_\alpha^a h)(t) = (T_\alpha^a f)(g(t))(T_\alpha^a g)(t)\Big(g(t) - a\Big)^{\alpha-1}.$$

$$2)\ If\ g(t) < a,\ then \quad (T_\alpha^a h)(t) = -({}^aT_\alpha f)(g(t))(T_\alpha^a g)(t)\Big(a - g(t)\Big)^{\alpha-1}.$$

Analogously, let $f, g : (-\infty, b] \to \mathbb{R}$ *be* $\alpha-$*differentiable functions and h(t)=(fog)(t). Then, h(t) is right* $\alpha-$*differentiable and for all* t *(*$t \neq b$ *and* $g(t) \neq b$*), we have*

$$1)\ If\ g(t) \leq b,\ then \quad ({}^bT_\alpha h)(t) = ({}^bT_\alpha f)(g(t))({}^bT_\alpha g)(t)\Big(b - g(t)\Big)^{\alpha-1}.$$

$$2)\ If\ g(t) > b,\ then \quad ({}^bT_\alpha h)(t) = -(T_\alpha^b f)(g(t))({}^bT_\alpha g)(t)\Big(g(t) - b\Big)^{\alpha-1}.$$

If $\Big(T_\alpha^a h\Big)(t)$*,* $\Big({}^bT_\alpha h\Big)(t)$ *exist on* $(a, +\infty)$ *and* $(-\infty, b)$*, respectively, then*

$$\Big(T_\alpha^a h\Big)(a) = \lim_{t \to a^+}\Big(T_\alpha^a h\Big)(t), \quad \Big({}^bT_\alpha h\Big)(a) = \lim_{t \to b^-}\Big({}^bT_\alpha h\Big)(t).$$

Proof. See [16]. □

3. The conformable fractional Laplace transform

The conformable fractional Laplace transform introduced by Abdeljawad [1] help us to solve some of the CFDEs. In this section we will investigate basic definitions and some useful Theorems about conformable fractional Laplace transform. Over the following set of functions [2].

$$A = \left\{u(t) : \exists M, \tau_1, \tau_2 > 0, |u(t)| < Me^{\frac{|t|}{\tau_j}}, if\ t \in (-1)^j[0, \infty), j = 1, 2\right\}. \tag{3.1}$$

The conformable fractional Laplace transform is defined as follows:

Definition 3.1. *The conformable fractional Laplace transform (CFLT) of function* $u : [0, \infty) \to \mathbb{R}$ *for* $t > 0$*, of order* $0 < \alpha \leq 1$*, starting from* a *of* u *is defined by*

$$L_\alpha^a\{u(t)\} = \int_a^\infty e^{-s\frac{(t-a)^\alpha}{\alpha}} u(t)(t-a)^{\alpha-1} dt = U_\alpha^a(s). \tag{3.2}$$

If a=0, we have

$$L_\alpha^0\{u(t)\} = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} u(t) t^{\alpha-1} dt = U_\alpha^0(s) = U_\alpha(s). \tag{3.3}$$

In particular, if $\alpha = 1$, then Eq. (3.3) is reduced to the definition of the Laplace transform

$$L\left\{u(t)\right\} = \int_0^{\infty} e^{-st}u(t)dt = U(s). \tag{3.4}$$

Theorem 3.2. *Let $u : [a, \infty) \to \mathbb{R}$ be differentiable real valued function and $0 < \alpha \leq 1$. Then*

$$L_\alpha^a\left\{{}_tT_\alpha^a(u)(t)\right\} = sU_\alpha^a(s) - u(a). \tag{3.5}$$

Proof. See [1]. □

Theorem 3.3. *Let u is piecewise continuous on $[0, \infty)$ and $L_\alpha^a\{u(t)\} = U_\alpha^a(s)$, then*

$$L_\alpha^0\left\{t^{n\alpha}u(t)\right\} = (-1)^n\alpha^n\frac{d^n}{ds^n}\left[U_\alpha^0(s)\right], \quad n \in \mathbb{N}.$$

Proof. See [12]. □

Theorem 3.4. *If u is piecewise continuous on $[0, +\infty)$ with $L_\alpha^0\left\{u(t)\right\} = U_\alpha(s)$ and $\lim_{t\to 0^+}\frac{u(t)}{t^\alpha} < \infty$ exist, then*

$$L_\alpha^0\left\{\frac{u(t)}{t^\alpha}\right\} = \frac{1}{\alpha}\int_s^{\infty} U(x)dx.$$

Proof. See [12]. □

Theorem 3.5. *(Translation Theorem) If u is piecewise continuous on $[0, +\infty)$ and $L_\alpha^a\left\{u(t)\right\} = U_\alpha^a(s)$, then $L_\alpha^a\left\{e^{b\frac{(t-a)^\alpha}{\alpha}}u(t)\right\} = U_\alpha^a(s-b), \quad s > b$.*

Proof. See [12]. □

4. Solution of certain ordinary CFDEs

In this section, we develop the fractional Laplace transformations for solving a class of CFDEs. The next theorems, play an important role to convert the CFDEs into ordinary differential equations.

Proposition 4.1. *the conformable type problem If $0 < \alpha \leq 1$, $s > 0$ and k is constant, then the conformable type problem*

$$k\left({}_tT_\alpha^a u(t)\right) = f(t), \tag{4.1}$$

has its solution given by

$$u(t) = \frac{1}{k}\int_a^t (z-a)^{\alpha-1}f(z)dz + u(a). \tag{4.2}$$

Proof. By using fractional Laplace transform, we have

$$kL_\alpha^a\left\{{}_tT_\alpha^a u(t)\right\} = L_\alpha^a\{f(t)\}.$$

From Theorem 3.2, we obtain

$$U_\alpha^a(s) = \frac{F_\alpha^a(s)}{ks} + \frac{u(a)}{s}.$$

Now, taking inverse Laplace transform, we have

$$u(t) = (L_\alpha^a)^{-1}\left\{U_\alpha^a(s)\right\} = (L_\alpha^a)^{-1}\left\{\frac{F_\alpha^a(s)}{ks}\right\} + (L_\alpha^a)^{-1}\left\{\frac{u(a)}{s}\right\},$$

which completes the proof. □

For example, by substituting $k = 1$, $f(t) = bt^{1-\alpha}\cos(bt)$ and $u(0) = 0$, we have ${}_tT_\alpha^0 u(t) = bt^{1-\alpha}\cos(bt)$, hence $u(t) = \int_0^t b\cos(bz)dz = \sin(bt)$.

Theorem 4.2. *Let $u : [a, \infty) \to \mathbb{R}$ be twice differentiable on (a, ∞) and $\alpha, \beta > 0$ such that $\alpha + \beta \leq 1$, then*

$$ {}_tT^a_\beta \left({}_tT^a_\alpha u(t)\right) = (1-\alpha)(t-a)^{1-(\alpha+\beta)}u'(t) + (t-a)^{2-(\alpha+\beta)}u''(t). $$

Proof. From Theorem 2.4, we have

$$ \begin{aligned} {}_tT^a_\beta \left({}_tT^a_\alpha u(t)\right) &= {}_tT^a_\beta \left((t-a)^{1-\alpha}u'(t)\right) \\ &= (t-a)^{1-\beta}\left((t-a)^{1-\alpha}u'(t)\right)' \\ &= (1-\alpha)(t-a)^{1-(\alpha+\beta)}u'(t) + (t-a)^{2-(\alpha+\beta)}u''(t). \end{aligned} $$

□

In particular, for $a = 0$, one has

$$ {}_tT_\beta \left({}_tT_\alpha u(t)\right) = (1-\alpha)t^{1-(\alpha+\beta)}u'(t) + t^{2-(\alpha+\beta)}u''(t). \tag{4.3} $$

Theorem 4.3. *Let $u : [a, \infty) \to \mathbb{R}$ be twice differentiable on (a, ∞) and $\alpha, \beta > 0$ such that $\alpha + \beta \leq 1$, then*

$$ \int_0^\infty e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}\left(tu''(t)\right)dt = u(0) + s\int_0^\infty t^{\alpha+\beta}e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}u'(t)dt - sU_{(\alpha+\beta)}(s), $$

$$ \int_0^\infty \left(1-\alpha+st^{\alpha+\beta}\right)e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}u'(t)dt = (\alpha-1)u(0) + \left(1-(2\alpha+\beta)\right)sU_{(\alpha+\beta)}(s) - (\alpha+\beta)s^2U'_{(\alpha+\beta)}(s). $$

Proof. By using integration by parts, we obtain

$$ \int_0^\infty e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}\left(tu''(t)\right)dt = u(0) + s\int_0^\infty t^{\alpha+\beta}e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}u'(t)dt - s\int_0^\infty t^{\alpha+\beta-1}e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}u(t)dt. $$

Therefore, from Definition 3.1, we arrive at the first formula. Next, Theorem 3.3 and integration by parts yields

$$ \begin{aligned} \int_0^\infty \left(1-\alpha+st^{\alpha+\beta}\right)e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}u'(t)dt &= (\alpha-1)u(0) + \left(1-(2\alpha+\beta)\right)s\int_0^\infty e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}t^{\alpha+\beta-1}u(t)dt \\ &\quad + s^2\int_0^\infty e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}\left(t^{\alpha+\beta}u(t)\right)t^{\alpha+\beta-1}dt. \end{aligned} $$

Thus, the second formula Theorem 4.3 holds. □

Theorem 4.4. *Let $u : [a, \infty) \to \mathbb{R}$ be twice differentiable on (a, ∞) and $\alpha, \beta > 0$ such that $\alpha + \beta \leq 1$, then*

$$ L^0_{(\alpha+\beta)}\left\{{}_tT_\beta({}_tT_\alpha u(t))\right\} = \alpha u(0) - (2\alpha+\beta)sU_{(\alpha+\beta)}(s) - s^2(\alpha+\beta)U'_{(\alpha+\beta)}(s). $$

Proof. From Eq. (4.3) and using fractional Laplace transformation, we get

$$ \begin{aligned} L^0_{(\alpha+\beta)}\left\{{}_tT_\beta({}_tT_\alpha u(t))\right\} &= L^0_{(\alpha+\beta)}\Big\{(1-\alpha)t^{1-(\alpha+\beta)}u'(t) + t^{2-(\alpha+\beta)}u''(t)\Big\}, \\ &= (1-\alpha)\int_0^\infty e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}\Big(t^{1-(\alpha+\beta)}u'(t)\Big)t^{(\alpha+\beta)-1}dt \\ &\quad + \int_0^\infty e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}\Big(t^{2-(\alpha+\beta)}u''(t)\Big)t^{(\alpha+\beta)-1}dt. \end{aligned} $$

Therefore, by simple calculations and Theorem 4.3, we have

$$ \begin{aligned} L^0_{(\alpha+\beta)}\left\{{}_tT_\beta({}_tT_\alpha u(t))\right\} &= \int_0^\infty \left(1-\alpha+st^{\alpha+\beta}\right)e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}u'(t)dt + u(0) - sU_{(\alpha+\beta)}(s), \\ &= \alpha u(0) - (2\alpha+\beta)sU_{(\alpha+\beta)}(s) - s^2(\alpha+\beta)U'_{(\alpha+\beta)}(s), \end{aligned} $$

and the theorem is thus proved. □

Lemma 4.5. *Let $s > 0$ and $\alpha, \beta > 0$ be such that $\alpha + \beta \leq 1$, then*

$$L^0_{\alpha+\beta}\{t^\alpha\} = \frac{(\alpha+\beta)^{\frac{\alpha}{\alpha+\beta}}}{s^{1+\frac{\alpha}{\alpha+\beta}}}\Gamma\left(1 + \frac{\alpha}{\alpha+\beta}\right). \tag{4.4}$$

Proof. By change of variable $z = \frac{t^\alpha}{\alpha}$, we have

$$L^0_\alpha\{t^p\} = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} t^p t^{\alpha-1} dt = \int_0^\infty e^{-sz}\left(\alpha^{\frac{1}{\alpha}} z^{\frac{1}{\alpha}}\right)^p dz = \frac{\alpha^{\frac{p}{\alpha}}}{s^{1+\frac{p}{\alpha}}}\Gamma\left(1 + \frac{p}{\alpha}\right).$$

By setting $p = \alpha$, $\alpha = \alpha + \beta$, the desired result for $L^0_{\alpha+\beta}\{t^\alpha\}$ can be concluded, i.e. (4.4) is valid. □

Proposition 4.6. *Suppose that $u(t)$ be twice differentiable on $(0, \infty)$ and $\alpha, \beta > 0$ be such that $\alpha + \beta \leq 1$, p is constant. Then, the following CFDE*

$$_tT_\beta\left(_tT_\alpha u(t)\right) = p(p-\alpha)t^{p-(\alpha+\beta)}, \quad u(0) = 0,$$

has its solution given by

$$u(t) = t^p + \frac{C}{(\alpha+\beta)^{\frac{\alpha}{\alpha+\beta}}\Gamma\left(1 + \frac{\alpha}{\alpha+\beta}\right)} t^\alpha, \tag{4.5}$$

where C is constant.

Proof. Applying $L^0_{\alpha+\beta}$ and Theorem 4.4 with Eq. (4.4) gives

$$\alpha u(0) - (2\alpha+\beta)sU_{\alpha+\beta}(s) - s^2(\alpha+\beta)U'_{\alpha+\beta}(s) = p(p-\alpha)\frac{(\alpha+\beta)^{\frac{p}{\alpha+\beta}-1}}{s^{\frac{p}{\alpha+\beta}}}\Gamma\left(\frac{p}{\alpha+\beta}\right).$$

Since $u(0) = 0$, the above differential equation can be written as

$$U'_{\alpha+\beta}(s) + \frac{(2\alpha+\beta)}{s(\alpha+\beta)}U_{\alpha+\beta}(s) = -\frac{p(p-\alpha)(\alpha+\beta)^{\frac{p}{\alpha+\beta}-2}}{s^{\frac{p}{\alpha+\beta}+2}}\Gamma\left(\frac{p}{\alpha+\beta}\right).$$

By solving the above ODE of first order, we have

$$U_{\alpha+\beta}(s) = s^{-\frac{\alpha}{\alpha+\beta}-1}\left(p(\alpha+\beta)^{\frac{p}{\alpha+\beta}-1}\Gamma\left(\frac{p}{\alpha+\beta}\right)s^{\frac{\alpha}{\alpha+\beta}-\frac{p}{\alpha+\beta}}\right) + Cs^{-\frac{\alpha}{\alpha+\beta}-1}.$$

Since $\frac{p}{\alpha+\beta}\Gamma\left(\frac{p}{\alpha+\beta}\right) = \Gamma\left(1 + \frac{p}{\alpha+\beta}\right)$, we can write

$$U_{\alpha+\beta}(s) = \Gamma\left(1 + \frac{p}{\alpha+\beta}\right)\frac{(\alpha+\beta)^{\frac{p}{\alpha+\beta}}}{s^{1+\frac{p}{\alpha+\beta}}} + \frac{C}{s^{1+\frac{\alpha}{\alpha+\beta}}}.$$

By applying $\left(L^0_{\alpha+\beta}\right)^{-1}$, we obtain

$$u(t) = \left(L^0_{\alpha+\beta}\right)^{-1}\{U_{\alpha+\beta}(s)\} = \left(L^0_{\alpha+\beta}\right)^{-1}\left\{\Gamma\left(1 + \frac{p}{\alpha+\beta}\right)\frac{(\alpha+\beta)^{\frac{p}{\alpha+\beta}}}{s^{1+\frac{p}{\alpha+\beta}}}\right\} + \left(L^0_{\alpha+\beta}\right)^{-1}\left\{\frac{C}{s^{1+\frac{\alpha}{\alpha+\beta}}}\right\},$$

and the solution (4.5) follows from Eq. (4.4). □

In particular, if $\alpha = \beta$, then function $u(t) = t^p + C\sqrt{\frac{2}{\pi\alpha}}t^\alpha$ is the solution to the following CFDE

$$_tT_\alpha\left(_tT_\alpha u(t)\right) = p(p-\alpha)t^{p-2\alpha}, \quad u(0) = 0.$$

Proposition 4.7. *Suppose that $u(t)$ be twice differentiable on $(0,\infty)$ and $\alpha,\beta>0$ be such that $\alpha+\beta\leq 1$, k is constant. Then, the following CFDE*

$$_tT_\beta\left(_tT_\alpha u(t)\right)+ku(t)=f(t),$$

has its solution given by

$$u(t)=\left(L^0_{\alpha+\beta}\right)^{-1}\left\{s^{-\frac{2\alpha+\beta}{\alpha+\beta}}e^{\frac{-k}{(\alpha+\beta)s}}\left(\int\frac{\alpha u(0)-F_{\alpha+\beta}(s)}{(\alpha+\beta)s^{\frac{\beta}{\alpha+\beta}}}e^{\frac{k}{(\alpha+\beta)s}}ds+C\right)\right\},$$

where C is constant.

Proof. Applying $L^0_{\alpha+\beta}$ and Theorem 4.4, we have

$$\alpha u(0)-(2\alpha+\beta)sU_{\alpha+\beta}(s)-s^2(\alpha+\beta)U'_{\alpha+\beta}(s)+kU_{\alpha+\beta}(s)=F_{\alpha+\beta}(s).$$

Hence, we have

$$U'_{\alpha+\beta}(s)+\frac{2\alpha+\beta-\frac{k}{s}}{s(\alpha+\beta)}U_{\alpha+\beta}(s)=\frac{\alpha u(0)-F_{\alpha+\beta}(s)}{s^2(\alpha+\beta)}.$$

By solving the above ODE, we have

$$U_{\alpha+\beta}(s)=s^{-\frac{2\alpha+\beta}{\alpha+\beta}}e^{\frac{-k}{(\alpha+\beta)s}}\left(\int\frac{\alpha u(0)-F_{\alpha+\beta}(s)}{(\alpha+\beta)s^{\frac{\beta}{\alpha+\beta}}}e^{\frac{k}{(\alpha+\beta)s}}ds+C\right).$$

Thus, solution $u(t)$ results from the CF inverse transform. □

In particular, when $\alpha=\beta=\frac{1}{2}$, the solution to the CFDE

$$_tT_{\frac{1}{2}}\left(_tT_{\frac{1}{2}}u(t)\right)+u(t)=\sqrt{t},\quad u(0)=0,$$

is given by

$$u(t)=L^{-1}\left\{\frac{\Gamma(\frac{3}{2})}{s^{\frac{3}{2}}}\right\}+CL^{-1}\left\{\frac{e^{\frac{-1}{s}}}{s^{\frac{3}{2}}}\right\}=\sqrt{t}+\frac{C}{\sqrt{\pi}}\sin\left(2\sqrt{t}\right).$$

Theorem 4.8. *Let $s>0$ and $\alpha,\beta>0$ be such that $\alpha+\beta\leq 1$, then*

$$L^0_{\alpha+\beta}\left\{_tT_{\alpha+\beta}u(t)\right\}=sU_{\alpha+\beta}(s)-u(0). \tag{4.6}$$

Proof. From the equality $_tT_{\alpha+\beta}u(t)=t^{-\beta}\left(_tT_\alpha u(t)\right)$, we calculate

$$L^0_{\alpha+\beta}\left\{_tT_{\alpha+\beta}u(t)\right\}=L^0_{\alpha+\beta}\left\{t^{-\beta}\left(_tT_\alpha u(t)\right)\right\}=L^0_{\alpha+\beta}\left\{t^{-\beta}\left(t^{1-\alpha}u'(t)\right)\right\}=\int_0^\infty e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}u'(t)dt.$$

Now, by using integration by parts, we obtain

$$\int_0^\infty e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}u'(t)dt=-u(0)+s\int_0^\infty e^{-s\frac{t^{\alpha+\beta}}{\alpha+\beta}}t^{\alpha+\beta-1}u(t)dt=sU_{\alpha+\beta}(s)-u(0),$$

and this completes the proof □

Proposition 4.9. *Assume $u(t)$ be twice differentiable on $(0,\infty)$ and $\alpha,\beta>0$ be such that $\alpha+\beta\leq 1$, k is constant. Then, the solution to the CFDE*

$$_tT_\beta\left(_tT_\alpha u(t)\right)+k\left(_tT_{\alpha+\beta}u(t)\right)=q(t),$$

is given by

$$u(t)=\left(L^0_{\alpha+\beta}\right)^{-1}\left\{s^{-\frac{2\alpha+\beta-k}{\alpha+\beta}}\left(\int\frac{(\alpha-k)u(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)s^{\frac{\beta+k}{\alpha+\beta}}}ds+C\right)\right\},$$

where C is constant.

Proof. Applying $L^0_{\alpha+\beta}$ and using Theorems 4.4, 4.8, we have

$$L^0_{\alpha+\beta}\left\{{}_tT_\beta\left({}_tT_\alpha u(t)\right)\right\}+kL^0_{\alpha+\beta}\left\{{}_tT_{\alpha+\beta}u(t)\right\}=L^0_{\alpha+\beta}\left\{q(t)\right\},$$

or, equivalently,

$$\alpha u(0)-(2\alpha+\beta)sU_{(\alpha+\beta)}(s)-s^2(\alpha+\beta)U'_{(\alpha+\beta)}(s)+k\left(sU_{\alpha+\beta}(s)-u(0)\right)=Q_{\alpha+\beta}(s).$$

Therefore, we can write

$$U'_{\alpha+\beta}(s)+\frac{2\alpha+\beta-k}{(\alpha+\beta)s}U_{\alpha+\beta}(s)=\frac{(\alpha-k)u(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)s^2}.$$

By solving the above ODE of first order, we have

$$U_{\alpha+\beta}(s)=s^{-\frac{2\alpha+\beta-k}{\alpha+\beta}}\left(\int\frac{(\alpha-k)u(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)s^{\frac{\beta+k}{\alpha+\beta}}}ds+C\right).$$

So, by taking $(L^0_{\alpha+\beta})^{-1}$ with respect to s, we obtain the explicit solution. □

In particular, for $\alpha=\beta=k=\frac{1}{2}$, the solution to the CFDE

$${}_tT_{\frac{1}{2}}\left({}_tT_{\frac{1}{2}}u(t)\right)+\frac{1}{2}t^{-\frac{1}{2}}\left({}_tT_{\frac{1}{2}}u(t)\right)=-3e^{-3t}+9te^{-3t},\quad u(0)=1,$$

is given by

$$u(t)=L^{-1}\left\{\frac{-1}{s}\right\}+L^{-1}\left\{\frac{1}{s+3}\right\}+L^{-1}\left\{\frac{C}{s}\right\}=-1+e^{-3t}+C.$$

Since $u(0)=1$, so $C=1$ and $u(t)=e^{-3t}$.

Proposition 4.10. *Let $u(t)$ be twice differentiable on $(0,\infty)$ and $\alpha,\beta>0$ be such that $\alpha+\beta\leq 1$, k is constant. Then, the following CFDE*

$${}_tT_\beta\left({}_tT_\alpha u(t)\right)+kt^{\alpha+\beta}u(t)=q(t),\tag{4.7}$$

has its solution given by

$$u(t)=\left(L^0_{\alpha+\beta}\right)^{-1}\left\{(s^2+k)^{-\frac{2\alpha+\beta}{2(\alpha+\beta)}}\left(\int\frac{\alpha u(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)(s^2+k)^{\frac{\beta}{2(\alpha+\beta)}}}ds+C\right)\right\},$$

where C is constant.

Proof. Applying the CF Laplace transform to the both sides of equation (4.7) yields

$$L^0_{\alpha+\beta}\left\{{}_tT_\beta\left({}_tT_\alpha u(t)\right)\right\}+kL^0_{\alpha+\beta}\left\{t^{\alpha+\beta}u(t)\right\}=L^0_{\alpha+\beta}\left\{q(t)\right\},$$

and from Theorems 4.4, 3.3, we get

$$\alpha u(0)-(2\alpha+\beta)sU_{\alpha+\beta}(s)-s^2(\alpha+\beta)U'_{\alpha+\beta}(s)-k(\alpha+\beta)U'_{\alpha+\beta}(s)=Q_{\alpha+\beta}(s).$$

So, we can write

$$U'_{\alpha+\beta}(s)+\frac{2\alpha+\beta}{(\alpha+\beta)(s^2+k)}sU_{\alpha+\beta}(s)=\frac{\alpha u(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)(s^2+k)}.$$

By solving the above ODE of first order, we have

$$U_{\alpha+\beta}(s)=(s^2+k)^{-\frac{2\alpha+\beta}{2(\alpha+\beta)}}\left(\int\frac{\alpha u(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)(s^2+k)^{\frac{\beta}{2(\alpha+\beta)}}}ds+C\right),$$

by taking $(L^0_{\alpha+\beta})^{-1}$ with respect to s, we obtain the explicit solution $u(t)$. □

For example, if $\alpha = \beta = \frac{1}{2}$, then the conformable type problem

$$ {}_tT_{\frac{1}{2}}\left({}_tT_{\frac{1}{2}}u(t)\right) + tu(t) = t^{\frac{3}{2}}, \quad u(0) = 0, $$

has the solution given by

$$ u(t) = L^{-1}\left\{\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}\right\} + L^{-1}\left\{\frac{C}{(s^2+1)^{\frac{3}{4}}}\right\} = \sqrt{t} + 1.216280215Ct^{\frac{1}{4}}J_0\left(\frac{t}{4}\right). $$

Theorem 4.11. *Let $\alpha, \beta > 0$ be such that $\alpha + \beta \leq 1$, $s > 0$, then the following relation is valid:*

$$ L^0_{\alpha+\beta}\left\{t^{\alpha+\beta}\left({}_tT_{\alpha+\beta}u(t)\right)\right\} = -(\alpha+\beta)\left(U_{\alpha+\beta}(s) + sU'_{\alpha+\beta}(s)\right). $$

Proof. From Theorem 4.8 and 3.3, it is clear that

$$ L^0_{\alpha+\beta}\left\{t^{\alpha+\beta}\left({}_tT_{\alpha+\beta}u(t)\right)\right\} = (-1)^1(\alpha+\beta)\frac{d}{ds}\left(sU_{\alpha+\beta}(s) - u(0)\right), $$

and this completes the proof. □

Remark 4.12. *Since $tu'(t) = t^{\alpha+\beta}\left({}_tT_{\alpha+\beta}u(t)\right)$, then we have*

$$ L^0_{\alpha+\beta}\left\{tu'(t)\right\} = -(\alpha+\beta)\left(U_{\alpha+\beta}(s) + sU'_{\alpha+\beta}(s)\right). \tag{4.8} $$

Proposition 4.13. *Assume $u(t)$ be twice differentiable on $(0, \infty)$ and $\alpha, \beta > 0$ be such that $\alpha + \beta \leq 1$, k is constant. Then, the solution to the CFDE*

$$ {}_tT_\beta\left({}_tT_\alpha u(t)\right) + kt^{\alpha+\beta}\left({}_tT_{\alpha+\beta}u(t)\right) = q(t), \tag{4.9} $$

is given by

$$ u(t) = \left(L^0_{\alpha+\beta}\right)^{-1}\left\{\frac{1}{s(s+k)^{\frac{\alpha}{\alpha+\beta}}}\left(\int\frac{\alpha u(0) - Q_{\alpha+\beta}(s)}{(\alpha+\beta)(s+k)^{\frac{\beta}{\alpha+\beta}}}ds + C\right)\right\}, $$

where C is constant.

Proof. Similarly, taking $L^0_{\alpha+\beta}$ to the both sides of equation (4.9) and using Theorems 4.4 and 4.11, we have

$$ U'_{\alpha+\beta}(s) + \left(\frac{\alpha}{(\alpha+\beta)(s+k)} + \frac{1}{s}\right)U_{\alpha+\beta}(s) = \frac{\alpha u(0) - Q_{\alpha+\beta}(s)}{(\alpha+\beta)(s^2+ks)}. $$

The above is an ODE, so we obtain

$$ U_{\alpha+\beta}(s) = \frac{1}{s(s+k)^{\frac{\alpha}{\alpha+\beta}}}\left(\int\frac{\alpha u(0) - Q_{\alpha+\beta}(s)}{(\alpha+\beta)(s^2+ks)}s(s+k)^{\frac{\alpha}{\alpha+\beta}}ds + C\right). $$

Thus, solution $u(t)$ results from the CF inverse transform. □

Proposition 4.14. *Assume $u(t)$ be twice differentiable on $(0, \infty)$ and $\alpha, \beta > 0$ be such that $\alpha + \beta \leq 1$, k is constant. Then, the solution to the CFDE*

$$ {}_tT_\beta\left({}_tT_\alpha u(t)\right) + ktu'(t) = q(t), \tag{4.10} $$

is given by

$$ u(t) = \left(L^0_{\alpha+\beta}\right)^{-1}\left\{\frac{1}{s(s+k)^{\frac{\alpha}{\alpha+\beta}}}\left(\int\frac{\alpha u(0) - Q_{\alpha+\beta}(s)}{(\alpha+\beta)(s+k)^{\frac{\beta}{\alpha+\beta}}}ds + C\right)\right\}, $$

where C is constant.

Proof. From Remark 4.12, the proof is clear. □

In particular, for $\alpha = \frac{1}{3}$, $\beta = \frac{2}{3}$, $k = 1$, the solution to the CFDE

$$ {}_tT_{\frac{2}{3}}\left({}_tT_{\frac{1}{3}}u(t)\right) + tu'(t) = \frac{2}{3}e^t + 2te^t, \quad u(0) = 1, $$

is given by

$$ \begin{aligned} u(t) &= \left(L^0_{\frac{1}{3}+\frac{2}{3}}\right)^{-1}\left\{U_{\frac{1}{3}+\frac{2}{3}}(s)\right\} \\ &= L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{C}{s(s+1)^{\frac{1}{3}}}\right\} \\ &= e^t + C\left(1 + \frac{\sqrt{3}\Gamma\left(\frac{2}{3}\right)}{6\pi t^{\frac{2}{3}}}\left(2\Gamma\Big(-\frac{2}{3}t\Big)t^{\frac{2}{3}} - 3e^{-t}\right)\right). \end{aligned} $$

On the other hand, since $u(0) = 1$, then $C = 0$ and $u(t) = e^t$.

Proposition 4.15. *Assume $u(t)$ be twice differentiable on $(0, \infty)$ and $\alpha, \beta > 0$ be such that $\alpha + \beta \leq 1$, k, m are constants. Then, the following CFDE*

$$ {}_tT_\beta\left({}_tT_\alpha u(t)\right) + ktu'(t) + mu(t) = q(t). \tag{4.11} $$

has its solution given by

$$ u(t) = \left(L^0_{\alpha+\beta}\right)^{-1}\left\{\frac{s^{\frac{m-k(\alpha+\beta)}{k(\alpha+\beta)}}}{(s+k)^{\frac{\alpha k+m}{k(\alpha+\beta)}}}\left(\int \frac{\alpha u(0) - Q_{\alpha+\beta}(s)}{s^{\frac{m}{k(\alpha+\beta)}}(\alpha+\beta)}(s+k)^{\frac{m-k\beta}{k(\alpha+\beta)}}ds + C\right)\right\}, $$

where C is constant.

Proof. Applying the CF Laplace transform to the both sides of equation (4.11) and from Theorems 4.4 and Remark 4.12, we calculate

$$ \begin{aligned} &\alpha u(0) - (2\alpha+\beta)sU_{\alpha+\beta}(s) - s^2(\alpha+\beta)U'_{\alpha+\beta}(s) - k(\alpha+\beta)U_{\alpha+\beta}(s) \\ &- k(\alpha+\beta)sU'_{\alpha+\beta}(s) + mU_{\alpha+\beta}(s) = Q_{\alpha+\beta}(s). \end{aligned} $$

Therefore, we can write

$$ U'_{\alpha+\beta}(s) + \left(\frac{\alpha}{(\alpha+\beta)(s+k)} + \frac{1}{s} - \frac{m}{s(s+k)(\alpha+\beta)}\right)U_{\alpha+\beta}(s) = \frac{\alpha u(0) - Q_{\alpha+\beta}(s)}{(s^2+ks)(\alpha+\beta)}. $$

By solving the above ordinary differential equation, we have

$$ U_{\alpha+\beta}(s) = \left\{\frac{s^{\frac{m-k(\alpha+\beta)}{k(\alpha+\beta)}}}{(s+k)^{\frac{\alpha k+m}{k(\alpha+\beta)}}}\left(\int \frac{\alpha u(0) - Q_{\alpha+\beta}(s)}{s^{\frac{m}{k(\alpha+\beta)}}(\alpha+\beta)}(s+k)^{\frac{m-k\beta}{k(\alpha+\beta)}}ds + C\right)\right\}. $$

Thus, solution $u(t)$ results from the CF inverse transform. □

For example, if $\alpha = \beta = \frac{1}{2}$, $k = m = 1$, then the conformable type problem

$$ {}_tT_{\frac{1}{2}}\left({}_tT_{\frac{1}{2}}u(t)\right) + tu'(t) + u(t) = \frac{1}{2} + 2t, \quad u(0) = 0, $$

has the solution given by

$$ \begin{aligned} u(t) &= \left(L^0_{\frac{1}{2}+\frac{1}{2}}\right)^{-1}\left\{U_{\frac{1}{2}+\frac{1}{2}}(s)\right\} \\ &= L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{-C}{(s+1)^{\frac{3}{2}}}\right\} = t - 2Ce^{-t}\sqrt{\frac{t}{\pi}}. \end{aligned} $$

5. Conclusion

The conformable fractional derivative is a new kind of fractional derivatives which needs to investigate more. We discussed about the fractional Laplace transform which is compatible with type of fractional derivatives. Some new results are reported which is useful in the theory of conformable fractional differential equations. Our representations of analytical solutions of CFDEs, explicitly reveal the complete reliability and efficiency of the presented method.

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