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The interior inverse boundary value problem for the impulsive Sturm-Liouville operator with the spectral boundary conditions

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Abstract

In this study, we discuss the inverse problem for the Sturm-Liouville operator with the impulse and with the spectral boundary conditions on the finite interval $(0, \pi)$. By taking the Mochizuki-Trooshin's method, we have shown that some information of eigenfunctions at some interior point and parts of two spectra can uniquely determine the potential function $q(x)$ and the boundary conditions.

Keywords. Inverse problem, Sturm-Liouville operator with the impulse, Spectral boundary condition, Spectrum.
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1. Introduction

We consider the boundary value problem $L = L(q, r, h_0, h_1, H_0, H_1)$ with the differential equation

$$-y'' + q(x)y = \lambda r(x)y, \quad x \in (0, \pi), \tag{1.1}$$

and the boundary conditions

$$U(y) := y'(0) - (h_1\rho + h_0)y(0) = 0, \tag{1.2}$$

$$V(y) := y'(\pi) + (H_1\rho + H_0)y(\pi) = 0. \tag{1.3}$$

Here the potential function $q(x)$ is a complex function in $L^2[0, \pi]$ and the weight function

$$r(x) = \begin{cases} 1, & x < \frac{\pi}{2}, \\ \omega^2, & x > \frac{\pi}{2}, \end{cases}$$

for $1 \neq \omega > 0$. The parameters h_0, h_1, H_0, H_1 are complex and $\lambda = \rho^2$ is a spectral parameter.

Discontinuous boundary value problems have great applications in various branches of natural sciences especially mathematical physics and quantum mechanics. Sturm-Liouville problems with spectral boundary conditions are also seen in various fields of sciences. Inverse problems for these problems have been investigated by many scholars in recent years (see [2, 3, 5, 9, 14, 18, 20–22]). To study discontinuous boundary value problems, some conditions under titles interface conditions, point interactions, transmission conditions and impulsive conditions are imposed in the discontinuous point. Among these discontinuities, the impulsive differential equations have been discussed in many references and a large number of authors surveyed the spectral theory of these equations (see for example [3, 9, 18]). In this work, we want to study the uniqueness solution for the impulsive Sturm-Liouville equation using two sets of spectra plus some information of eigenfunctions at some interior point. Inverse problems for second order

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Sturm-Liouville operators have been originated in the studies of Ambarzumian since 1929 [1] and have been gradually completed in the next years (see, e.g., [7, 13, 15–17, 19, 21, 22, 25]). Some aspects of the spectral theory for the Sturm-Liouville problem of high order have also been surveyed in [10–12]. Interior inverse problems as one of the methods to solve the inverse problem were also studied by several scientists and the readers can see the published results in the works [15, 21, 22, 25]. This method which is originated by Mochizuki and Trooshin shows that a set of values of eigenfunctions at some interior point and parts of two spectra can uniquely determine the potential $q(x)$ [13]. To the best of my knowledge, interior inverse problems for the boundary value problem (1.1)-(1.3) are not considered so far. The impulsive condition and spectral boundary condition cause to be difficult to study the spectral properties of the operator. In this paper, we discuss the inverse problem for the impulsive Sturm-Liouville operator with the spectral boundary conditions on the finite interval $(0,\pi)$. By improving the Mochizuki-Trooshin's method [13], we will prove that the boundary value problem L is uniquely determined by spectral data of a kind: some information of eigenfunctions at some interior point and parts of two spectra.

This paper is organized as follows. We present some preliminaries in Sec. 2. In Sec. 3, using the Mochizuki and Trooshin's method, we discuss the interior inverse problem for the impulsive Sturm-Liouville problem (1.1)-(1.3) and state main results. Finally, Sec. 4 contains some conclusion.

2. Preliminaries

Suppose that $y(x,\rho)$ is the solution of the equation (1.1) satisfying the initial conditions

$$y(0,\rho)=1, \quad y'(0,\rho)=h_1\rho+h_0. \tag{2.1}$$

It is trivial that $U(y)=0$. Denote

$$\Delta(\rho):=V(y(x,\rho)), \tag{2.2}$$

as the characteristic function of L. The zeros of $\Delta(\rho)$ coincide with the eigenvalues of L [6].

From [3, 23, 25], we know that for each fixed $x\in(0,\pi)$, the function $y(x,\rho)$ and their derivatives with respect to x are entire in ρ, and for sufficiently large ρ, one has

$$y(x,\rho)=\cos\rho x+h_1\sin\rho x+O\left(\frac{1}{\rho}\exp(|\Im\rho|x)\right), \qquad x<\frac{\pi}{2}, \tag{2.3}$$

$$\begin{aligned} y(x,\rho) \;\; = \;\; & \frac{\omega+1}{2\omega}(\cos\rho\gamma(x)+h_1\sin\rho\gamma(x)) \\ & +\frac{\omega-1}{2\omega}\left(\cos\rho\left(2\gamma\left(\frac{\pi}{2}\right)-\gamma(x)\right)+h_1\sin\rho\left(2\gamma\left(\frac{\pi}{2}\right)-\gamma(x)\right)\right) \\ & +O\left(\frac{1}{\rho}\exp(|\Im\rho|\gamma(x))\right), \qquad x>\frac{\pi}{2}, \end{aligned} \tag{2.4}$$

uniformly in x, where $\gamma(x)=\int_0^x\sqrt{r(t)}dt$.

It follows from the relations (1.3), (2.2) and (2.4) that for sufficiently large ρ,

$$\begin{aligned} \Delta(\rho)=&\frac{\rho}{2}\left(\frac{\delta(1+\omega)}{H_1+h_1\omega}\cos\left(\frac{\rho(1+\omega)\pi}{2}-\sigma_+\right)-\frac{\delta(1-\omega)}{H_1-h_1\omega}\cos\left(\frac{\rho(1-\omega)\pi}{2}-\sigma_-\right)\right) \\ &+O\left(\exp(|\Im\rho|\gamma(\pi))\right), \end{aligned} \tag{2.5}$$

where $\delta^2=(H_1^2+\omega^2)(1+h_1^2)$ and $\sigma_\pm=\frac{1}{2i}ln\frac{(H_1\pm h_1\omega)i-h_1H_1\pm\omega}{(H_1\pm h_1\omega)i+h_1H_1\mp\omega}$. Applying the Rouche theorem [4], we can write that the roots of (2.5) have the following asymptotic form for sufficiently large n,

$$\rho_n=\frac{1}{1+\omega}\left(2n+1+\frac{2}{\pi}\sigma_+\right)+O(n^{-1}). \tag{2.6}$$

In the next section, we discuss the interior inverse problem for the boundary value problem L. For this reason, we consider the following boundary value problem $\widetilde{L}:=L(\widetilde{q},r,\widetilde{h}_0,\widetilde{h}_1,\widetilde{H}_0,\widetilde{H}_1)$ defined by

$$-y''+\widetilde{q}(x)y=\lambda r(x)y, \quad x\in(0,\pi), \tag{2.7}$$

and the boundary conditions

$$U(y) := y'(0) - (\widetilde{h}_1\rho + \widetilde{h}_0)y(0) = 0, \tag{2.8}$$

$$V(y) := y'(\pi) + (\widetilde{H}_1\rho + \widetilde{H}_0)y(\pi) = 0. \tag{2.9}$$

It should be noted that λ_n and $y_n(x,\rho)$ are the eigenvalues and corresponding eigenfunctions of L, respectively.

3. Main result

In this section, we state and prove the main theorem of this paper. To prove the uniqueness theorem we use the Mochizuki and Trooshin's method. In the case $b \neq \frac{\pi}{2}$, by one full spectrum, a part of the second spectrum and some information of the eigenfunctions at $x = b$, we can state the uniqueness theorem to the problem L.

First, we give the eigenfunctions and the product of eigenfunctions that are important in proving the main result.

Suppose that $A(x,t)$ and $B(x,t)$ are bounded functions. The following integral representation holds [6, 24],

$$y(x,\rho) = \cos\rho x + h_1 \sin\rho x + \int_0^x A(x,t)\cos\rho t dt + \int_0^x B(x,t)\sin\rho t dt,\ x < \frac{\pi}{2}.$$

Therefore

$$\begin{aligned} y(x,\rho)\widetilde{y}(x,\rho) \quad &= \quad \frac{1+h_1\widetilde{h}_1}{2} + \frac{1-h_1\widetilde{h}_1}{2}\cos 2\rho x + \frac{h_1+\widetilde{h}_1}{2}\sin 2\rho x \\ &+\frac{1}{2}\int_0^x A'(x,t)\cos 2\rho t dt + \frac{1}{2}\int_0^x B'(x,t)\sin 2\rho t dt,\ x < \frac{\pi}{2}, \end{aligned} \tag{3.1}$$

where $A'(x,t)$ and $B'(x,t)$ are bounded functions.

Consider two sequences $l(n)$ and $r(n)$ as follows:

$$l(n) = \frac{n}{\sigma_1}(1+\epsilon_{1n}), \qquad r(n) = \frac{n}{\sigma_2}(1+\epsilon_{2n}); \quad 0 < \sigma_k \leq 1, \quad \epsilon_{kn} \longrightarrow 0, \tag{3.2}$$

for $k = 1,2$ and let μ_n be the eigenvalues of $L_1 = L(q,r,h_0,h_1,\mathcal{H}_0,\mathcal{H}_1)$, $\mathcal{H}_s \neq H_s, \mathcal{H}_s \in \mathbb{C}$ for $s = 0,1$.

In the following theorem we state the main uniqueness result.

Theorem 3.1. *Let $l(n)$ and $r(n)$ be two sequences in (3.2). Suppose that $b \in \left(\frac{\pi}{2},\pi\right)$ such that $\sigma_1 > \frac{4b}{(1+\omega)\pi} - 1$ and $\sigma_2 > \frac{4\omega}{1+\omega} - \frac{4\omega b}{(1+\omega)\pi}$. If for any n, have*

$$\lambda_n = \widetilde{\lambda}_n, \quad \mu_{l(n)} = \widetilde{\mu}_{l(n)},$$

$$< y_{r(n)}, \widetilde{y}_{r(n)} >_{x=b} = 0,$$

where $< y,z > := yz' - y'z$, then $q(x) = \widetilde{q}(x)$ a.e. on $[0,\pi]$ and

$$h_0 = \widetilde{h}_0,\ h_1 = \widetilde{h}_1,\ H_0 = \widetilde{H}_0,\ H_1 = \widetilde{H}_1.$$

We express the following lemma which help us to prove Theorem 3.1.

Lemma 3.2. *Consider a sequence $m(n)$ as follows:*

$$m(n) = \frac{n}{\sigma}(1+\epsilon_n), \quad 0 < \sigma \leq 1, \quad \epsilon_n \longrightarrow 0.$$

(1) Choose $b \in \left(0,\frac{\pi}{2}\right)$ such that $\sigma > \frac{4b}{(1+\omega)\pi}$. If for any n,

$$\lambda_{m(n)} = \widetilde{\lambda}_{m(n)}, \quad < y_{m(n)}, \widetilde{y}_{m(n)} >_{x=b} = 0,$$

then $q(x) = \widetilde{q}(x)$ *a.e. on* $[0, b]$ *and* $h_1 = \widetilde{h}_1$, $h_0 = \widetilde{h}_0$.
(2) Choose $b \in \left(\frac{\pi}{2}, \pi\right)$ *such that* $\sigma > \frac{4\omega}{1+\omega} - \frac{4\omega b}{(1+\omega)\pi}$. *If for any* n,

$$\lambda_{m(n)} = \widetilde{\lambda}_{m(n)}, \quad < y_{m(n)}, \widetilde{y}_{m(n)} >_{x=b} = 0,$$

then $q(x) = \widetilde{q}(x)$ *a.e. on* $[b, \pi]$ *and* $H_0 = \widetilde{H}_0$, $H_1 = \widetilde{H}_1$.

Proof. Consider $y(x)$ as the solution to the equation

$$-y'' + q(x)y = \lambda r(x)y, \quad x \in (0, \pi), \tag{3.3}$$

with the initial conditions $y(0, \rho) = 1$ and $y'(0, \rho) = h_1\rho + h_0$, and $\widetilde{y}(x)$ as the solution of the equation

$$-\widetilde{y}'' + \widetilde{q}(x)\widetilde{y} = \lambda r(x)\widetilde{y}, \quad x \in (0, \pi), \tag{3.4}$$

with the initial conditions $\widetilde{y}(0, \rho) = 1$ and $\widetilde{y}'(0, \rho) = \widetilde{h}_1\rho + \widetilde{h}_0$. Multiplying (3.3) by $\widetilde{y}(x, \rho)$ and (3.4) by $y(x, \rho)$ and subtracting the resulted equations, then integrating on $[0, b]$, we obtain

$$\begin{aligned} G_b(\rho) \quad &:= \quad \int_0^b \big(q(x) - \widetilde{q}(x)\big) y(x)\widetilde{y}(x)dx + (h_1 - \widetilde{h}_1)\rho + h_0 - \widetilde{h}_0 \\ &= (y'(x)\widetilde{y}(x) - y(x)\widetilde{y}'(x))|_{x=b}. \end{aligned} \tag{3.5}$$

From the assumptions of the theorem, we can write

$$G_b(\rho_{m(n)}) = 0.$$

Now it needs to be proved that $G_b(\rho) = 0$ for all $\rho \neq \rho_n$.

Since from (3.1)

$$|y(x, \rho)\widetilde{y}(x, \rho)| \leq M_1 exp(2|\Im\rho|x),$$

we can give that

$$|G_b(\rho)| \leq M_2 \exp(2b\mathfrak{e}|sin\theta|). \tag{3.6}$$

Define

$$h(\theta) := \limsup_{\mathfrak{e}\to\infty} \frac{ln(|G_b(\mathfrak{e}\exp(i\theta))|)}{\mathfrak{e}}, \tag{3.7}$$

as an indicator of the function $G_b(\rho)$. By virtue of (3.6) and (3.7), we give that

$$h(\theta) \leq 2b|sin\theta|,$$

and so

$$\frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta \leq \frac{b}{\pi}\int_0^{2\pi} |sin\theta|d\theta = \frac{4b}{\pi}. \tag{3.8}$$

Suppose that $n(\mathfrak{e})$ denotes the number of roots of the function $G_b(\rho)$ in the disk $|\rho| \leq \mathfrak{e}$. We have

$$n(\mathfrak{e}) \geq 2 \sum_{\frac{2n}{\sigma(1+\omega)}(1+O(n^{-1}))<\mathfrak{e}} 1 = \mathfrak{e}\sigma(1+\omega)[1 + \epsilon(\mathfrak{e})], \tag{3.9}$$

from the assumption of Lemma 3.2 and the asymptotic form of eigenvalues (2.6) for sufficiently large $\mathfrak{e}$. From $\sigma > \frac{4b}{(1+\omega)\pi}$, one gives

$$\lim_{\mathfrak{e}\to\infty} \frac{n(\mathfrak{e})}{\mathfrak{e}} \geq \sigma(1+\omega) \geq \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta. \tag{3.10}$$

For any nonzero entire function $G_b(\rho)$ of exponential type, we have

$$\lim_{\mathfrak{e}\to\infty} \frac{n(\mathfrak{e})}{\mathfrak{e}} \leq \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta \tag{3.11}$$

(see [8] for more details). Taking (3.10) and (3.11), we can give that $G_b(\rho) = 0$ on the whole complex plane.
Put $Q(x) = q(x) - \widetilde{q}(x)$. By substituting (3.1) into (3.5) and taking $G_b(\rho) = 0$ for all values ρ, we infer that

$$\begin{aligned}(h_1 - \widetilde{h}_1)\rho + (h_0 - \widetilde{h}_0) \quad &+ \quad \int_0^b Q(x)\left[\frac{1 + h_1\widetilde{h}_1}{2} + \frac{1 - h_1\widetilde{h}_1}{2}\cos 2\rho x + \frac{h_1 + \widetilde{h}_1}{2}\sin 2\rho x\right]dx \\ &+ \quad \int_0^b Q(x)\left[\int_0^x A'(x,t)\cos 2\rho t dt\right]dx \\ &+ \quad \int_0^b Q(x)\left[\int_0^x B'(x,t)\sin 2\rho t dt\right]dx = 0,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}(h_1 - \widetilde{h}_1)\rho + (h_0 - \widetilde{h}_0) \quad &+ \quad \int_0^b \frac{1 + h_1\widetilde{h}_1}{2} Q(x)dx \\ &+ \quad \int_0^b \cos 2\rho t\left[\frac{1 - h_1\widetilde{h}_1}{2}Q(t) + \int_t^b A'(x,t)Q(x)dx\right]dt \\ &+ \quad \int_0^b \sin 2\rho t\left[\frac{h_1 + \widetilde{h}_1}{2}Q(t) + \int_t^b B'(x,t)Q(x)dx\right]dt = 0.\end{aligned}$$

The Riemann-Lebesgue lemma concludes that for $|\rho| \to \infty$,

$$\begin{cases} \int_0^b \cos 2\rho t\left[Q(t) + \int_t^b A''(x,t)Q(x)dx\right]dt = 0, \\ \\ \int_0^b \sin 2\rho t\left[Q(t) + \int_t^b B''(x,t)Q(x)dx\right]dt = 0, \end{cases} \tag{3.12}$$

and

$$\begin{cases} (h_0 - \widetilde{h}_0) + \int_0^b \frac{1+h_1\widetilde{h}_1}{2}Q(x)dx = 0, \\ \\ (h_1 - \widetilde{h}_1)\rho = 0, \end{cases} \tag{3.13}$$

for bounded functions $A''(x,t)$ and $B''(x,t)$. Using the completeness of the functions " cos " and " sin " [6], one gets

$$Q(t) + \int_t^b A''(x,t)Q(x)dx = 0 = Q(t) + \int_t^b B''(x,t)Q(x)dx.$$

Since these equations are homogeneous Volterra integral equations and have only zero solution, it results that $Q(x) = 0$ for $x \in (0,b)$. So, $q(x) = \widetilde{q}(x)$ a.e. on $[0,b]$. Furthermore, from (3.13), it is easily shown that $h_0 = \widetilde{h}_0$ and $h_1 = \widetilde{h}_1$.
If we take the change of variable $x \to \pi - x$, the interval (b,π) is converted to the interval $(0, \pi - b)$. To prove the problem on (b,π), we have to repeat the obtained arguments in the pervious section for the supplementary problem $\widehat{L}$

$$-y'' + q_1(x)y = \lambda r_1(x)y, \quad x \in (0,\pi), \tag{3.14}$$

$$U(y) := y'(0) - (H_1\rho + H_0)y(0) = 0, \tag{3.15}$$

$$V(y) := y'(\pi) + (h_1\rho + h_0)y(\pi) = 0, \tag{3.16}$$

where $q_1(x) = q(\pi - x)$ and $r_1(x) = r(\pi - x)$. It is easily seen that the assumptions of theorem 3.1 are satisfied to $\widehat{L}$. Repeating the previous discussions, we get $Q_1(x) = Q(\pi - x) = 0$ on $(0, \pi - b)$. So $q(x) = \widetilde{q}(x)$ a.e. on $[b,\pi]$ and $H_0 = \widetilde{H}_0$, $H_1 = \widetilde{H}_1$. The proof is completed. $\square$

Proof of Theorem 3.1. The equality $\lambda_n = \widetilde{\lambda}_n$ gives us that $\lambda_{r(n)} = \widetilde{\lambda}_{r(n)}$. Since $< y_{r(n)}, \widetilde{y}_{r(n)} >_{x=b}= 0$, this equality and Lemma 3.2 infer that $\widetilde{q}(x) = q(x)$ on $x \in [b, \pi]$ and $H_0 = \widetilde{H}_0$, $H_1 = \widetilde{H}_1$. To complete the proof, we have to show that $\widetilde{q}(x) = q(x)$ for $x \in [0, b]$ and $h_0 = \widetilde{h}_0$, $h_1 = \widetilde{h}_1$.

Considering (3.5) in the case $b \in [\frac{\pi}{2}, \pi]$, we can infer that

$$\begin{aligned} \mathbb{G}_b(\rho) \quad := \quad & \int_0^b \big(q(x) - \widetilde{q}(x)\big) y(x)\widetilde{y}(x)dx + (h_1 - \widetilde{h}_1)\rho + h_0 - \widetilde{h}_0 \\ & = (y'(x)\widetilde{y}(x) - y(x)\widetilde{y}'(x))|_{\frac{\pi}{2}-0} - (y'(x)\widetilde{y}(x) - y(x)\widetilde{y}'(x))|^b_{\frac{\pi}{2}+0}. \end{aligned} \tag{3.17}$$

Since $y_n(x)$ and $\widetilde{y}_n(x)$ have the same condition at the point $x = \pi$ and $\widetilde{q}(x) = q(x)$ on $x \in [b, \pi]$, we get for any n and constants α_n,

$$y_n(x) = \alpha_n \widetilde{y}_n(x), \quad x \in [b, \pi]. \tag{3.18}$$

Together with (3.17) and equality $< y, z > |_{x=\frac{\pi}{2}-0} =< y, z > |_{x=\frac{\pi}{2}+0}$, this yields $\mathbb{G}_b(\lambda_n) = 0$ and analogously $\mathbb{G}_b(\mu_{l(n)}) = 0$.

We see $1 + \mathfrak{e}(1 + \omega)[1 + \epsilon(\mathfrak{e})]$ of λ_n and $1 + \mathfrak{e}\sigma_1(1 + \omega)[1 + \epsilon(\mathfrak{e})]$ of $\mu_{l(n)}$ inside the disc of radius $\mathfrak{e}$. So, the total of these roots becomes $n(\mathfrak{e}) = 2 + \mathfrak{e}(1 + \omega)[1 + \sigma_1 + \epsilon(\mathfrak{e})]$. Condition $\sigma_1 > \frac{4b}{(1+\omega)\pi} - 1$ implies that

$$\lim_{\mathfrak{e}\to\infty} \frac{n(\mathfrak{e})}{\mathfrak{e}} \geq (1 + \omega)(1 + \sigma_1) \geq \frac{1}{2\pi} \int_0^{2\pi} h(\theta)d\theta. \tag{3.19}$$

For any nonzero entire function $\mathbb{G}_b(\lambda)$ of exponential type, we have

$$\lim_{\mathfrak{e}\to\infty} \frac{n(\mathfrak{e})}{\mathfrak{e}} \leq \frac{1}{2\pi} \int_0^{2\pi} h(\theta)d\theta \tag{3.20}$$

(see [8] for more details). Together with (3.19), this yields that $\mathbb{G}_b(\lambda) = 0$.

From $\mathbb{G}_b(\lambda) = 0$ and taking the same method as in the proof of Lemma 3.2, we can prove that $q(x) = \widetilde{q}(x)$ a.e. on $[0, b]$ and $h_0 = \widetilde{h}_0$, $h_1 = \widetilde{h}_1$. The proof is completed. □

4. Conclusion

In the present paper, we investigated the interior inverse problems for the impulsive Sturm-Liouville operator with eigenparameter dependent boundary conditions. By taking one full spectrum, a part of the second spectrum and some information of the eigenfunctions at $x = b \in \left(\frac{\pi}{2}, \pi\right)$, we proved the Mochizuki and Trooshin theorem for this inverse problem.

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