Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 10, No. 1, 2022, pp. 121-143
DOI:10.22034/cmde.2021.41150.1787

Numerical solution for solving fractional parabolic partial differential equations

Jalil Rashidinia* **and Elham Mohmedi**
School of Mathematics, Iran University of Science and Technology, Tehran, Iran.

Abstract

In this paper, a reliable numerical scheme is developed and reviewed in order to obtain approximate solution of time fractional parabolic partial differential equations. The introduced scheme is based on Legendre tau spectral approximation and the time fractional derivative is employed in the Caputo sense. The L^2 convergence analysis of the numerical method is analysed. Numerical results for different examples are examined to verify the accuracy of spectral method and justification the theoretical analysis, and to compare with other existing methods in the literatures.

Keywords. Time fractional parabolic partial differential equations, Caputo derivative, Shifted Legendre tau method.
2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

The theory of fractional calculus was first proposed in the year 1695 by Marquis L Hopital and from then on many studies were done and many important books were published on this field, among which we can especially mention the study of Oldham and Spanier [50] . Fractional calculus surely has a long history, dating back to famous mathematician such as Euler, Laplace, Fourier, Liouville and Riemann, The first applications in physics were dealt with by Abel and Heaviside. Such interest has been stimulated by the applications that this calculus finds in different areas of physics and engineering, possibly including fractal phenomena [9]. The link between mathematics and physics should not surprise the reader, we can think of mathematics as the only way to describe the physics, and not just as a useful tool. This does not means that this two disciplines can not develop independently but important milestones are reached through and intense interplay of this two camps, as pointed in [9]. Fractional calculus is a field of applied mathematics which have attracted increasing attention because they have applications in various fields of science and engineering [17, 21, 42]. For instance they can describe fluid flow, rheology, diffusion, relaxation, oscillation, anomalous diffusion, reaction-diffusion, turbulence, diffusive transport akin to diffusion, electric networks, polymer physics, chemical physics, electro chemistry of corrosion, relaxation processes in complex systems, propagation of seismic waves, dynamical processes in self-similar and porous structures, nonlinear optics and many other physical processes [31]. The application of FDEs in signal processing, control engineering, electromagnetism, biosciences, fluid mechanics, electrochemistry, diffusion processes, complex physical systems and control , also many phenomena in fluid mechanics, viscoelasticity, chemistry, physics, finance and other sciences can be described very successfully models using mathematical tools from fractional calculus [18, 28, 52, 65]. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [23], and fluid-dynamic traffic model with fractional derivatives [24] can eliminate the deficiency arising from the assumption of continuum traffic flow. Also, the fractional cable model is widely used in several fields of science to describe the propagation of signals. A relevant medical and biological example is the anomalous subdiffusion in spiny neuronal dendrites observed in several studies of the last decade [9]. In [66], different applications of derivatives and integrals of fractional order in physics, chemistry, engineering, astrophysics, and so on

Received: 06 August 2020 ; Accepted: 21 February 2021.
∗ Corresponding author. Email: rashidinia@iust.ac.ir.

have been presented. In [25, 26, 52] , we can find some applications of fractional differential equations in classical mechanics, quantum mechanics, nuclear physics, hadron spectroscopy, and quantum field theory. The spectral tau method is particular type of spectral schemes which is more appropriate and widely used to solve many types of the differential equations [57, 75, 77]. There are many researchers which are interesting to develop numerical methods for fractional partial differential equations [6, 19, 30, 36, 38, 41, 43, 51, 58, 64, 67, 74]. Explicit finite difference [59, 61, 76, 81]and implicit finite difference methods [3, 33, 35, 71], compact finite difference method [7, 8, 68, 69, 79], finite element method [20, 29, 40], spline scheme [1, 63], radial basis functions [2, 22, 54], wavelets method [27, 34, 60], sinc-radial basis functions method [53, 55], and local radial basis functions method [44–49]. The time fractional convection-diffusion-wave equation is a mathematical model of important physical phenomena. In physics, process involving the phenomena of diffusion and wave propagation have great relevance, these physical process are governed, from mathematical point of view, by differential equations of order 1 and 2 in time. By introducing a fractional derivatives of order β in time, with $0 < \beta < 1$ or $1 \leq \beta \leq 2$, we lead to process in mathematical physics which we may refer to as fractional phenomena, this is not merely a phenomenological procedure providing an additional fit parameter. In this paper, we consider the following time fractional parabolic partial differential equations with variable coefficients

$$\begin{aligned}&\alpha_2 {}_0^cD_t^\beta u(x,t) + \alpha_1 u_t(x,t) = -e(x)u(x,t) - c(x)u_x(x,t) + b(x)u_{xx}(x,t)\\&-a(x)u_{xxxx}(x,t) + z(x,t),\end{aligned} \tag{1.1}$$

where $T = [0,\tau], \Omega = [0,\ell]$ and $\mathrm{O} = \Omega \times T$ with initial condition

$$\begin{aligned}&u(x,0) = p_1(x),\\&u_t(x,0) = p_2(x),\end{aligned} \tag{1.2}$$

and boundary conditions

$$\begin{aligned}&u(0,t) = q_1(t),\\&u(\ell,t) = q_2(t),\\&u_{xx}(0,t) = q_3(t),\\&u_{xx}(\ell,t) = q_4(t),\end{aligned} \tag{1.3}$$

where, α_1, α_2 are known parametrs with $\alpha_1 \geq 0, \alpha_2 > 0, \alpha_1 + \alpha_2 \neq 0$. The functions $p_1(x), p_2(x), q_1(t), q_2(t), q_3(t), q_4(t)$, $c(x), b(x), e(x), a(x)$ and $z(x,t)$ prescribed known functions.

If we consider in (1.1), $\alpha_1 = 0, b(x) = K_\beta, e(x) = c\prime(x)$ and $a(x) = 0$, we can obtain the following time fractional Fokker-Planck equation [56]

$${}_0^cD_t^\beta u(x,t) = K_\beta \frac{\partial^2 u}{\partial x^2}(x,t) - c(x)\frac{\partial u}{\partial x}(x,t) - c'(x)u(x,t) + z(x,t), \tag{1.4}$$

subject to the initial

$$\begin{aligned}&u(x,0) = p_1(x),\\&u_t(x,0) = p_2(x),\end{aligned} \tag{1.5}$$

and boundary conditions

$$\begin{aligned}&u(0,t) = q_1(t),\\&u(\ell,t) = q_2(t),\end{aligned} \tag{1.6}$$

where K_β shows the anomalous diffusion coefficient and $c(x) = -\dfrac{v'(x)}{m\eta_\beta}$ represents an external force field governed by the potential by the potential $v'(x)$(a prime stands for the derivative with respect to the space coordinate x), with m being the mass of the diffusion particle and η_β the generalized friction coefficient.

Yuan-Min Wang proposed compact finite difference method for time fractional convection-diffusion-wave equations with variable cofficients [72, 73]. Also, wavlet method has been presented for a class of fractional convection-diffusion equation with variable coefcients in [11]. Recently, Dahaghin and Hassani [16] presented optimization method based on generalized polynomials for solving a class of time fractional convection-diffusion- wave equations with variable coefficients. In [15], compact exponential scheme has been proposed for the time fractional convection diffusion reaction equations with variable coefficients. Efficient compact finite difference methods [69] has been introduced for a class of time-fractional convection reaction diffusion equations with variable coefficients. Also, a fourth-order extrapolated compact difference method [70] for time-fractional convection-reaction-diffusion equations with spatially variable coefficients has been reported. In this paper the introduced scheme is based on Legendre tau spectral approximation. This paper is organized as follows: In section 2 we propose some basic priliminiries and defintions of fractional calculus. Section 3 dedicate to the concept and some properties of orthogonal Legendre polynomial. We developed spectral method for approximating the solution of time fractional parabolic partial differential equations with variable coefficients in section 4. The convergence analysis of the presented approach is discussed in section 5. We test our method on six examples in section 6 to confirm convergence behavior and to demonstrate the efficiency and accuracy of the presented scheme. At the end concluding remarks are given.

2. Priliminiries and notations

In this section, we give some necessary definitions of the fractional calculus [52] which are needed for developing spectral method.

Definition 2.1. The integral of order $\beta \geq 0$ (fractional) according to Riemann-Liouville is given by

$$I^{\beta}u(x,t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}u(x,\tau)d\tau,$$

where $t > 0, \beta > 0$, $I^0u(x,t) = u(x,t)$,

$$\Gamma(v) = \int_0^{\infty} e^{-x}x^{v-1}dx,$$

is the Gamma function.

The operator I^{β} has the following properties

$$(I^{\beta}I^{\alpha}u)(x,t) = (I^{\beta+\alpha}u)(x,t),$$

$$I^{\beta}x^{v} = \frac{\Gamma(v+1)}{\Gamma(\beta+v+1)} \times x^{\beta-v},$$

$$\alpha, \beta \geq 0, v > -1.$$

Definition 2.2. The caputo fractional derivative, $D_t^{\beta}u(x,t)$ of order α with respect to time is defined as

$$D_t^{\beta}u(x,t) = \frac{\partial^{\beta}u(x,t)}{\partial t^{\beta}} = \begin{cases} \frac{1}{\Gamma(n-\beta)}\int_0^t \frac{1}{\Gamma(t-\tau)^{\beta-n+1}}\frac{\partial^n u(x,\tau)}{\partial t^n}d\tau, n-1<\beta<n, n\in \mathrm{N}, \\ \frac{\partial^n u(x,t)}{\partial t^n}, \beta = n, n \in \mathrm{N}. \end{cases}$$

Definition 2.3. The Riemann-Liouville fractional derivative operator which is defined as

$$D_t^{1-\beta}v(x,t) = \frac{1}{\Gamma(\beta)}\frac{\partial}{\partial t}\int_0^t \frac{v(x,s)}{(t-s)^{1-\beta}}ds,$$

where $\Gamma(.)$ is the Gamma function.

The usable and effective relation between the Riemann-Liouville operator and caputo operator is given by the following expression.

$$(I^{\beta}D^{\beta}v)(x,t) = v(x,t) - \sum_{k=0}^{n-1} v^{(k)}(0^{+})\frac{t^k}{k!}, t > 0. \tag{2.1}$$

3. Shifted Legendre polynomials

It is clear that the classical Legendre polynomials [62], are defined on $[-1,1]$ by the three-term recurrence relation

$$(j+1)L^*_{j+1}(y) = (2j+1)yL^*_j(y) - iL^*_{j-1}(y), j = 1,2,\dots,$$

$$L^*_0(y) = 1, L^*_1(y) = y.$$

By means of $y = (2x/\ell) - 1$, Legendre polynomials are defined on the interval $[0,\ell]$ and we introduce the shifted Legendre polynomials $L_i((2x/\ell)-1)$, which has recurrence formula

$$L_{j+1}(x) = \tfrac{2j+1}{j+1}(2x-1)L_j(x) - \tfrac{j}{j+1}L_{j-1}x, j = 1,2,\dots,$$

$$L_0(x) = 1, L_1 = 2x-1.$$

The shifted Legendre polynomails [62], have weight function $w_L(x) = 1$ over $\Omega = [0,\ell]$ with the orthogonality property

$$\delta_{i,j} = \int_0^\ell L_i(x)L_j(x)dx = \begin{cases} \frac{1}{2j+1}, & i = j, \\ 0, & i \neq j, \end{cases} \tag{3.1}$$

and following analytic form:

$$L_j(x) = \sum_{k=0}^{j} (-1)^{j+k} \frac{(j+k)!x^k}{(j-k)!(k!)^2},$$

$$L_j(0) = (-1)^j, L_j(1) = 1.$$

Suppose that $u(x)$ is a square integrable function with the Legendre shifted polynomial

$$u(x) = \sum_{j=0}^{\infty} c_j L_{\ell,j}(x),$$

from which the coefficient c_j are obtained by

$$c_j = (2j+1)\int_0^\ell u(x)L_j(x)dx, j = 0,1,\dots,S.$$

If we approximate $u(x)$ by the first $(S+1)$ terms, then we can write

$$u_S(x) = \sum_{j=0}^{S} u_j L_{\ell,j}x,$$

which has the matrix form

$$u_S(x) = U^T\psi_{\ell,S}(x),$$

$$U^T = [u_0, u_1, \dots, u_S],$$

with

$$\psi_{L,S}(x) = [L_{\ell,0}(x), L_{\ell,1}(x), \dots, L_{\ell,S}(x)],$$

similarly, let $u(x,t)$ be an infinitely differential function defined on $0 \le x \le \ell$ and $0 \le t \le \tau$, then it can be express as

$$u_{S,S}(x,t) = \sum_{i=0}^{S}\sum_{j=0}^{S} u_{ij}L_{\tau,i}(t)L_{\ell,j}(x) = {\psi^T}_{\tau,S}(t)U\psi_{\ell,S}(x),$$

$$U = \begin{pmatrix} u_{00} & u_{01} & \ldots, & u_{0S} \\ u_{10} & u_{11} & \ldots, & u_{1S} \\ \vdots & \vdots & \ldots, & \vdots \\ u_{S0} & u_{S1} & u_{_{S2}} & u_{SS} \end{pmatrix},$$

$$u_{ij} = \frac{1}{\delta_{i,i}\delta_{j,j}} \int_0^\tau \int_0^\ell u(x,t) L_{\tau,i}(t) L_{\ell,j}(x) dx dt,$$

where u_{ij} are the entries of unknown matrix U

$$u_{ij} = \tfrac{(2i+1)}{\tau} \tfrac{(2j+1)}{\ell} \int_0^\tau \int_0^\ell u(x,t) L_{\tau,i}(t) L_{\ell,j}(x) dx dt,$$

$i = 0, 1, \ldots, S,$
$j = 0, 1, \ldots, S.$

The first derivative of shifted Legendre vector $u_S(x)$ can be expressed as

$$\frac{du_S(x)}{dx} = Du_S(x), \tag{3.2}$$

where the operational D is the operational matrix and thr $(S+1)(S+1)$ entries of this defined as

$$D = (d_{ij}) = \begin{cases} \zeta(i,j), & i > j, \\ 0, & \texttt{otherwise}, \end{cases}$$

$$\zeta(i,j) = \tfrac{(i+1)(i+2)_j (j+2)_{i-j-1} \Gamma(j+1)}{(i-j-1)! \Gamma(2j+1)} \times {}_3F_2 \left(\begin{matrix} -i+1+j, & i+j+2 & j+1 \\ j+2 & 2j+2 & \end{matrix} ; 1 \right),$$

(The ${}_3F_2$ is the Hyper-geometric function). The above eq. (3.2) can be written:

$$\frac{d^m u_{L,S}(x)}{dx^m} = D^m u_{L,m}(x), \tag{3.3}$$

where m is a natural number .

Theorem 3.1. *Let $u_{\tau,S}(t)$ be the shifted Legendre vector and $\beta > 0$ then*

$$I^\beta u_{\tau,S}(t) \simeq P_\beta u_{\tau,S}(t), \tag{3.4}$$

where P_β is the $(S+1)(S+1)$ operational matrix of fractional integration of order β in the Riemann- Liouville sense and is defined as follows:

$$\begin{pmatrix} \Phi_\beta(0,0) & \Phi_\beta(0,1) & \ldots, & \Phi_\beta(0,S) \\ \Phi_\beta(1,0) & \Phi_\beta(1,1) & \ldots, & \Phi_\beta(1,S) \\ \vdots & \vdots & \vdots & \vdots \\ \Phi_\beta(i,0) & \Phi_\beta(i,1) & \ldots, & \Phi_\beta(i,S) \\ \vdots & \vdots & \vdots & \vdots \\ \Phi_\beta(S,0) & \Phi_\beta(S,1) & \ldots, & \Phi_\beta(S,S) \end{pmatrix},$$

$$\Phi_\beta(i,j) = \sum_{k=0}^{i} \tfrac{(-1)^{i-k} \Gamma(i+1) \Gamma(i+k+1) \tau^\beta}{\Gamma(k+1) \Gamma(i+1) (i-k)! \Gamma(k+\beta+1)},$$

$$\times \sum_{s=0}^{j} \tfrac{(-1)^{j-s} \Gamma(j+s+1) \Gamma(s+k+\beta+1)(2j+1) j!}{\Gamma(j+1)(j-s)! (s!)^2 \Gamma(s+k+\beta+2)}.$$

For proof see [5].

4. The spectral method

In this section, the spectral tau scheme combined with the shifted Legendre polynomials is developed for solving the class of time-fractional parabolic partial differential equations with variable coefficients. In Eq. (1.1), by employing the Riemann-Liouville time fractional integration of order β on both sides of Eq. (1.1) and making use of (2.1), we obtain the integrated form of Eq. (1.1) as follows

$$\begin{aligned} &\alpha_2(u(x,t)-h(x,t))+\alpha_1 I_t^{\beta-1}u(x,t)=I_t^{\beta}[b(x)u_{xx}(x,t)]-I_t^{\beta}[c(x)u_x(x,t)] \\ &-I_t^{\beta}[d(x)u(x,t)]-I_t^{\beta}[e(x)u_{xxxx}(x,t)]+I_t^{\beta}[z(x,t)], \end{aligned} \tag{4.1}$$

where $h(x,t)=p_1(x)+tp_2(x)$ due to use of spectral tau method with the shifted Legendre operational matrix for solving Eq. (1.1), we suppose that $z(x,t)$, $b(x),c(x),a(x)$ and $e(x)$ are smooth and continuous functions. Thus, $u(x,t),z(x,t),h(x,t)$, $b(x),c(x),a(x)$ and $e(x)$ can be expanded in terms of shifted Legendre polynomails as

$$\begin{aligned} &u_{S,S}(x,t)={\psi^T}_{\tau,S}(t)U\psi_{\ell,S}(x), \\ &z_{S,S}(x,t)={\psi^T}_{\tau,S}(t)Z\psi_{\ell,S}(x), \\ &h_{S,S}(x,t)={\psi^T}_{\tau,S}(t)H\psi_{\ell,S}(x), \\ &b_S(x)=B^T\psi_{\ell,S}(x), \\ &c_S(x)=C^T\psi_{\ell,S}(x), \\ &a_S(x)=A^T\psi_{\ell,S}(x), \\ &e_S(x)=E^T\psi_{\ell,S}(x), \end{aligned} \tag{4.2}$$

where U is the unknown coefficient $(S+1)(S+1)$ matrix and Z,H, B^T,C^T,A^T,E^T are known matrices and vectors that are in the following forms

$$Z=\begin{pmatrix} z_{00} & z_{01} & \dots, & z_{0S} \\ z_{10} & z_{11} & \dots, & z_{1S} \\ \vdots & \vdots & \dots, & \vdots \\ z_{S0} & z_{S1} & z_{_{S2}} & z_{SS} \end{pmatrix},$$

$$H=\begin{pmatrix} h_{00} & h_{01} & \dots, & h_{0S} \\ h_{10} & h_{11} & \dots, & h_{1S} \\ \vdots & \vdots & \dots, & \vdots \\ h_{S0} & h_{S1} & h_{_{S2}} & h_{SS} \end{pmatrix}.$$

$$\begin{aligned} B^T&=[b_0,b_1,\dots,b_S], \\ C^T&=[c_0,c_1,\dots,c_S], \\ A^T&=[a_0,a_1,\dots,a_S], \\ E^T&=[e_0,e_1,\dots,e_S], \end{aligned}$$

The coefficients $b_j, c_j, a_j, e_j, j = 0, 1, \ldots, S$ can be determined by

$$b_j = \frac{1}{\delta_{j,j}} \int_0^\ell b(x) L_{\ell,j}(x) dx, \qquad j = 0, 1, \ldots, S,$$

$$c_j = \frac{1}{\delta_{j,j}} \int_0^\ell c(x) L_{\ell,j}(x) dx, \qquad j = 0, 1, \ldots, S,$$

$$a_j = \frac{1}{\delta_{j,j}} \int_0^\ell a(x) L_{\ell,j}(x) dx, \qquad j = 0, 1, \ldots, S,$$

$$e_j = \frac{1}{\delta_{j,j}} \int_0^\ell e(x) L_{\ell,j}(x) dx, \qquad j = 0, 1, \ldots, S.$$

For a general form of the function $b(x)$ one is unable to compute exactly the above integrals for example can be determined, the coefficients b_j can be approximated by using the discrete inner product at the Legendre - Gauss nodes [5] to obtain

$$b_j = \frac{1}{\delta_{j,j}} \sum_{i=0}^{S} b(x_{S,i}) L_{S,j}(x_{S,i}), j = 0, 1, \ldots, S$$

where $x_{S,i}, 0 \le i \le S$ are the zeros of Legendre Gauss quadrature in the interval $(0, \ell)$. Similarly, the coefficients c_j, a_j, e_j z_{ij}, and h_{ij} are obtained by

$$c_j = \frac{1}{\delta_{j,j}} \sum_{i=0}^{S} c(x_{S,i}) L_{S,j}(x_{S,i}),$$

$$a_j = \frac{1}{\delta_{j,j}} \sum_{i=0}^{S} a(x_{S,i}) L_{S,j}(x_{S,i}),$$

$$e_j = \frac{1}{\delta_{j,j}} \sum_{i=0}^{S} e(x_{S,i}) L_{S,j}(x_{S,i}),$$

$$z_{ij} = \frac{1}{\delta_{i,i}\delta_{j,j}} \int_0^\tau \int_0^\ell z(x,t) L_{\tau,i}(t) L_{\ell,j}(x) dx dt,$$

$$h_{ij} = \frac{1}{\delta_{i,i}\delta_{j,j}} \int_0^\tau \int_0^\ell h(x,t) L_{\tau,i}(t) L_{\ell,j}(x) dx dt.$$

Using relations (3.2), (3.3), (4.1) and (4.2), the following relations are obtained

$$I_t^\beta [\frac{\partial^2 u(x,t)}{\partial x^2}] = \psi_{\tau,S}^T(t) P_\beta^T U D^{(2)} \psi_{\ell,S}(x), \tag{4.3}$$

$$I_t^\beta h(x,t) = \psi_{\tau,S}^T P_\beta^T H \psi_{\ell,S}(x), \tag{4.4}$$

$$I_t^{\beta-1} u(x,t) = \psi_{\tau,S}^T(t) P_{\beta-1}^T U \psi_{\ell,S}(x), \tag{4.5}$$

$$I_t^{\beta}[b(x)\frac{\partial u(x,t)}{\partial x}] \simeq (B^T\psi_{\ell,S}(x))(I_t^{\beta}\psi_{\tau,S}^T(t)U(D\psi_{\ell,S}(x))$$
$$= (B^T\psi_{\ell,S}(x))(\psi_{\tau,S}^T(t)P_{\beta}^TUD\psi_{\ell,S}(x)) \tag{4.6}$$
$$= \psi_{\tau,S}^T(t)P_{\beta}^TUD\psi_{\ell,S}(x)\psi_{\ell,S}^T(x)B,$$

$$I_t^{\beta}[c(x)\frac{\partial u(x,t)}{\partial x}] \simeq (C^T\psi_{\ell,S}(x))(I_t^{\beta}\psi_{\tau,S}^T(t)U(D\psi_{\ell,S}(x))$$
$$= (C^T\psi_{\ell,S}(x))(\psi_{\tau,S}^T(t)P_{\beta}^TUD\psi_{\ell,S}(x)) \tag{4.7}$$
$$= \psi_{\tau,S}^T(t)P_{\beta}^TUD\psi_{\ell,S}(x)\psi_{\ell,S}^T(x)C,$$

$$I_t^{\beta}[a(x)\frac{\partial u(x,t)}{\partial x}] \simeq (A^T\psi_{\ell,S}(x))(I_t^{\beta}\psi_{\tau,S}^T(t)U(D\psi_{\ell,S}(x))$$
$$= (A^T\psi_{\ell,S}(x))(\psi_{\tau,S}^T(t)P_{\beta}^TUD\psi_{\ell,S}(x)) \tag{4.8}$$
$$= \psi_{\tau,S}^T(t)P_{\beta}^TUD\psi_{\ell,S}(x)\psi_{\ell,S}^T(x)A,$$

$$I_t^{\beta}[e(x)\frac{\partial u(x,t)}{\partial x}] \simeq (D^T\psi_{\ell,S}(x))(I_t^{\beta}\psi_{\tau,S}^T(t)U(D\psi_{\ell,S}(x))$$
$$= (E^T\psi_{\ell,S}(x))(\psi_{\tau,S}^T(t)P_{\beta}^TUD\psi_{\ell,S}(x)) \tag{4.9}$$
$$= \psi_{\tau,S}^T(t)P_{\beta}^TUD\psi_{\ell,S}(x)\psi_{\ell,S}^T(x)E,$$

put

$$\psi_{\ell,S}(x)\psi_{\ell,S}^T(x)B = R_1^T\psi_{\ell,S}(x), \tag{4.10}$$

$$\psi_{\ell,S}(x)\psi_{\ell,S}^T(x)C = R_2^T\psi_{\ell,S}(x), \tag{4.11}$$

$$\psi_{\ell,S}(x)\psi_{\ell,S}^T(x)A = R_3^T\psi_{\ell,S}(x), \tag{4.12}$$

$$\psi_{\ell,S}(x)\psi_{\ell,S}^T(x)E = R_4^T\psi_{\ell,S}(x), \tag{4.13}$$

where R_1, R_2, R_3, R_4 are vectors.
Then (4.10) can be written in the form

$$\sum_{n=0}^{S} b_k L_{l,n}(x)L_{l,j}(x) = \sum_{n=0}^{S} R_{1kj}L_{l,n}(x), j = 0, 1, \dots, S \tag{4.14}$$

Multiplying both sides by $L_{\ell,i}, i = 0, 1, \dots, S$ and integrating 0 to ℓ, the following relation is given.

$$\sum_{n=0}^{S} b_k \int_0^l L_{l,i}(x)L_{l,n}(x)L_{l,j}(x)dx = \sum_{n=0}^{S} R_{1ij} \int_0^l L_{l,n}(x)L_{l,i}(x)dx, \tag{4.15}$$

By using (3.1), the following relation has been obtained.

$$R_{1ij} = \frac{1}{\delta_{j,j}} \sum_{n=0}^{S} (f_k \int_0^l L_{l,i}(x)L_{l,n}(x)L_{l,j}(x)dx), i, j = 0, 1, \dots, S. \tag{4.16}$$

Also, in the first term on RHS of (4.1) and by using (4.6) and (4.10), we have

$$I_t^{\beta}[b(x)\frac{\partial^2 u(x,t)}{\partial^2 x}] = \psi_{\tau,S}^T(t)P_{\beta}^T U D^2 R_1^T \psi_{\ell,S}(x). \tag{4.17}$$

Similarly, for other terms on RHS of (4.1) by using (4.6), (4.11),(4.12) and (4.13), we get

$$I_t^{\beta}[c(x)\frac{\partial u(x,t)}{\partial x}] = \psi_{\tau,S}^T(t)P_{\beta}^T U D R_2^T \psi_{\ell,S}(x). \tag{4.18}$$

$$I_t^{\beta}[e(x)u(x,t)] = \psi_{\tau,S}^T(t)P_{\beta}^T U R_3^T \psi_{\ell,S}(x). \tag{4.19}$$

$$I_t^{\beta}[a(x)\frac{\partial^4 u(x,t)}{\partial^4 x}] = \psi_{\tau,S}^T(t)P_{\beta}^T U D^4 R_4^T \psi_{\ell,S}(x). \tag{4.20}$$

Now by substituting equations (4.17) , (4.18) ,(4.19) and (4.20) in the residual $\zeta_{S,S}(x,t)$ for Eq. (4.1), the following relation has been obtained.

$$\begin{aligned} &RR_{S,S}(x,t) = \psi_{\tau,S}^T(t)[\alpha_2(U-H) + \alpha_1 P_{\beta-1}^T + \\ &P_{\beta}^T U D^2 R_1^T + P_{\beta}^T U D R_2^T + P_{\beta}^T U R_3^T + P_{\beta}^T U D^4 R_4^T - P_{\beta}^T Z]\psi_{\ell,S}(x), \end{aligned} \tag{4.21}$$

By minimizing (4.21), the $(S+1)(S-3)$ linear algebraic equations in the unknown expansion coefficients u_{ij}, $i = 0,1,\ldots,S, j = 0,1,\ldots,S$, can be arising as

$$\begin{aligned} &\int_0^{\tau}\int_0^{l} RR_{S,S}(x,t)L_{\tau,i}(t)L_{l,j}(x)dxdt = 0, \\ &i = 0,1,\ldots,S, \\ &j = 0,1,\ldots,S-3, \end{aligned} \tag{4.22}$$

the $(S+1)(S-3)$ linear algebraic equations in the unknown coefficients, $u_{ij}, i = 0,1,\ldots,S, j = 0,1,\ldots,S-3$ have been obtained. The boundary conditions are in the following form

$$\begin{aligned} &\psi_{\tau,S}^T(t)U\psi_{\ell,S}(0) = q_1(t), \\ &\psi_{\tau,S}^T(t)U\psi_{\ell,S}(\ell) = q_2(t), \\ &\psi_{\tau,S}^T(t)UD^2 d\psi_{\ell,S}(0) = q_3(t), \\ &\psi_{\tau,S}^T(t)UD^2\psi_{\ell,S}(\ell) = q_4(t), \end{aligned} \tag{4.23}$$

Eqs. (4.23) are collocated at $S+1$ points, then $4(S+1)$ linear algebraic equations are obtained. The unknown coefficients $u_{ij}, i = 0,1,\ldots,S, j = 0,1,\ldots,S$ can be determined by solving the system of equations Eqs. (4.22) associated with (4.23). Then the approximate solution $u_{S,S}(x,t)$ can be determined by using (4.2).

5. Convergence analysis

In this section, we analyze the convergence of the spectral Legendre-tau method for the Eq. (1.1). We first give some definitions and lemmas about function spaces.

Definition 5.1. The Sobolev space $H^m(\Omega), m \geq 0$ is defined as

$$\mathrm{H}^m(\Omega) = \{u \in L^2(\Omega) \mid \frac{\partial^p u}{\partial x^p} \in L^2(\Omega), 0 \leq p \leq m\}, \tag{5.1}$$

The semi-norm and norm associated with $\mathrm{H}^m(\Omega)$ are

$$|u|_{\mathrm{H}^m(\Omega)} = (\left\|\frac{\partial^m u}{\partial x^m}\right\|_{L^2(\Omega)}^2)^{\frac{1}{2}}, \|u\|_{\mathrm{H}(\Omega)} = (\sum_{p=0}^{m}\left\|\frac{\partial^p u}{\partial x^p}\right\|_{L^2(\Omega)}^2)^{\frac{1}{2}}. \tag{5.2}$$

Theorem 5.2. *Define the norm for $r >= 0$*

$$\|u\|_{r,0} = (\int_I \|u(.,t)\|^2_{\mathrm{H}^r(\Omega)} dt)^{\frac{1}{2}}, \tag{5.3}$$

of the space $\mathrm{H}^{r,0}(\mathrm{O}) = \mathrm{H}^r(\Omega; L^2(I))$, *and the norm*

$$\|u\|_{0,r} = (\sum_{p=0}^{r} \left\| \frac{\partial^p u}{\partial t^p} \right\|^2_{L^2(\mathrm{O})})^{\frac{1}{2}}, \tag{5.4}$$

of the space $H^{0,r} = L^2(\Omega; \mathrm{H}^r(I))$.

Definition 5.3. Now the Hilbert space is defined in the following form

$$\mathrm{H}^{r,s}(\mathrm{O}) = \mathrm{H}^r(\Omega; \mathrm{H}^s(I)) = \{u \in L^2(\mathrm{O}); \frac{\partial^{p+q} u}{\partial x^p \partial t^q}, \in L^2(\mathrm{O}), 0 \le p \le r, o \le q \le s\}, \tag{5.5}$$

dedicate with the norm

$$\|u\|_{r,s} = (\sum_{p=0}^{r}\sum_{q=0}^{s} \left\| \frac{\partial^{p+q} u(x,t)}{\partial x^p \partial t^q} \right\|^2_{L^2(\mathrm{O})})^{\frac{1}{2}}. \tag{5.6}$$

Now we neeed to recall the following Lemmas in [4],

Lemma 5.4. *For all $u \in \mathrm{H}^m(\Omega), m \ge 0$, then*

$$\begin{aligned} \left\|u - P^S u\right\|_{\mathrm{H}^l(\Omega)} &\le C S^{2l-\frac{1}{2}-SS} |u|_{\mathrm{H}^{SS,S}(\Omega)}, \\ \left\|u - P^S u\right\|_{L^2(\Omega)} &\le C S^{-SS} |u|_{\mathrm{H}^{SS,S}(\Omega)}, \end{aligned} \tag{5.7}$$

where semi-norm on the right-hand side is defined as

$$|u|_{\mathrm{H}^{SS,S}(\Omega)} = (\sum_{p=\min(SS,S+1)}^{SS} \left\| \frac{\partial^p u}{\partial x^p} \right\|^2_{L^2(\Omega)})^{\frac{1}{2}}, \tag{5.8}$$

when $S \ge SS - 1$, we have

$$|u|_{\mathrm{H}^{SS,N}(\Omega)} = \left\| \frac{\partial^{SS} u}{\partial x^{SS}} \right\|_{L^2(\Omega)} = |u|_{\mathrm{H}^{SS}(\Omega)}. \tag{5.9}$$

Proof of this lemma is given in [4]

Lemma 5.5. *For all, $r > 0, s > 0$ we have*

$$\|u - P_{S,S}\|_{L^2(\mathrm{O})} \le C_1 S^{-r} \|u\|_{r,0} + C_2 S^{-s} \|u\|_{0,s}. \tag{5.10}$$

proof of this lemma is given in [4].

Now we will analyzed the convergence of our presented method by the following Theorem

Theorem 5.6. *Assume that $u \in L^2(\mathrm{O})$ is the exact solution of (1.1) and $u_{S,S}$ is the approximation solution of* (1.1)*. Then for sufficiently large S, we have*

$$\|u_{S,S}(x,t) - u(x,t)\|_{L^2(\mathrm{O})} \to 0. \tag{5.11}$$

Proof. We define the error functions $\zeta^u_{S,S}(x,t) = u(x,t) - u_{S,S}(x,t)$ and $\zeta^u_{P_{S,S}}(x,t) = u(x,t) - P_{S,S}u(x,t)$. By using presented tau method, the following relation is obtained.

$$\begin{aligned} u_{S,S}(x,t) &= \lambda_1 P_{S,S}(I_t^{\beta-1} u_{S,S}(x,t)) - \lambda_2 P_{S,S}(I_t^{\beta}(b(x)\tfrac{\partial^2}{\partial x^2} u_{S,S}(x,t)) \\ &- \lambda_2 P_{S,S}(I_t^{\beta}(c(x)\tfrac{\partial}{\partial x} u_{S,S}(x,t)) - \lambda_2 P_{S,S}(I_t^{\beta}(e(x) u_{S,S}(x,t)) \\ &- \lambda_2 P_{S,S}(I_t^{\beta}(a(x)\tfrac{\partial^4}{\partial x^4} u_{S,S}(x,t)) + \lambda_2 P_{S,S} Z(x,t) + P_{S,S} h(x,t) \end{aligned} \tag{5.12}$$

where $\frac{-\alpha_1}{\alpha_2} = \lambda_1, \frac{1}{\alpha_2} = \lambda_2, \alpha_2 \neq 0,$

if we subtract (5.12) from (4.1), we get error equation as follows

$$
\begin{aligned}
&\zeta_{S,S}(x,t) = \lambda_1 P_{S,S}(I_t^{\beta-1} u_{S,S}(x,t)) - \lambda_1 I_t^{\beta-1} u(x,t) \\
&+\lambda_2 P_{S,S}(I_t^{\beta}(b(x)\tfrac{\partial^2 u_{S,S}(x,t)}{\partial x^2})) - \lambda_2 I_t^{\beta}(b(x)(\tfrac{\partial^2 u_{S,S}(x,t)}{\partial x^2})) \\
&+\lambda_2 P_{S,S}(I_t^{\beta}(c(x)\tfrac{\partial u_{S,S}(x,t)}{\partial x})) - \lambda_2 I_t^{\beta}(c(x)\tfrac{\partial u_{S,S}(x,t)}{\partial x}) \\
&\lambda_2 P_{S,S}(I_t^{\beta} e(x) u_{S,S}(x,t) - \lambda_2 I_t^{\beta}(e(x) u_{S,S}(x,t)) \\
&+\lambda_2 P_{S,S}(I_t^{\beta}(a(x)\tfrac{\partial^4 u_{S,S}(x,t)}{\partial x^4})) - \lambda_2 I_t^{\beta}(a(x)(\tfrac{\partial^4 u_{S,S}(x,t)}{\partial x^4})) \\
&+\lambda_2 \zeta^{z}_{S,S}(x,t) + \zeta^{h}_{S,S}(x,t),
\end{aligned}
\tag{5.13}
$$

then we can write

$$
\begin{aligned}
&\zeta^u_{S,S} = \zeta^h_{P_{S,S}} + \lambda_2 \zeta^z_{P_{S,S}} + \lambda_1 \zeta^{E^{\beta-1}}_{P_{S,S}} + \lambda_2 \zeta^{E^{\beta}}_{P_{S,S}} + \lambda_2 \zeta^{{E_1}^{\beta}}_{P_{S,S}} + \lambda_2 \zeta^{{E_2}^{\beta}}_{P_{S,S}} + \lambda_2 \zeta^{{E_3}^{\beta}}_{P_{S,S}} + \lambda_2 \zeta^{{E_4}^{\beta}}_{P_{S,S}} \\
&+\lambda_1 P_{S,S}(I_t^{\beta-1}\zeta^u_{S,S}) + \lambda_2 P + \lambda_2 P_{S,S}(I_t^{\beta} b(x)\tfrac{\partial^2 \zeta^u_{S,S}}{\partial x^2}) + \lambda_2 P_{S,S}(I_t^{\beta} c(x)\tfrac{\partial \zeta^u_{S,S}}{\partial x}) + \\
&\lambda_2 P_{S,S}(I_t^{\beta} e(x)\zeta^u_{S,S}) + \lambda_2 P_{S,S}(I_t^{\beta} a(x)\tfrac{\partial^4 \zeta^u_{S,S}}{\partial x^4}) = \zeta^u_{P_{S,S}} + \lambda_1 P_{S,S}(I_t^{\beta-1}\zeta^u_{S,S}) + \\
&\lambda_2 P_{S,S}(I_t^{\beta} b(x)\tfrac{\partial^2 \zeta^u_{S,S}}{\partial x^2}) + \lambda_2 P_{S,S}(I_t^{\beta} c(x)\tfrac{\partial \zeta^u_{S,S}}{\partial x}) + \lambda_2 P_{S,S}(I_t^{\beta} e(x)\zeta^u_{S,S}) + \lambda_2 P_{S,S}(I_t^{\beta} a(x)\tfrac{\partial^4 \zeta^u_{S,S}}{\partial x^4}),
\end{aligned}
\tag{5.14}
$$

where

$$E^{\beta-1} = I_t^{\beta-1} u, E_1^{\beta} = I_t^{\beta}\frac{\partial^2 u}{\partial x^2}, E_2^{\beta} = I_t^{\beta}\frac{\partial u}{\partial x}, E_3^{\beta} = I_t^{\beta} u, E_4^{\beta} = I_t^{\beta}\frac{\partial^4 u}{\partial x^4}.$$

According to lemma (5.5) the following relation is obtained.

$$
\begin{aligned}
&\left\|\zeta^u_{S,S}\right\|_{L^2(\mathrm{O})} \le G_4 S^{-r}\|u\|_{r,0} + G_5 S^{-s}\|u\|_{0,s} \\
&\left\|G6\tfrac{t^{2-\beta}}{\tau(3-\beta)}\right\|_{L^1(O)}\left\|\zeta^u_{S,S}(x,\eta)\right\|_{L^2(O)} \\
&\left\|+G_7\tfrac{M_1 t^{(1-\beta)}}{\Gamma(2-\beta)}\right\|_{L^1(\mathrm{O})}\left\|\tfrac{\partial^2 \zeta^u_{S,S}(x,\eta)}{\partial x^2}\right\|_{L^2(\mathrm{O})} \\
&+\left\|G_8\tfrac{M_2 t^{1-\beta}}{\Gamma(2-\beta)}\right\|_{L^1(\mathrm{O})}\left\|\tfrac{\partial \zeta^u_{S,S}(x,\eta)}{\partial x}\right\|_{L^2(\mathrm{O})} \\
&\left\|G_9\tfrac{M_3 t^{2-\beta}}{\tau(3-\beta)}\right\|_{L^1(O)}\left\|\zeta^u_{S,S}(x,\eta)\right\|_{L^2(O)} \\
&\left\|+G_{10}\tfrac{M_4 t^{(1-\beta)}}{\Gamma(2-\beta)}\right\|_{L^1(\mathrm{O})}\left\|\tfrac{\partial^4 \zeta^u_{S,S}(x,\eta)}{\partial x^4}\right\|_{L^2(\mathrm{O})},
\end{aligned}
\tag{5.15}
$$

by using Youngs inequality, we get

$$\begin{aligned}
&\left\|\zeta^u_{S,S}\right\|_{L^2(O)} \leq G_4 S^{-r}\|u\|_{r,0} + G_5 S^{-s}\|u\|_{0,s} + \left\|G_6 \tfrac{t^{2-\beta}}{\Gamma(3-\beta)}\right\|_{L^1(O)} \left\|\zeta^u_{S,S}(x,\eta)\right\|_{L^2(O)} \\
&\left\|G_7 \tfrac{M_1 t^{(1-\beta)}}{\Gamma(2-\beta)}\right\|_{L^1(O)} \left\|\tfrac{\partial^2 \zeta^u_{S,S}}{\partial x^2}\right\|_{L^2(O)} + \left\|G_8 \tfrac{M_2 t^{1-\beta}}{\Gamma(2-\beta)}\right\|_{L^1(O)} \left\|\tfrac{\partial \zeta^u_{S,S}(x,\eta)}{\partial x}\right\|_{L^2(O)} + \\
&\left\|G_9 \tfrac{M_3 t^{1-\beta}}{\Gamma(2-\beta)}\right\|_{L^1(O)} \left\|\zeta^u_{S,S}(x,\eta)\right\|_{L^2(O)} + \left\|G_{10} \tfrac{M_4 t^{(1-\beta)}}{\Gamma(2-\beta)}\right\|_{L^1(O)} \left\|\tfrac{\partial^4 \zeta^u_{S,S}(x,\eta)}{\partial x^4}\right\|_{L^2(O)} \leq \\
&G_4 S^{-r}\|u\|_{r,0} + G_5 S^{-s}\|u\|_{0,s} + G_{11}\left\|\zeta^u_{S,S}\right\|_{L^2(O)} + G_{12}\left\|\tfrac{\partial^2 \zeta^u_{S,S}}{\partial x^2}\right\| + G_{13}\left\|\tfrac{\partial \zeta^u_{S,S}}{\partial x}\right\|_{L^2(O)} + \\
&G_{14}\left\|\zeta^u_{S,S}\right\|_{L^2(O)} + G_{15}\left\|\tfrac{\partial^4 \zeta^u_{S,S}}{\partial x^4}\right\|_{L^2(O)} \leq G_4 S^{-r}\|u\|_{r,0} + G_5 S^{-s}\|u\|_{0,s} G G_4 S^{-r}\|u\|_{r,0} + \\
&G G_5 S^{-s}\|u\|_{0,s} + G G_6 \left\|S^u_{S,S}\right\|_{2,0} + G G_r \left\|\zeta^u_{S,S}\right\|_{1,0} + G G_8 \left\|\zeta^u_{S,S}\right\|_{4,0}
\end{aligned} \tag{5.16}$$

where $M_1 = max(b(x)), M_2 = max(c(x)), M_3 = max(e(x)), M_4 = max(a(x))$. According to (5.1) , we can conclude that for sufficiently large S

$$\left\|\zeta^u_{S,S}\right\|_{L^2(\mathrm{O})} \to 0. \tag{5.17}$$

The above result indicate that the convergence rate of the proposed method depends on how many times $u(x,t)$ is differentiable with respect to x and t.

□

6. Numerical illustrations

Example 6.1. Consider time fractional parabolic partial differential problem with variable coefficients

$$\begin{aligned}
&\tfrac{\partial^\beta u(x,t)}{\partial t^\beta} = \tfrac{x\partial^2 u(x,t)}{\partial x^2} - \sin(x)\tfrac{\partial u(x,t)}{\partial x} - x^2 u(x,t) + \\
&\tfrac{2t^{2-\beta}}{\Gamma(3-\beta)}(1-x)\sin(x) + t^2[(1-x)\sin(x)(x+\cos(x)+x^2) + \\
&2x\cos(x) - \sin^2(x)], (0 < x < 1, 0 < t < 1), \\
&u(x,0) = u_t(x,0) = 0, 0 \leq x \leq 1, \\
&u(0,t) = u(1,t) = 0, 0 \leq t \leq 1.
\end{aligned} \tag{6.1}$$

Exact solution of this problem is $u(x,t) = t^2(1-x)sin(x)$. This problem has been solved by our method and L^∞ errors in the solution of $t = 1, \beta = 0.3, 0.5, 0.8$ and $S = 6$ has been tabulated in Table 1. According to this results, we can confirm that Legendre tau method is valid and acceptable.

In Table 1, we illustrate comparison between obtained results of present scheme with Finite difference method [32]. The results of finite difference method [32] are obtained in Table 1 by solving linear system of (17×17), although our results are achived by solving a system of (7×7). This table show that the results of our method is more accurate than the proposed method in [32]. Also, we can conclude that for obtaining accurate and stable results even small values of S is sufficient.

Table 1. L^∞ errors for solution of equation 6.1 at $t = 1$.

β	L^∞ of our spectral method for $S = 6$	Method [32] in the $M = N = 16$
0.3	4.8161×10^{-6}	4.3279×10^{-4}
0.5	1.3468×10^{-5}	3.7417×10^{-4}
0.8	1.5167×10^{-5}	3.855×10^{-4}

Also this problem has been solved by our method and L^∞ errors in the solution of $t = 1, \beta = 0.3, 0.5, 0.8$ and values of $S = 4, 6, 8, 10, 12, 14$ has been tabulated in Table 2. According to this results, we can confirm that Legendre tau method is valid and acceptable.

TABLE 2. The L^∞ errors at $t = 1$ for 6.1

S	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.8$
4	1.2345×10^{-4}	1.5303×10^{-4}	1.5901×10^{-4}
6	4.8161×10^{-6}	1.3468×10^{-5}	1.5167×10^{-5}
8	1.6419×10^{-6}	4.8971×10^{-6}	5.5087×10^{-6}
10	7.1374×10^{-7}	2.1747×10^{-6}	2.4455×10^{-6}
12	3.5480×10^{-7}	1.0954×10^{-6}	1.2360×10^{-6}
14	3.5480×10^{-7}	6.0466×10^{-7}	1.2360×10^{-6}

Example 6.2. Consider the following problem time fractional Fokker-Planck equation in [56]

$$\begin{cases} {}_0^cD_t^\beta u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + (1+11x)\frac{\partial u(x,t)}{\partial x} + 11u(x,t) + z(x,t), \\ u(x,0) = x(2-x), x \in (0,1), \\ u(0,t) = 0, u(1,t) = 1 + t^{3+\beta}, t \in (0,1], \end{cases} \tag{6.2}$$

where

$$z(x,t) = \frac{\Gamma(4+\beta)}{\Gamma(4)} t^3(2x - x^2) + 2(1 + t^{3+\beta}) - (1 + t^{3+\beta})(2 - 2x)(1 + 11x) - 11(1 + t^{3+\beta})(2x - x^2).$$

The exact solution of this problem is given by $u(x,t) = (1 + t^{3+\beta})(2x - x^2)$.

TABLE 3. Comparing the L^2 errors for different values of S at $t = 1$ of present scheme with method in [56] for Example 6.2

S	Δt	$\beta = \frac{5}{10}$	$\beta = \frac{5}{10}$ [56]	$\beta = \frac{8}{10}$	$\beta = \frac{8}{10}$ [56]
6	$\frac{1}{10}$	1.4675×10^{-5}	7.1183×10^{-3}	3.2661×10^{-6}	1.0741×10^{-2}
7	$\frac{1}{20}$	2.1108×10^{-5}	1.8086×10^{-3}	9.2258×10^{-6}	2.7135×10^{-3}
8	$\frac{1}{40}$	1.3927×10^{-6}	4.5593×10^{-4}	7.8968×10^{-7}	6.8238×10^{-4}
9	$\frac{1}{80}$	2.4656×10^{-6}	1.1434×10^{-4}	9.4116×10^{-7}	1.7110×10^{-4}
10	$\frac{1}{160}$	19057×10^{-7}	2.8455×10^{-5}	1.6558×10^{-7}	4.2699×10^{-5}

In Table 3, we make a comparison of the introduced scheme at various values of S with the presented scheme in [56]. Furthermore the results of a high-order compact difference method [56] by solving linear system of (101×101) whereas our results are achieved by solving a system of (6×6). We see that, the present method is more accurate than presented method in ([56]). Also, for obtaining good results even small choices of S are needed.

Example 6.3. Consider the following problem time fractional Fokker-Planck equation in [56]

$$\begin{cases} {}_0^cD_t^\beta u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + e^{(x-0.5)^2}\frac{\partial u(x,t)}{\partial x} + 2(x-0.5)e^{(x-0.5)^2}u(x,t) + z(x,t) \\ u(x,0) = 0, x \in [0,1], \\ u(0,t) = u(1,t) = 0, t \in (0,1], \end{cases} \tag{6.3}$$

where

$$z(x,t) = \frac{\Gamma(3)}{\Gamma(3-\beta)}t^{2-\beta}\sin(\pi x) - t^2 e^{(x-0.5)^2}(2(x-0.5)\sin(\pi x) + \pi\cos(\pi x)) + \pi^2 t^2 \sin(\pi x).$$

The exact solution of this problem is given by $u(x,t) = t^2 sin(\pi x)$.

TABLE 4. Comparing the L^2 errors for different values of S at $t = 1$ of present scheme with method in [56] for Example 6.3

S	Δt	$\beta = \frac{2}{10}$	$\beta = \frac{2}{10}$ [56]	$\beta = \frac{5}{10}$	$\beta = \frac{5}{10}$ [56]
6	$\frac{1}{10}$	2.3629×10^{-5}	6.031×10^{-4}	1.4675×10^{-5}	1.2542×10^{-3}
7	$\frac{1}{20}$	1.7121×10^{-5}	1.4752×10^{-4}	2.1108×10^{-5}	3.1291×10^{-4}
8	$\frac{1}{40}$	3.5787×10^{-6}	3.6631×10^{-5}	1.3927×10^{-6}	7.8234×10^{-5}
9	$\frac{1}{80}$	2.2933×10^{-6}	9.1544×10^{-6}	2.4656×10^{-6}	1.9558×10^{-5}
10	$\frac{1}{160}$	537.80×10^{-7}	2.2870×10^{-6}	1.9057×10^{-7}	4.8880×10^{-6}

In Table 4, we make a comparison of the introduced scheme at various values of S with the presented scheme in [56]. We see that, the present method is more accurate than presented method in ([56]). Also, for obtaining good results even small choices of S are needed.

Example 6.4. In this example, we consider fourth order time fractional parabolic partial differential equation with variable coefficents

$$\begin{aligned} &D_t^\beta u(x,t) = \frac{x\partial^2 u(x,t)}{\partial x^2} - \sin(x)\frac{\partial u(x,t)}{\partial x} - x^2 u(x,t) - x^3\frac{\partial^4 u(x,t)}{\partial x^4} + \\ &\frac{2t^{2-\beta}}{\Gamma(3-\beta)}(1-x)\sin(x) + t^2x^3(4\cos(x) + sin(x) - x\sin(x)) + \\ &t^2(2x\cos(x) + (1-x)(x + x^2 + \cos(x) + \sin(x) - \sin^2(x)), \end{aligned} \tag{6.4}$$

where

$$\begin{aligned} &u(x,0) = 0, \\ &u_t(x,0) = 0, \end{aligned} \tag{6.5}$$

and boundary conditions

$$\begin{aligned} &u(0,t) = 0, \\ &u(\ell,t) = 0, \\ &u_{xx}(0,t) = -2t^2, \\ &u_{xx}(\ell,t) = -2cos(1)t^2. \end{aligned} \tag{6.6}$$

The exact solution of this example is $t^2(1-x)sin(x)$.

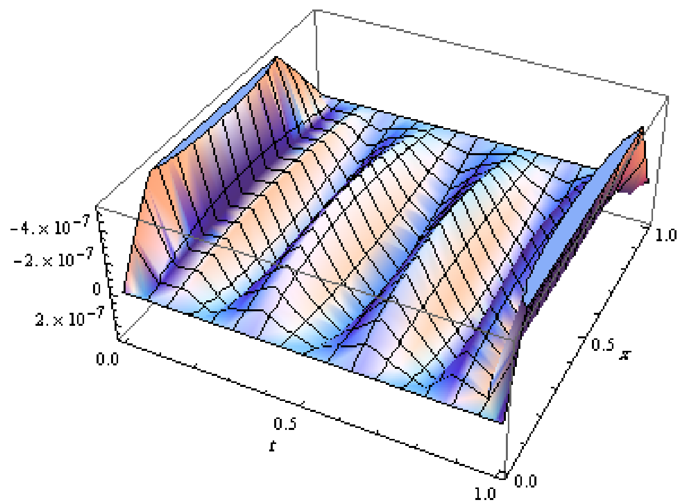

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FIGURE 1. Error function at $\beta = 0.9, S = 10$ for 6.4

We see from Fig. 1 that the error function at $\beta = 0.9, S = 10$ for 6.4.

TABLE 5. The L^∞ errors for different values of S at $t = 1$ for example 6.4

S	$\beta = \frac{3}{10}$	$\beta = \frac{7}{10}$	$\beta = \frac{9}{10}$
4	8.4082×10^{-3}	8.3261×10^{-3}	8.2759×10^{-3}
5	1.0978×10^{-3}	1.0772×10^{-3}	1.0456×10^{-3}
6	2.9635×10^{-5}	1.6179×10^{-5}	2.1504×10^{-5}
7	7.5408×10^{-7}	7.9073×10^{-6}	3.2674×10^{-6}
8	1.3664×10^{-6}	6.3069×10^{-6}	3.2574×10^{-6}
9	6.01152×10^{-7}	4.0993×10^{-6}	2.1006×10^{-6}
10	2.5647×10^{-7}	2.7833×10^{-6}	1.4324×10^{-6}
11	4.2383×10^{-7}	1.9519×10^{-6}	1.0122×10^{-6}
12	3.06027×10^{-7}	1.4063×10^{-6}	7.3572×10^{-7}

This example has been solved by our method with different values of $S = 4, 5, 6, \ldots, 12$.

Table 6 illustrates the L^2 errors by applying tau spectral scheme for $\beta = 0.3, 0.7, 0.9$ and different values of S at $t = 1$. The reported results are accurate and stable. According to the reported results in Table 7, the effectiveness, appropriateness and high validity of the presented method has been obseved.

Example 6.5. Consider the following problem :

$$D_t^\beta u(x,t) + u_t(x,t) = x\frac{\partial^2 u(x,t)}{\partial x^2} - \cos(x)\frac{\partial u(x,t)}{\partial x} - x^2 u(x,t) - \sin(x)\frac{\partial^4 u(x,t)}{\partial x^4} + z(x,t) \tag{6.7}$$

TABLE 6. The L^2 errors for different values of S at $t = 1$ for example 6.4

S	$\beta = \frac{3}{10}$	$\beta = \frac{7}{10}$	$\beta = \frac{9}{10}$
4	1.8766×10^{-1}	1.8587×10^{-1}	1.8457×10^{-1}
5	2.4160×10^{-2}	2.3687×10^{-2}	2.2962×10^{-2}
6	6.5013×10^{-4}	3.1794×10^{-4}	4.5223×10^{-2}
7	1.1807×10^{-5}	1.7476×10^{-4}	1.5209×10^{-4}
8	3.08814×10^{-5}	1.4245×10^{-4}	7.04409×10^{-5}
9	1.3590×10^{-5}	9.2666×10^{-5}	7.3662×10^{-5}
10	2.5647×10^{-5}	6.2921×10^{-5}	4.7500×10^{-5}
11	9.5847×10^{-6}	4.4130×10^{-5}	2.2872×10^{-5}
12	6.9235×10^{-6}	3.18027×10^{-5}	1.6622×10^{-5}

TABLE 7. The absolute errors for different values of β, x, t at $S = 8$ for example 6.4

(x,t)	$\beta = \frac{3}{10}$	$\beta = \frac{5}{10}$	$\beta = \frac{7}{10}$
$(0.1, 0.1)$	7.9757×10^{-9}	2.6687×10^{-8}	4.0596×10^{-8}
$(0.2, 0.2)$	4.40011×10^{-8}	1.5217×10^{-7}	2.9330×10^{-7}
$(0.3, 0.3)$	1.0619×10^{-7}	3.85333×10^{-7}	7.0433×10^{-7}
$(0.4, 0.4)$	3.7096×10^{-8}	1.0681×10^{-7}	1.4709×10^{-7}
$(0.5, 0.5)$	1.2739×10^{-7}	4.2178×10^{-7}	7.7305×10^{-7}
$(0.6, 0.6)$	5.8552×10^{-8}	2.4012×10^{-7}	3.9543×10^{-7}
$(0.7, 0.7)$	5.6652×10^{-8}	2.4882×10^{-7}	5.0197×10^{-7}
$(0.8, 0.8)$	9.5847×10^{-6}	3.487×10^{-7}	2.8051×10^{-7}
$(0.9, 0.9)$	2.5647×10^{-7}	1.3178×10^{-7}	1.8608×10^{-7}

where

$$z(x,t) = \frac{6t^{3-\beta}}{\Gamma(4-\beta)} + 3t^{2} e^{x} + e^{x} t^{3}(-x + \cos(x) + x^{2} + \sin(x)).$$

The initial conditions are

$$u(x,0) = 0, u_t(x,0) = 0, \tag{6.8}$$

and the boundary conditions are

$$u(0,t) = t^3, u(1,t) = et^3, \qquad u_{xx}(0,t) = t^3, u_{xx}(1,t) = et^3. \tag{6.9}$$

TABLE 8. The L^{∞}errors for different values of β, S at $t = 1$ for example 6.5

S	$\beta = \frac{1}{4}$	$\beta = \frac{1}{2}$	$\beta = \frac{3}{4}$
3	2.1982×10^{-2}	2.1982×10^{-2}	2.1982×10^{-2}
6	1.4410×10^{-6}	1.7595×10^{-6}	4.0997×10^{-6}
9	8.2491×10^{-8}	2.0718×10^{-7}	5.9067×10^{-7}
12	2.6462×10^{-9}	2.9504×10^{-8}	1.0482×10^{-7}
15	4.2767×10^{-10}	5.9511×10^{-9}	2.9504×10^{-8}

In Table8, the absolute errors have been listed for $\beta = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and $S = 3, 6, 9, 12, 15$ at $t = 1$. From this results, we can conclude that this method is sable and acceptable and accurate.

Table9 shows the L^2 errors for $\beta = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and $S = 3, 6, 9, 12, 15$ at $t = 1$.

TABLE 9. The L^2 errors for different values of β, S at $t = 1$ for example 6.5

S	$\beta = \frac{1}{4}$	$\beta = \frac{1}{2}$	$\beta = \frac{3}{4}$
3	4.9345×10^{-1}	2.1982×10^{-2}	4.9345×10^{-1}
6	2.3566×10^{-5}	3.201×10^{-5}	9.5237×10^{-5}
9	1.8586×10^{-6}	4.7297×10^{-6}	1.3521×10^{-5}
12	6.1480×10^{-8}	6.8382×10^{-7}	2.4390×10^{-6}
15	1.0171×10^{-8}	1.4053×10^{-7}	6.8382×10^{-7}

Example 6.6. In order to make some numerical comparison of tau method with the compact finite difference scheme [72] and finite difference scheme given in [41]. This problem is governed by the equation 1.1 with the boundary and initial conditions in the domain $[0, 1] \times [0, 1]$, where $\alpha_2 = \alpha_1 = b(x) = c(x) = 1$ and $a(x) = e(x) = 0$

$$
\begin{aligned}
&z(x,t) = 3t^2\left(1 + \tfrac{2t^{1-\beta}}{\Gamma(4-\beta)}\right)e^x, \\
&u(0,t) = t^3, u(1,t) = et^3, t \in (0,1], \\
&u(x,0) = 0, u_t(x,0) = 0, x \in [0,1],
\end{aligned}
\tag{6.10}
$$

The exact analytical solution is given by $u(x, t) = t^3 e^x$.

TABLE 10. L^∞ errors for different vlues of S Example at $t = 1$ for example 6.6

S	L^∞ of spectral method	Δt	$L^\infty$of CFDS [72], $h = \Delta t^{\frac{3-\beta}{4}}$	L^∞ of FDS [41]
4	2.5935×10^{-4}	$\frac{1}{5}$	4.2933×10^{-3}	1.0770×10^{-1}
5	5.5871×10^{-5}	$\frac{1}{10}$	1.2343×10^{-3}	5.5461×10^{-2}
6	2.4610×10^{-5}	$\frac{1}{20}$	1.2343×10^{-3}	5.5461×10^{-2}
7	1.2169×10^{-5}	$\frac{1}{40}$	3.5439×10^{-4}	2.8107×10^{-2}
8	6.5619×10^{-6}	$\frac{1}{80}$	1.0501×10^{-4}	1.4146×10^{-2}
9	8.264×10^{-6}	$\frac{1}{160}$	3.0381×10^{-5}	7.0916×10^{-3}
10	2.7887×10^{-6}	$\frac{1}{320}$	8.80021×10^{-6}	8.80021×10^{-3}

Regarding problem (5.2), in [72], in Table 10 the best result is achieved at $t = 1, L = 1$ and $\beta = \frac{5}{4}$ with $\Delta t = \frac{1}{320}, h = \Delta t^{\frac{3-\beta}{4}}$, 8.80021×10^{-6}. The L^∞ error is 2.7887×10^{-6} for $S = 10$.

Table 12 indicates that the best result for problem (5.2) in [72] is obtained at $t = 1, L = 1$ and $\beta = \frac{7}{4}$ with $\Delta t = \frac{1}{320}, h = \Delta t^{\frac{3-\beta}{4}}$, 4.3076×10^{-4}. The L^∞ error is 1.5920×10^{-5} for $S = 10$.

TABLE 11. L^∞ errors of various choices of S at $t = 1$ for Example 6.6

S	L^∞ of spectral method	Δt	L^∞ of CFDS [72] , $h = \Delta t^{\frac{3-\beta}{4}}$	L^∞ of FDS [41]
4	3.9208×10^{-4}	$\frac{1}{5}$	3.3451×10^{-2}	2.3124×10^{-1}
5	1.3332×10^{-4}	$\frac{1}{10}$	1.1268×10^{-2}	1.2214×10^{-1}
6	5.7607×10^{-5}	$\frac{1}{20}$	3.9164×10^{-3}	6.2310×10^{-2}
7	2.9562×10^{-5}	$\frac{1}{40}$	1.3546×10^{-3}	3.1320×10^{-2}
8	1.5920×10^{-5}	$\frac{1}{80}$	4.7160×10^{-4}	1.5621×10^{-2}
9	9.4946×10^{-6}	$\frac{1}{160}$	1.6910×10^{-4}	7.8014×10^{-3}
10	7.4056×10^{-6}	$\frac{1}{320}$	5.9194×10^{-5}	3.8877×10^{-3}

TABLE 12. L^∞ errors of various choices of S at $t = 1$for Example 6.6

S	L^∞ of spectral method	Δt	L^∞ of CFDS [72], $h = \Delta t^{\frac{3-\beta}{4}}$	L^∞ of FDS [41]
4	3.3713×10^{-4}	$\frac{1}{5}$	7.7052×10^{-2}	2.8182×10^{-1}
5	1.3826×10^{-4}	$\frac{1}{10}$	3.1823×10^{-2}	1.5058×10^{-1}
6	6.6585×10^{-5}	$\frac{1}{20}$	1.3339×10^{-2}	7.6981×10^{-2}
7	3.7787×10^{-5}	$\frac{1}{40}$	5.7027×10^{-3}	3.8569×10^{-2}
8	1.9981×10^{-5}	$\frac{1}{80}$	2.3903×10^{-3}	1.9107×10^{-2}
9	1.5920×10^{-5}	$\frac{1}{160}$	1.0222×10^{-3}	9.4327×10^{-3}
10	1.5920×10^{-5}	$\frac{1}{320}$	4.3076×10^{-4}	4.6528×10^{-3}

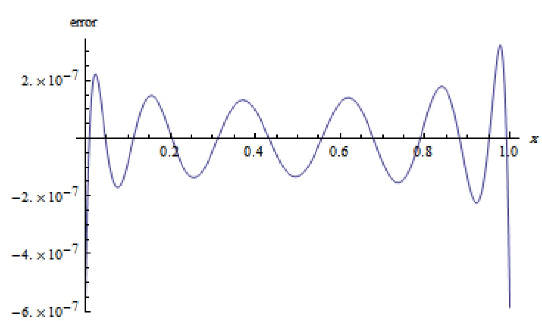

FIGURE 2. Errors at $t = 0.8, S = 12$ and $\beta = 1.5$ for Example 6.6

According to problem (5.2), in [72], Table 11 shows that the best result is obtained at $t = 1, L = 1$ and $\beta = \frac{3}{2}$ with $\Delta t = \frac{1}{320}, h = \Delta t^{\frac{3-\beta}{4}}$, 5.9194×10^{-5}. The L^∞ error is 7.4056×10^{-6} for $S = 10$.

TABLE 13. L^∞ errors of different values of S at $t = 1$ for Example 6.6

S	L^∞ of spectral method	h	L^∞ of CFDS [72] ,$\Delta T = \frac{4}{3-\beta}$	L^∞ of FDS [41]
4	4.0100×10^{-5}	$\frac{\pi}{4}$	9.0363×10^{-4}	2.3767×10^{-1}
8	1.0518×10^{-6}	$\frac{\pi}{8}$	5.3626×10^{-5}	1.31825×10^{-1}
12	1.2017×10^{-7}	$\frac{\pi}{16}$	3.2122×10^{-6}	6.8692×10^{-2}
16	2.4584×10^{-8}	$\frac{\pi}{32}$	9398×10^{-7}	3.5036×10^{-2}
18	1.2693×10^{-8}	$\frac{\pi}{64}$	1.1843×10^{-8}	1.7656×10^{-2}

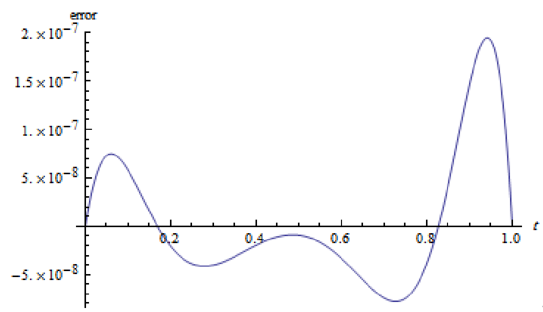

FIGURE 3. Errors at $x = 0.1$ and $S = 12$ and $\beta = 1.5$ for Example 6.6

Fig. 3 shows the error function $u(0.1, t) - u_{12,12}(0.1, t)$ with $\beta = 1.5$.

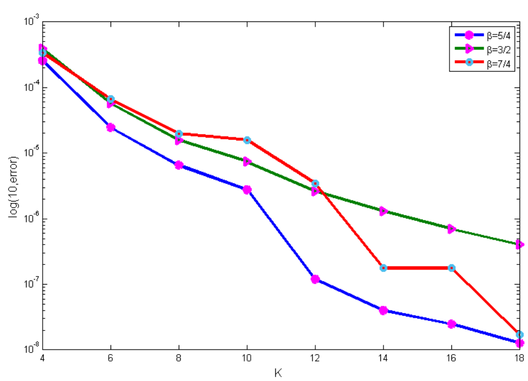

FIGURE 4. Convergence of problem 6.6 at $\beta = 1.25, 1.5, 1.75, t = 1$

The logarithmic graphs of MAEs (log_{10}Error) at $\tau = 1, t = 1, \beta = 1.25, 1.5, 1.75$ with different choices of S is illustrated in Fig. 4. We conclude that, by using our scheme that the numerical solution is more precise when β tends to its integer values and numerical errors for all values of β decrease rapidly as Ss increase.

We compare L^∞ errors in the solution of our method with compact finite difference scheme [72] and finite difference scheme given in [41] and the errors in the solution tabulated in Tables 10, 11 and 12 at $t = 1, \beta = 1.25, 1.5, 1.75$ and different values of $S = 4, 5, 6, 7, 8, 9, 10$. These tables show that, the result of our presented method are precise and acceptable even with small values of S. Also, in table 11, L^∞, L^2 and L^1-errors at $t = 0.5, \beta = 1.5$ and various choices of $S = 4(2)18$ is demonstrated. Results of this table shows that perfect agreement with what was expected for a spectral method. Also, these results illustrate the Legendre-tau method can converge reasonably, the precision and validity can be achieved. As well as, the scheme is effective and useful.

7. Conclusions

In this paper, the tau spectral scheme together with shifted Legendre polynomials for solving a class of fourth order time-fractional parabolic partial differential equations with variable coefficients has been developed. Convergence of tau method is discussed. The comparison of numerical results with the results of some well-known schemes in the literature illustrates the validity, high precision and efficiency of this method. The numerical approach in this paper can be extended to other fractional partial differential equations with the Caputo- fractional partial derivative.

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