Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 10, No. 2, 2022, pp. 431-444
DOI:10.22034/cmde.2020.42106.1818

Numerical solution of space fractional diffusion equation using shifted Gegenbauer polynomials

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Abstract

This paper is concerned with numerical approach for solving space fractional diffusion equation using shifted Gegenbauer polynomials, where the fractional derivatives are expressed in Caputo sense. The properties of Gegenbauer polynomials are exploited to reduce space fractional diffusion equation to a system of ordinary differential equations, that are then solved using finite difference method. Some selected numerical simulations of space fractional diffusion equations are presented and the results are compared with the exact solution, also with the results obtained via other methods in the literature. The comparison reveals that the proposed method is reliable, effective and accurate. All the computations were carried out using Matlab package.

Keywords. Gegenbauer polynomial, Caputo derivative, Fractional diffusion equation, Finite difference method.

2010 Mathematics Subject Classification. 35R11, 41A10, 65D05, 65M06, 65N06.

1. Introduction

In the recent past, considerable attention has been given to Fractional Differential Equations (FDEs) due to their frequent appearance in various applications in fluid mechanics, physics, biology and engineering [1, 20, 26, 34]. One of the advantages of fractional order derivatives over classical integer order model is that, it provides an excellent tool for the description of memory and hereditary properties of many materials and processes [9–11]. Fractional diffusion equation is one of the powerful concepts in the initial and boundary value problems and have been used in groundwater contaminant transport [6], modeling turbulent flow [8], applications in biology [14], and its applications can also be found in other fields [7, 23, 35]. In some instances where general integer order partial differential equation (PDE) is used in the prediction of COVID cases [31, 32] and creating a model of the shape of COVID-19 [4], fractional order PDE has been reported to give more accurate and reliable results in complex cases [5].

Mathematical aspects of the boundary-value problems for the time-fractional diffusion equation and their applications in physics have been treated in [13, 15]. On the other hand, the space fractional diffusion equation is obtained by replacing the second order space derivative in the diffusion equation by an inverse Riesz potential of order $\beta > 0$ [31]. A further generalization of the classical diffusion equation is the so-called space-time fractional diffusion equation (SFDE), where the first order time derivative is replaced with a Caputo-time fractional derivative of order $0 < \beta < 2$ and second order space derivative with Riesz-Feller space-fractional derivative of order $0 < \alpha < 2$ (see [16] and the references therein).

The present work is focused on the numerical solution of space fractional diffusion equation through exploiting the accuracy of Gegenbauer polynomials, and the fact that its solution generalizes the results obtained through the use of Legendre polynomials and Chebyshev polynomials of certain kinds. Many scientists that used other orthogonal

Received: 04 October 2020 ; Accepted: 30 December 2020.

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polynomials such as the four kinds of Chebyshev polynomials and their shifted forms include [3, 28, 29, 33], modified Legendre collocation method was reported in [19] and shifted Gegenbauer-Gauss collocation method for solving fractional-differential equations with proportional delay was reported in [17]. In this work, Gegenbauer polynomials together with the fractional derivative interpreted in Caputo sense is used to transform SFDE into a system of first order differential equations that are reduced to linear algebraic equations using finite difference method (FDM). The resulting algebraic equations are then solved for the unknown constants.
The remaining part of this paper is organized as follows. An overview of some orthogonal polynomials is discussed in section 2 (see 2.1, 2.2 and 2.3). The mathematical formulation of the scheme for the space finite fractional differential equation is presented in section 3, as part of section 3, we present properties of shifted Gegenbauer polynomials in 3.2, shifted Gegenbauer finite difference method in 3.3, we present the scheme used in solving the class of problem in sections 3.4, while the convergence and stability of the proposed method is discussed in 3.5. In section 4, numerical experiments are presented with the tables of results which are equally illustrated with $2D$ and $3D$ graphs to show the efficiency and accuracy of the method. Finally, in section 5, we discuss the results with concluding remarks.

2. Overview of some orthogonal polynomials

A system of real functions $\{\varphi_n(x)\},\ n = 0, 1, \cdots$ is said to be **orthogonal** with respect to the weight function $w(x)$ over $[a,\ b]$, if

$$\langle \varphi_i(x), \varphi_j(x)\rangle = \int_a^b \varphi_i(x)\varphi_j(x)w(x)dx = \begin{cases} 0, & i \neq j \\ \lambda_n, & i = j \end{cases}, \tag{2.1}$$

where

$$\varphi_i(x) = \sum_{r=0}^{i} a_r x^r. \tag{2.2}$$

Some of these orthogonal polynomials $\varphi_i(x)$ are discussed in the sequel.

2.1. **Chebyshev polynomials.** The four prominent kinds of Chebyshev polynomials are listed below with their corresponding weight functions $w(x)$ in the interval $[-1,\ 1]$ are found in [22] as:

$$\varphi_j = \begin{cases} T_j = \cos(jx),\ w(x) = \dfrac{1}{\sqrt{1-x^2}}, \\[2ex] U_j = \dfrac{\sin(j+1)\,x}{\sin(x)},\ w(x) = \sqrt{1-x^2}, \\[2ex] V_j = \dfrac{\cos\left(j+\frac{1}{2}\right)x}{\cos\left(\frac{x}{2}\right)},\ w(x) = \sqrt{\dfrac{1+x}{1-x}}, \\[2ex] W_j = \dfrac{\sin\left(j+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)},\ w(x) = \sqrt{\dfrac{1-x}{1+x}}, \end{cases} \tag{2.3}$$

The Chebyshev polynomials of the third and fourth kinds are orthogonal in the interval $[-1,\ 1]$, if

$$\langle \varphi_i, \varphi_j\rangle = \begin{cases} \langle V_i, V_j\rangle = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} V_i(x)V_j(x)dx = \begin{cases} 0, & i \neq j \\ \pi, & i = j \end{cases}, \\[3ex] \langle W_i, W_j\rangle = \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} W_i(x)W_j(x)dx = \begin{cases} 0, & i \neq j \\ \pi, & i = j \end{cases}, \end{cases} \tag{2.4}$$

The fifth kind of Chebyshev polynomial was reported in [2, 24]

2.2. **Legendre polynomials** $P_j(x)$**.** These are also orthogonal polynomials with weight function $w(x) = 1$ and recurrence relation

$$P_{j+1}(x) = \frac{2j+1}{j+1} x P_j(x) - \frac{j}{j+1} P_{j-1}(x),\ j \geq 1 \text{ where } P_0(x) = 1,\ P_1(x) = x, \tag{2.5}$$

see [21, 27] for details.

2.3. **Jacobi polynomials.** The well-known Jacobi polynomials $P_j^{(\alpha,\beta)}(x),\ j = 0,\ 1, \ldots,$ with parameters $\alpha,\ \beta > -1$ and weight function $w(x) = (1-x)^\alpha (1+x)^\beta$, the explicit form of which was used in [12, 30] takes the form

$$P_j^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+j+1)}{j!\Gamma(\alpha+\beta+j+1)} \sum_{i=0}^{j} \frac{j!\Gamma(\alpha+\beta+i+j+1)}{(j-i)!i!\Gamma(\alpha+i+1)} \left(\frac{x-1}{2}\right)^i, \tag{2.6}$$

where $P_0^{(\alpha,\beta)}(x) = 1,\ P_1^{(\alpha,\beta)}(x) = (\alpha+1) + (\alpha+\beta+2)\left(\frac{x-1}{2}\right)$. When $\alpha = \beta = 0$, then its reduces to Legendre polynomials ,while choosing $\alpha = \beta = \frac{1}{2}$ gives Chebyshev polynomials of second kind and also when $\alpha = \beta = 1$ gives Gegenbauer polynomials of the form $C_j^{(\frac{3}{2})}(x)$, that is $P_j^{(1,1)}(x) = C_j^{(\frac{3}{2})}(x)$.

Another orthogonal polynomial of importance in the present work is Gegenbauer polynomials $C_n^{(\alpha)}(x)$, which shall later be discussed in this paper.

3. Mathematical formulation of the scheme for SFDE

3.1. **Definition:** The Caputo fractional-order derivative operator D^ν of order ν is defined in [25] as:

$$D^\nu f(x) = \frac{1}{\Gamma(n-\nu)} \int_0^x \frac{f^n(t)}{(x-t)^{\nu+1-n}} dt,\ n-1 < \nu \leq n,\ n \in N, \tag{3.1}$$

where $\nu > 0$ is the order of the derivative. The linearity property of the Caputo fractional differentiation is demonstrated by the following theorem.

Theorem 3.1. *Let $g(x)$ and $h(x)$ be two functions defined on $[a,\ b]$ such that $D_a^\nu g(x)$ and $D_a^\nu h(x)$ exist almost everywhere. Also, let μ and $\lambda \in \mathbb{R}$. Then, $D_a^\nu (\mu g(x) + \lambda h(x))$ exists almost everywhere, and*

$$D_a^\nu (\mu g(x) + \lambda h(x)) = \mu D_a^\nu g(x) + \lambda D_a^\nu h(x) \tag{3.2}$$

The proofing of the theorem is contained in [25].

$$D^\nu S = 0,\ \text{S is a constant}$$

$$D^\nu x^i = \begin{cases} 0,\ \text{for } i \in \mathbb{N}_0 \text{ and } i < \lceil \nu \rceil \\ \\ \frac{\Gamma(i+1)}{\Gamma(i+1-\nu)} x^{i-\nu},\ \text{for } i \in \mathbb{N}_0 \text{ and } i \geq \lceil \nu \rceil \end{cases}. \tag{3.3}$$

The function $\lceil \nu \rceil$ denotes smallest integer greater than or equal to ν, also $\mathbb{N} = \{1,\ 2\ \ldots\}$ and $\mathbb{N}_0 = \{0,\ 1,\ 2\ \ldots\}$.
The main aim of this paper is to find another elegant approximate solution of space fractional order diffusion equation based on the shifted Gegenbauer polynomials for one-dimensional space fractional diffusion equation of the form:

$$\frac{\partial u(x,t)}{\partial t} = a(x) \frac{\partial^\nu u(x,t)}{\partial x^\nu} + b(x,t),\ 0 < x < \zeta,\ 0 \leq t \leq \tau, 1 < \nu \leq 2, \tag{3.4}$$

with initial condition

$$u(x,0) = g(x),\ 0 < x < \zeta \tag{3.5}$$

and boundary conditions

$$\begin{aligned} u(0,t) &= \gamma_0(t),\ 0 \leq t \leq \tau \\ u(\zeta,t) &= \gamma_1(t),\ 0 \leq t \leq \tau. \end{aligned} \tag{3.6}$$

Equation (3.4) becomes classical diffusion equation when $\nu = 2$, that is:

$$\frac{\partial u(x,t)}{\partial t} = a(x)\frac{\partial^2 u(x,t)}{\partial x^2} + b(x,t). \tag{3.7}$$

The existence and uniqueness of solution of Eqs. (3.4-3.6) had earlier been discussed in [14].

3.2. **Properties of shifted Gegenbauer polynomials.** Gegenbauer polynomials $C_n^{(\alpha)}(z)$ is an orthogonal polynomials of degree n in x defined on the interval $[-1,\ 1]$ with respect to weight function $w(z) = (1-z^2)^{(\alpha-\frac{1}{2})}$ and can be determined using

$$C_n^{(\alpha)}(z) = \sum_{k=0}^{n} \frac{(-1)^k \Gamma(2\alpha+2n-k)\Gamma(\alpha+\frac{1}{2})}{(n-k)!\Gamma(k+1)\Gamma(n-k+\alpha+\frac{1}{2})\Gamma(2\alpha)} z^{n-k}, \tag{3.8}$$

and its recurrence relation is

$$C_n^{(\alpha)}(z) = \frac{1}{n}\left[2(n+\alpha-1)zC_{n-1}^{(\alpha)}(z) - (n+2\alpha-2)C_{n-2}^{(\alpha)}(z)\right],\ n \geq 2, \tag{3.9}$$

where $C_0^{\alpha}(z) = 1,\ C_1^{\alpha}(z) = 2\alpha z$. To use the polynomials on the interval $x \in [a,\ b]$, we define the so-called shifted Gegenbauer polynomials by introducing the change of variable $z = \frac{2x-(a+b)}{b-a}$. The shifted Gegenbauer polynomials in x are therefore obtained as:

$$C_n^{(\alpha)*}(x) = \frac{1}{n}\left[2(n+\alpha-1)\left(\frac{2x-(a+b)}{b-a}\right)C_{n-1}^{(\alpha)*}(x) - (n+2\alpha-2)C_{n-2}^{(\alpha)*}(x)\right],\ n \geq 2, \tag{3.10}$$

similar to what was reported in [18]. Where $C_0^{(\alpha)*}(x) = 1,\ C_1^{(\alpha)*}(x) = 2\alpha\left(\frac{2x-(a+b)}{b-a}\right)$. The analytic form of the shifted Gegenbauer polynomials $C_n^{(\alpha)*}(x)$ of degree n on the interval $[0,\ 1]$ is given by:

$$C_n^{(\alpha)*}(x) = \sum_{k=0}^{n} \frac{(-1)^k \Gamma(2\alpha+2n-k)\Gamma(\alpha+\frac{1}{2})}{(n-k)!\Gamma(k+1)\Gamma(n-k+\alpha+\frac{1}{2})\Gamma(2\alpha)} x^{n-k}. \tag{3.11}$$

The orthogonality condition is

$$\langle C_m^{(\alpha)*}(x),\ C_n^{(\alpha)*}(x)\rangle = \int_0^1 \left(x-x^2\right)^{(\alpha-\frac{1}{2})} C_m^{(\alpha)*}(x)C_n^{(\alpha)*}(x)dx = \begin{cases} 0,\ \text{for } m \neq n \\ \frac{\pi 2^{1-4\alpha}\Gamma(n+2\alpha)}{n![\Gamma(\alpha)]^2(n+\alpha)},\ \text{for } m = n \end{cases} \tag{3.12}$$

Let $g(x)$ be a square integrable function on the interval $[0,\ 1]$ and can be expressed in terms of shifted Gegenbauer polynomials $C_n^{(\alpha)*}(x)$ as:

$$g(x) = \sum_{n=0}^{\infty} \beta_n C_n^{(\alpha)*}(x) \tag{3.13}$$

where the coefficients $\beta_n,\ n = 0,\ 1, \ldots$ are given by:

$$\beta_n = \frac{n!\left[\Gamma(\alpha)\right]^2(n+\alpha)}{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)} \int_{-1}^{1} \left(1-x^2\right)^{\alpha-\frac{1}{2}} g\left(\frac{x+1}{2}\right) C_n^{(\alpha)}(x)dx \tag{3.14}$$

or

$$\beta_n = \frac{n!\left[\Gamma(\alpha)\right]^2(n+\alpha)}{\pi 2^{1-4\alpha}\Gamma(n+2\alpha)} \int_{0}^{1} \left(x-x^2\right)^{\alpha-\frac{1}{2}} g(x)C_n^{(\alpha)*}(x)dx. \tag{3.15}$$

Practically, only the first $(N+1)-$terms of shifted Gegenbauer polynomials are considered in the approximation. Therefore, Eq. (3.13) now becomes

$$g(x) = \sum_{n=0}^{N} \beta_n C_n^{(\alpha)*}(x). \tag{3.16}$$

3.3. **Shifted Gegenbauer finite difference method.** Here, we derive an approximate formula for fractional derivative $g(x)$. Assume $\nu > 0$, now using Caputo linearity property, Eq. (3.16) becomes

$$D^{\nu} g_N(x) = \sum_{n=0}^{N} \beta_n D^{\nu} \left(C_n^{(\alpha)*}(x) \right). \tag{3.17}$$

Moreover, using properties of linearity of the Caputo derivative together with Eq. (3.3) we obtain:

$$D^{\nu} \left(C_n^{(\alpha)*}(x) \right) = 0,\ n = 0,\ 1, \dots, \lceil \nu \rceil - 1,\ \nu > 0 \tag{3.18}$$

$$D^{\nu} \left(C_n^{(\alpha)*}(x) \right) = \sum_{k=0}^{n} \frac{(-1)^k \Gamma(2\alpha + 2n - k)\Gamma(\alpha + \frac{1}{2})}{(n-k)!\Gamma(k+1)\Gamma(n-k+\alpha+\frac{1}{2})\Gamma(2\alpha)} D^{\nu} x^{n-k},\ n \geq \lceil \nu \rceil. \tag{3.19}$$

Applying Eq. (3.3) to Eq. (3.19), we obtain a similar result to that obtained in [12]

$$D^{\nu} \left(C_n^{(\alpha)*}(x) \right) = \sum_{k=0}^{n-\lceil \nu \rceil} \frac{(-1)^k \Gamma(2\alpha + 2n - k)\Gamma(\alpha + \frac{1}{2})}{\Gamma(k+1)\Gamma(n-k+\alpha+\frac{1}{2})\Gamma(2\alpha)\Gamma(n+1-k-\mu)} x^{n-k-\nu}. \tag{3.20}$$

Substituting Eq. (3.20) in Eq. (3.17) we obtain:

$$D^{\nu} \left(g_N(x) \right) = \sum_{n=\lceil \nu \rceil}^{N} \sum_{k=0}^{n-\lceil \nu \rceil} \beta_n \frac{(-1)^k \Gamma(2\alpha + 2n - k)\Gamma(\alpha + \frac{1}{2})}{\Gamma(k+1)\Gamma(n-k+\alpha+\frac{1}{2})\Gamma(2\alpha)\Gamma(n+1-k-\mu)} x^{n-k-\nu}. \tag{3.21}$$

Eq. (3.21) can be rewritten as:

$$D^{\nu} \left(g_N(x) \right) = \sum_{n=\lceil \nu \rceil}^{N} \sum_{k=0}^{n-\lceil \nu \rceil} \beta_n N_{n,k} x^{n-k-\nu}, \tag{3.22}$$

where

$$N_{n,k} = \frac{(-1)^k \Gamma(2\alpha + 2n - k)\Gamma(\alpha + \frac{1}{2})}{\Gamma(k+1)\Gamma(n-k+\alpha+\frac{1}{2})\Gamma(2\alpha)\Gamma(n+1-k-\mu)}, \tag{3.23}$$

and β_n is given by Eq. (3.15).

3.3.1. *Simple illustration:* Consider $g(x) = x^2$ with $N = 3$ and $\nu = 1.5$, we obtain from Eq. (3.3):

$$D^{1.5} x^2 = \frac{\Gamma(2+1)}{\Gamma(2+1-1.5)} x^{\frac{1}{2}} = \frac{4}{\Gamma\left(\frac{1}{2}\right)} x^{\frac{1}{2}} \tag{3.24}$$

Now, using the proposed method in Eq. (3.22) we get:

$$D^{1.5} x^2 = \beta_2 x^{\frac{1}{2}} N_{2,0} + \beta_3 \left(x^{\frac{3}{2}} N_{3,0} + x^{\frac{1}{2}} N_{3,1} \right) \tag{3.25}$$

The constants β_2 and β_3 can be determined using Eq. (3.14) or (3.15) and $N_{2,0}$, $N_{3,0}$ and $N_{3,1}$ from Eq. (3.23). For $\alpha = \frac{1}{2}$: $N_{2,0} = \frac{24}{\Gamma(\frac{1}{2})}$, $N_{3,0} = \frac{160}{\Gamma(\frac{1}{2})}$, $N_{3,1} = -\frac{120}{\Gamma(\frac{1}{2})}$, $\beta_2 = \frac{1}{6}$, $\beta_3 = 0$, then substitute in Eq.(3.25), we obtain $D^{1.5} x^2 = \frac{4}{\Gamma(\frac{1}{2})} x^{\frac{1}{2}}$.

For $\alpha = 1$: $N_{2,0} = \frac{64}{\Gamma(\frac{1}{2})}$, $N_{3,0} = \frac{512}{\Gamma(\frac{1}{2})}$, $N_{3,1} = -\frac{384}{\Gamma(\frac{1}{2})}$, $\beta_2 = \frac{1}{16}$, $\beta_3 = 0$, then substituting in Eq.(3.25), we obtain $D^{1.5} x^2 = \frac{4}{\Gamma(\frac{1}{2})} x^{\frac{1}{2}}$, therefore it goes for all values of α.

3.4. **Numerical scheme.** Considering Eq. (3.4) together with initial and boundary conditions given in Eqs. (3.5) and (3.6) respectively, the approximate solution $u_N(x,t)$ corresponding to the exact solution $u(x,t)$ using Gegenbauer collocation method is derived as follows:

$$u_N(x,t) = \sum_{n=0}^{N} u_n(t) C_n^{(\alpha)*}(x). \tag{3.26}$$

Substituting Eq.(3.26) in Eq. (3.4) and applying Eq.(3.22) on the fractional part, we obtain:

$$\sum_{n=0}^{N} \frac{du_n(t)}{dt} C_n^{(\alpha)*}(x) = a(x) \sum_{n=\lceil \nu \rceil}^{N} \sum_{k=0}^{n-\lceil \nu \rceil} u_n(t) N_{n,k}^{(\nu)} x^{n-k-\nu} + b(x,t). \tag{3.27}$$

Now, collocating Eq. (3.27) at $x = x_i,\ i = 0,\ 1, \ldots, (N+1-\lceil \nu \rceil)$ using the roots of shifted Gegenbauer polynomials $C_{m+1-\lceil \nu \rceil}^{(\alpha)*}(x)$ we obtain:

$$\sum_{n=0}^{N} \frac{du_n(t)}{dt} C_n^{(\alpha)*}(x_i) = a(x_i) \sum_{n=\lceil \nu \rceil}^{N} \sum_{k=0}^{n-\lceil \nu \rceil} u_n(t) N_{n,k}^{(\nu)} x_i^{n-k-\nu} + b(x_i,t), \tag{3.28}$$

Moreover, Eq. (3.15) is used to determine the constants $u_n(t_0)$ in the initial case at ($t = 0$ that is at t_0), also we substitute the boundary conditions given in Eq. (3.6) and obtain $\lceil \nu \rceil$ equations. The boundary conditions given in Eqs. (3.6) in the range $0 < x < 1$ are:

$$\sum_{n=0}^{N} \frac{(-1)^n \Gamma(n+2\alpha)}{n!\Gamma(2\alpha)} u_n(t) = \gamma_0, \ \sum_{n=0}^{N} \frac{\Gamma(n+2\alpha)}{n!\Gamma(2\alpha)} u_n(t) = \gamma_1. \tag{3.29}$$

Eqs.(3.28), together with $\lceil \nu \rceil$ equations of the boundary conditions Eq.(3.29), give $(N+1)$ ordinary differential equations. Then, we solve this system using FDM to get $u_n(t_j),\ n = 0,\ ,1,\ \ldots, N$ and $j = 1,\ 2, \ldots, M$. Rewriting Eq.(3.4) using finite difference method, where

$$\frac{du_n(t_m)}{dt} = \frac{u_n(t_m) - u_n(t_{m-1})}{\triangle t} = \frac{u_n^m - u_n^{m-1}}{\triangle t}, \tag{3.30}$$

we have

$$\frac{u_n^m - u_n^{m-1}}{\triangle t} = a(x_n) D_x^{\nu} u_n^m + b(x_n, t_m) \tag{3.31}$$

which is the same as

$$u_n^m - \triangle t\ a(x_n) D_x^{\nu} u_n^m = u_n^{m-1} + \triangle t\ b_n^m, \tag{3.32}$$

where $D_x^{\nu} u = \dfrac{\partial^{\nu} u(x,t)}{\partial x^{\nu}}$ is fractional diffusion operator.

3.5. **Convergence and stability of the proposed method.** This section is devoted to two main subjects of convergence and stability of the proposed method. Let θ be an open bound domain in $\mathbb{R}^2$, $L_2(\theta)$ be an Hilbert space with the inner product

$$\langle u(x),\ w(x) \rangle = \int_{\theta} u(x) w(x) dx, \tag{3.33}$$

together with the Euclidean norm $\|u(x)\|^2 = \langle u(x),\ u(x) \rangle$ and Sobolev space

$$\Omega^{\kappa}(\theta) = \{u \in L_2(\theta), \frac{d^{\kappa} u}{dx^{\kappa}} \in L_2(\theta)\}. \tag{3.34}$$

To investigate the stability and convergence of the method, we use the assumption $a(x) > 0$ for $0 \le x \le 1$ and begin with the following lemmas.

Lemma 3.2. *For any $u, w \in \Omega^{\nu/2}(\theta),\ 1 < \nu < 2$, we have*

$$\langle {}_aD_x^{\nu}u,\ w\rangle = \langle {}_aD_x^{\frac{\nu}{2}}u,\ {}_xD_b^{\frac{\nu}{2}}w\rangle$$
$$\langle {}_xD_b^{\nu}u,\ w\rangle = \langle {}_xD_b^{\frac{\nu}{2}}u,\ {}_aD_x^{\frac{\nu}{2}}w\rangle$$
$$\langle {}_aD_x^{\nu}u,\ {}_xD_b^{\nu}u\rangle = \cos(\nu\pi)\|{}_aD_x^{\nu}u\|^2 = \cos(\nu\pi)\|{}_xD_b^{\nu}u\|^2$$

Proof. The proof can be found in [14] □

Lemma 3.3. *For the functions $h(x)$ and ${}_aD_x^{\nu}h(x) \in \Omega^{\nu/2}(\theta)$ $\exists \triangle t \geq 0$ sufficiently small and $1 < \nu < 2$, then*

$$\|h(x) - \triangle t\ a(x_n){}_aD_x^{\nu}h(x)\| \geq \|h(x)\|$$

Proof. From Lemma 3.2, we have

$$\begin{aligned}\|h(x) - \triangle t\ a(x_n){}_aD_x^{\nu}h(x)\| \geq \|h(x)\|^2 &= \langle h(x) - \triangle t\ a(x_n){}_aD_x^{\nu}h(x), h(x) - \triangle t\ a(x_n){}_aD_x^{\nu}h(x)\rangle \\ &= \|h(x)\|^2 - 2\triangle t\ a(x_n)\langle {}_aD_x^{\nu/2}h(x),\ {}_xD_b^{\nu/2}h(x)\rangle \\ &\ + (\triangle t)^2\ a(x_n)^2\|{}_aD_x^{\nu}h(x)\|^2 \\ &= \|h(x)\|^2 - 2\triangle t\ a(x_n)\|{}_aD_x^{\nu/2}h(x)\|^2\cos\left(\frac{\nu}{2}\pi\right) \\ &\ + (\triangle t)^2\ a(x_n)^2\|{}_aD_x^{\nu}h(x)\|^2,\end{aligned}$$

since $1 < \nu < 2 \Rightarrow \cos\left(\frac{\nu}{2}\pi\right) < 0$, therefore

$$-2\triangle t\ a(x_n)\|{}_aD_x^{\nu/2}h(x)\|^2\cos\left(\frac{\nu}{2}\pi\right) + (\triangle t)^2\ a(x_n)^2\|{}_aD_x^{\nu}h(x)\|^2 \geq 0,$$

hence the proof. □

Lemma 3.4. *Let $U^m \in \Omega^1(\theta),\ m = 1, \ldots, M$ and U^0 be the solution and initial condition, respectively, then*

$$\|U^m\| \leq \|U^{m-1}\| + \max_{0\leq n\leq N}\triangle t\ b_n^m, \qquad \text{where } U^m = u(x, t_m). \tag{3.35}$$

Proof. Using mathematical induction on m. Put $m = 1$ in Eq. (3.32), we have

$$U^1 - \triangle t\ a(x_n)D_x^{\nu}U^1 = U^0 + \triangle t\ b^1, \tag{3.36}$$

multiply Eq. (3.36) by U^1 and integrate the resulting equation on θ, we obtain

$$\|U^1\|^2 - \triangle t\ a(x_n)\langle {}_aD_x^{\nu}U^1,\ U^1\rangle = \langle U^0,\ U^1\rangle + \triangle t\ \langle b^1,\ U^1\rangle, \tag{3.37}$$

using Lemma 3.2, that is $\cos\left(\frac{\nu}{2}\pi\right) < 0$

$$\langle {}_aD_x^{\nu}U^1,\ U^1\rangle = \langle {}_aD_x^{\nu}U^1,\ {}_xD_b^{\nu}U^1\rangle = \cos\left(\frac{\nu}{2}\pi\right)\|{}_aD_x^{\nu/2}U^1\| < 0. \tag{3.38}$$

Left hand side of Eq. (3.37) becomes

$$\|U^1\|^2 - \triangle t\ a(x_n)\langle {}_aD_x^{\nu}U^1,\ U^1\rangle \geq \|U^1\|^2, \tag{3.39}$$

from Eqs. (3.37) and (3.39), we obtain

$$\|U^k\| \leq \|U^{k-1}\| + \max_{0\leq n\leq N}\triangle t\ b_n^m. \tag{3.40}$$

Assume that Eq. (3.35) is true for all $k = 1,\ 2, \ldots, m1$ as

$$\|U^k\| \leq \|U^{k-1}\| + \max_{0\leq n\leq N}\triangle t\ b_n^k. \tag{3.41}$$

Again, multiply Eq. (3.32) by U^m and integrate the resulting equation on θ, we obtain

$$\|U^m\|^2 - \triangle t\ a(x_n)\langle {}_aD_x^{\nu}U^m,\ U^m\rangle = \langle U^{m-1},\ U^m\rangle + \triangle t\ \langle b^m,\ U^m\rangle, \tag{3.42}$$

using the same procedure as above, Eq.(3.42) leads to

$$\|U^m\| \leq \|U^{m-1}\| + \max_{0\leq n\leq N} \triangle t\ b_n^m, \text{where } U^m = u(x, t_m), \tag{3.43}$$

hence the proof. □

Theorem 3.5. *The numerical scheme introduced in section 3.4 precisely Eq. (3.32) is unconditionally stable for* $\nu > 0$.

Proof. Suppose $U_n^m,\ m = 1,\ 2, \ldots, M$ is the Gegenbauer approximant of the method obtained by Eq.(3.32) subject to the boundary conditions $U_n^0 = u(x_n, 0)$, then the error $\xi^m = u(x_n, t_m) - U_n^m$ satisfies

$$\xi^m - \triangle t\ a(x_n)_a D_x^\nu \xi^m = \xi^{m-1} \tag{3.44}$$

According to Lemma 3.4, we have

$$\|\xi^m\| \leq \|\xi^{m-1}\|,\ m = 1,\ 2, \ldots, M, \tag{3.45}$$

and that completes the proof of the unconditional stability of the scheme. □

4. Illustrative Examples

In this section, we implement the proposed method on some selected problems from the literature. For ease of comparison, we compute the maximum error E_N of the proposed method using

$$E_N = \max_{0\leq n\leq N} |U(x_n, T) - u_N(x_n, T)| \tag{4.1}$$

and compare with the existing methods.

Example 4.1. Consider the following space fractional order diffusion problem [27, 29]

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} = \Gamma(1.2)x^{1.8}\frac{\partial^{1.8}u(x,t)}{\partial x^{1.8}} + (6x^3 - 3x^2)e^{-t}\\
&u(x,\ 0) = x^2 - x^3,\\
&u(0,\ t) = u(1,\ t) = 0,\ t > 0.
\end{aligned}$$

The exact solution is $u(x,t) = x^2(1-x)e^{-t}$.

For approximate solution of degree 3, that is $N = 3$, Eq. (3.26) becomes

$$u_3(x,t) = \sum_{n=0}^{3} u_n(t) C_n^{(\alpha)*}(x). \tag{4.2}$$

Using Eq.(3.28), we obtain

$$\sum_{n=0}^{3} \frac{du_n(t)}{dt} C_n^{(\alpha)*}(x_i) = a(x_i)\sum_{n=2}^{3}\sum_{k=0}^{n-2} u_n(t) N_{n,k}^{(\nu)} x_i^{n-k-1.8} + b(x_i, t),\ i = 0, 1, \tag{4.3}$$

where $a(x) = \Gamma(1.2)x^{1.8}$ and $b(x) = (6x^3 - 3x^2)e^{-t}$. The x_i are the roots of shifted Gegenbauer polynomial equation $C_2^{(\alpha)*}(x) = 0$.
Therefore, Eq. (4.3) becomes

$$\begin{aligned}
&\frac{du_0(t)}{dt}C_0^{(\alpha)*}(x_0) + \frac{du_1(t)}{dt}C_1^{(\alpha)*}(x_0) + \frac{du_2(t)}{dt}C_2^{(\alpha)*}(x_0) + \frac{du_3(t)}{dt}C_3^{(\alpha)*}(x_0)\\
&= a(x_0)x_0^{0.2}N_{2,0}u_2(t) + u_3(t)\left(a(x_0)x_0^{1.2}N_{3,0} + a(x_0)x_0^{0.2}N_{3,1}\right) + b(x_0, t)
\end{aligned} \tag{4.4}$$

and

$$\frac{du_0(t)}{dt}C_0^{(\alpha)*}(x_1) + \frac{du_1(t)}{dt}C_1^{(\alpha)*}(x_1) + \frac{du_2(t)}{dt}C_2^{(\alpha)*}(x_1) + \frac{du_3(t)}{dt}C_3^{(\alpha)*}(x_1)$$
$$= a(x_1)x_1^{0.2}N_{2,0}u_2(t) + u_3(t)\left(a(x_1)x_1^{1.2}N_{3,0} + a(x_1)x_1^{0.2}N_{3,1}\right) + b(x_1,t). \tag{4.5}$$

Also, applying Eq. (3.29) on Eq.(3.26), we obtain

$$u_0(t) - \frac{\Gamma(1+2\alpha)}{\Gamma(2\alpha)}u_1(t) + \frac{\Gamma(2+2\alpha)}{2\Gamma(2\alpha)}u_2(t) - \frac{\Gamma(3+2\alpha)}{6\Gamma(2\alpha)}u_3(t) = \gamma_0, \tag{4.6}$$

and

$$u_0(t) + \frac{\Gamma(1+2\alpha)}{\Gamma(2\alpha)}u_1(t) + \frac{\Gamma(2+2\alpha)}{2\Gamma(2\alpha)}u_2(t) + \frac{\Gamma(3+2\alpha)}{6\Gamma(2\alpha)}u_3(t) = \gamma_0. \tag{4.7}$$

Using finite difference method to solve the system of Eqs. (4.4)-(4.7) with the following notations:

$$T = T_{final},\ 0 < t_m \leq T,\ \triangle t = \frac{T}{M},\ t_m = m\triangle t,\ \text{for } m = 0,\ 1,\ \ldots, M \tag{4.8}$$

Also,

$$\frac{du_n(t_m)}{dt} = \frac{u_n(t_m) - u_n(t_{m-1})}{\triangle t} = \frac{u_n^m - u_n^{m-1}}{\triangle t} \tag{4.9}$$

where $u_n(t_m) = u_n^m$. With the adoption of Eq.(4.9) and taking $b(x_n,t)$ evaluated at t_m as $b(x_n,t_m) = b_n^m$, Eqs. (4.4)-(4.7) reduces to a system of algebraic equations, which are then solved to obtain $u_n,\ n = 0,\ 1,\ 2,\ 3,$ that are later substituted in Eq.(4.2) to get the required approximate solution.
Table 1 displays the results obtained for corresponding values of α and figure 4 show the relationship between the exact solution and its corresponding approximate solution at $\alpha = 0.5$ with $\triangle t = \frac{T}{M} = \frac{1}{1000}$ and figure 1 is the relationship between the exact and approximate solutions at $T = 1$ with $\alpha = 1.5,\ N = 3$ and step length $\left(\triangle x = \frac{1}{100}\right)$.

Example 4.2. Consider the following space fractional order diffusion problem [29]

$$\frac{\partial u(x,t)}{\partial t} = \frac{\Gamma(2.2)}{6}x^{2.8}\frac{\partial^{1.5}u(x,t)}{\partial x^{1.5}} - (x^3+x^4)e^{-t}$$
$$u(x,\ 0) = x^3,$$
$$u(0,\ t) = 0,\ u(1,\ t) = e^{-t},\ t > 0.$$

The exact solution is $u(x,t) = x^3e^{-t}$.

Table 2 displays the errors with respect to the values of α and the results from the literature. Figure 2 is the corresponding figure with $\alpha = 1.5$ and $T = 1$, while figure 5 is the relationship between the exact and the approximate solutions with $\alpha = 1,\ \triangle x = 0.05$ and $\triangle t = 0.05$

Example 4.3. Consider the following space fractional order diffusion problem [27]:

$$\frac{\partial u(x,t)}{\partial t} = \Gamma(1.5)x^{0.5}\frac{\partial^{1.5}u(x,t)}{\partial x^{1.5}} - 2x\sin(t+1) + (x^2+1)\cos(t+1)$$
$$u(x,\ 0) = (x^2+1)\sin(1),$$
$$u(0,\ t) = \sin(t+1),\ u(1,\ t) = 2\sin(t+1),\ t > 0.$$

The exact solution is $u(x,t) = (x^2+1)\sin(t+1)$.

Table 3 displays the errors with respect to the values of α and the results from the literature as it compares favourably with the existing results. Figure 3 is the corresponding figure with $\alpha = 0.5$ and $T = 1$, while figure 5 is the relationship between the exact and the approximate solutions with $\alpha = 1.5,\ \triangle x = 0.05$ and $\triangle t = 0.05$

TABLE 1. Absolute errors for Example 4.1 and its corresponding values of α

x	$\alpha = 0.5$ $(P_j(x))$	$\alpha = 1$ $(U_j^*(x))$[29]	$\alpha = 1.5(P_j^{(1,1)}(x))$	$\alpha = 2.5(P_j^{(2,2)}(x)))$
0	0	0	0	0
0.1	5.33×10^{-6}	5.46×10^{-6}	4.70×10^{-6}	4.14×10^{-6}
0.2	8.06×10^{-6}	8.51×10^{-6}	16.66×10^{-6}	5.52×10^{-6}
0.3	8.72×10^{-6}	9.60×10^{-6}	6.50×10^{-6}	4.82×10^{-6}
0.4	7.84×10^{-6}	9.18×10^{-6}	4.87×10^{-6}	2.75×10^{-6}
0.5	5.96×10^{-6}	7.69×10^{-6}	2.42×10^{-6}	6.72×10^{-6}
0.6	3.59×10^{-6}	5.60×10^{-6}	2.35×10^{-6}	2.77×10^{-6}
0.7	1.29×10^{-6}	3.33×10^{-6}	2.44×10^{-6}	4.84×10^{-6}
0.8	4.32×10^{-7}	1.34×10^{-6}	2.56×10^{-6}	5.52×10^{-6}
0.9	1.04×10^{-6}	8.39×10^{-8}	2.96×10^{-6}	4.14×10^{-6}
1.0	0	0	0	0

TABLE 2. Absolute errors for Example 4.2 and its corresponding values of α

x	$\alpha = 0.5$ $(P_n(x))$	$\alpha = 1$ $(U_n^*(x))$[29]	$\alpha = 1.5(P_j^{(1,1)}(x))$
0	0	0	0
0.1	3.35×10^{-7}	1.51×10^{-7}	1.03×10^{-8}
0.2	1.02×10^{-6}	1.10×10^{-6}	11.04×10^{-6}
0.3	1.90×10^{-6}	2.53×10^{-6}	2.77×10^{-6}
0.4	2.81×10^{-6}	4.14×10^{-6}	4.76×10^{-6}
0.5	3.59×10^{-6}	5.61×10^{-6}	6.62×10^{-6}
0.6	4.09×10^{-6}	6.63×10^{-6}	7.95×10^{-6}
0.7	4.13×10^{-6}	8.38×10^{-6}	8.35×10^{-6}
0.8	3.58×10^{-7}	6.08×10^{-6}	7.42×10^{-6}
0.9	2.25×10^{-6}	3.89×10^{-8}	4.77×10^{-6}
1.0	0	0	0

TABLE 3. Absolute errors for Example 4.3 and its corresponding values of α

x	$\alpha = 0.5$ $(P_n(x))$	$\alpha = 1$ $(U_n^*(x))$	$\alpha = 1.5(P_j^{(1,1)}(x))$	[27] with $N = 7$
0	0	0	0	0
0.1	7.16×10^{-7}	6.65×10^{-6}	6.40×10^{-6}	4.66×10^{-5}
0.2	1.17×10^{-5}	1.09×10^{-5}	11.05×10^{-5}	7.74×10^{-5}
0.3	1.41×10^{-5}	1.31×10^{-5}	1.27×10^{-5}	5.00×10^{-5}
0.4	1.46×10^{-5}	1.36×10^{-5}	1.32×10^{-5}	2.30×10^{-5}
0.5	1.36×10^{-5}	1.28×10^{-5}	1.24×10^{-5}	2.74×10^{-5}
0.6	1.16×10^{-5}	1.10×10^{-5}	1.07×10^{-5}	4.38×10^{-5}
0.7	8.81×10^{-6}	8.41×10^{-6}	8.23×10^{-6}	3.87×10^{-5}
0.8	5.71×10^{-6}	5.50×10^{-6}	5.42×10^{-6}	1.01×10^{-5}
0.9	2.65×10^{-6}	2.59×10^{-8}	2.57×10^{-6}	3.35×10^{-6}
1.0	0	0	0	0

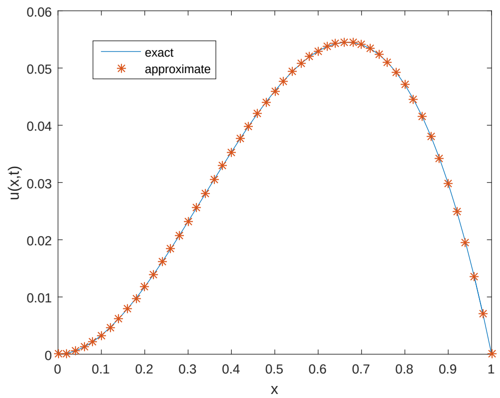

FIGURE 1. Relationship between exact and approximate solution for Example 4.1 at $N = 3,\ \alpha = 1.5$ and $T = 1$

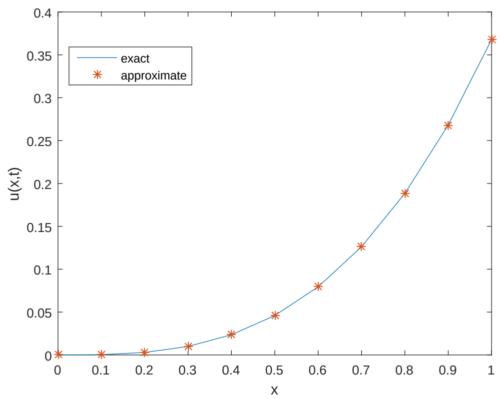

FIGURE 2. Relationship between exact and approximate solution for Example 4.2 at $N = 3,\ \alpha = 1$ and $T = 1$

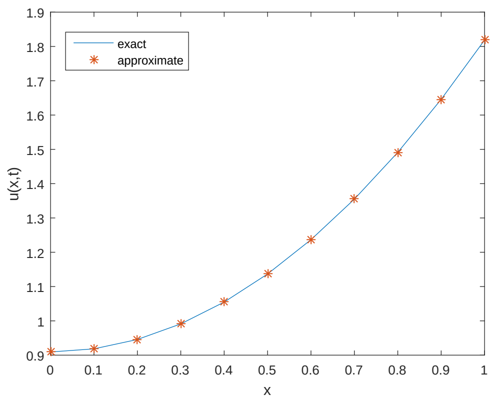

FIGURE 3. Relationship between exact and approximate solution for Example 4.3 at $N = 3,\ \alpha = 0.5$ and $T = 1$

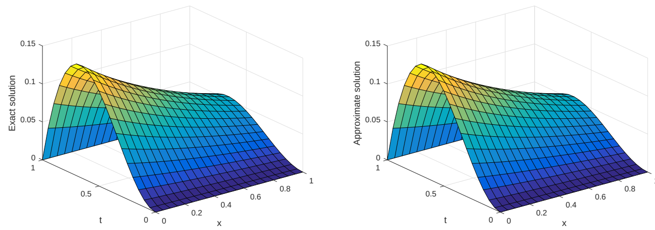

FIGURE 4. Exact solution and its corresponding approximate solution for Example 4.1 at $N = 3,\ \alpha = 0.5$

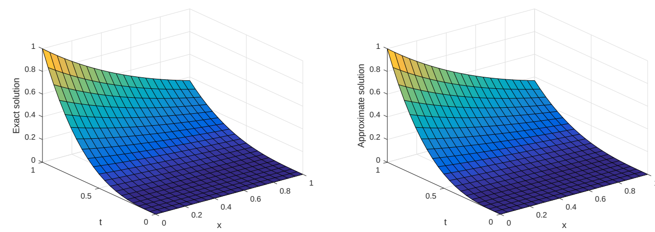

FIGURE 5. Exact solution and its corresponding approximate solution for Example 4.2 at $N = 3,\ \alpha = 1$

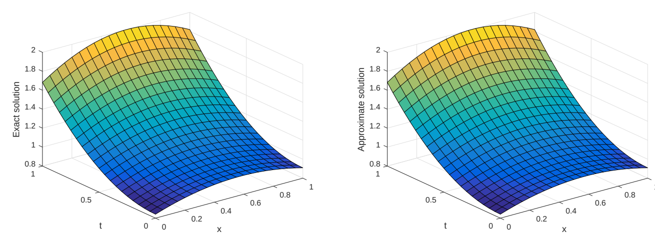

FIGURE 6. Exact solution and its corresponding approximate solution for Example 4.3 at $N = 3,\ \alpha = 1.5$

5. Discussion of results and conclusion

5.1. **Discussion of Results.** Column 2 of Table 1 gives better results compared to the results obtained in [27] despite the two methods using Legendre polynomials for approximation while column 3 of the same table is in complete agreement with the results obtained in [29]. It is therefore rewarding to use Gegenbauer polynomials as it generates results of other methods by changing the value of α in each case. The same explanation holds for Tables 2 and 3. The $2D$ graphs have elegantly illustrated the results tabulated, see Figures 1-3. Figure 1 corresponds to the approximate solution at $\alpha = 1.5$ which is Jacobi$(n, 1, 1, x)$ that is $P_j^{(1,1)}(x)$. . Figures 2 and 3 correspond to the approximate solutions at $\alpha = 1$ and $\alpha = 0.5$ respectively. Figures 4, 5 and 6 show the relationship between the exact and approximate solutions with step length $\triangle x = 0.05$ and $\triangle t = 0.05$ using $\alpha = 0.5,\ 1$ and 1.5 respectively.

6. Conclusion

In this paper, we proposed a numerical scheme based on shifted Gegenbauer finite difference method to solve the space fractional order diffusion equation, using shifted Gegenbauer polynomials properties. The finite difference method transforms the system of ordinary differential equations to a system of algebraic equations that are solved to obtain the desired approximate solution. The proposed method was implemented on some selected problems from the literature. Obviously, from the numerical results, the proposed method is effective and accurate as shown in the Tables of results. All computations were carried out using the Matlab program.

Acknowledgment

The authors really appreciate the thorough work done by the unanimous reviewers which have tremendously improved the quality of this work. Their various suggestions have contributed in no small measure to what we have as the final version of the paper.

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