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Some new soliton solutions for the nonlinear the fifth-order integrable equations

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Abstract

In this work, we established some exact solutions for the $(1+1)$-dimensional and $(2+1)$-dimensional fifth-order integrable equations ($(1+1)$D and $(2+1)$D FOIEs) which is considered based on the improved $\tanh(\phi(\xi)/2)$-expansion method (IThEM), by utilizing Maple software. We obtained new periodic solitary wave solutions. The obtained solutions include soliton, periodic, kink, kink-singular wave solutions. Comparing our new results with Wazwaz results, namely, the Hereman-Nuseri method shows that our results give further solutions. Many other such types of nonlinear equations arise in fluid dynamics, plasma physics, and nonlinear physics.

Keywords. Improved $\tanh(\phi/2)$-expansion method; Fifth-order integrable equations; Soliton wave solution.
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1. Introduction

Consider the $(1+1)$-dimensional and $(2+1)$-dimensional fifth-order integrable equations, respectively, given as follows

$$u_{ttt} + u_{txxxx} - \alpha(u_x u_t)_{xx} - \beta(u_x u_{xt})_x = 0, \tag{1.1}$$

$$u_{ttt} + u_{tyyyy} - u_{txx} - \alpha(u_y u_{yt})_y = 0, \tag{1.2}$$

where were established by Wazwaz [34, 35] and give multiple kink solutions. Kink solutions for three new fifth order nonlinear equations was investigated by Wazwaz [36]. Also the Hirota's direct method is used to derive multiple kink solutions for Eq. (1.1) for the case $\alpha = \beta = 4$, and only two soliton solutions for Eq. (1.2) for $\alpha = 4$, have been investigated by Wazwaz [36]. In [37], the Bäcklund transformation and the simplified Hirota's method were be used to study the derived couplings by Wazwaz. Wazwaz and Ebaid [38] studied the couplings of the fifth-order integrable Sawada-Kotera and Lax equations. Some new fifth-order nonlinear equations for obtaining the exact solutions have used the G'/G expansion and rational sine-cosine methods by Qawasmeh and Alquran [33].
Nonlinear partial differential equations (NLPDEs) play important roles in many areas such as biology, physics, chemistry, fluid mechanics and many engineering and science applications among others. Furthermore, the approaches of solving these types of equations alongside nonlinear PDEs ranging from analytical to numerical methods are very important in many engineering and sciences applications. Some of these methods include finding the exact solutions by using the special techniques in which can be manifested to new works with vigorous references ([11–15]).
Our objective here is to find exact solutions of the $(1+1)$-dimensional and $(2+1)$-dimensional fifth-order integrable equations under consideration the improved $\tanh(\phi(\xi)/2)$-expansion method for obtaining the new exact solitary wave

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solutions. Discussion about the improved $\tanh(\phi(\xi)/2)$-expansion method is given. The solitary wave solutions are investigated and derived exact solutions. In the continuation, we will present the graphical illustrations of some solutions of the aforementioned model. After that, we will deal with the investigation of solutions and we will end by a conclusion.

1.1. **Description of the IThEM.** The current method described here is the IThEM utilized to find traveling wave solutions of fifth-order equations which can be understood through the following steps:
We consider the following nonlinear partial differential equation of the form:

$$\mathcal{N}(u, u_x, u_t, u_{xx}, u_{tt}, ...) = 0. \tag{1.3}$$

In transforming Eq. (1.3); we make use of the wave transformation: which can be converted to an ODE

$$\xi = kx + wt, \tag{1.4}$$

where k and w are nonzero constants. Substitution of transformation (1.4) into (1.3) resulted to an ordinary differential equation of the form

$$\mathcal{Q}(u, ku', wu', k^2u'', w^2u'', ...) = 0. \tag{1.5}$$

Furthermore, the solution of Eq. (1.5) is assumed to be of the finite series form given by:

$$u(\xi) = S(\phi) = \sum_{k=-m}^{m} A_k \left[p + \tanh(\phi/2)\right]^k, \tag{1.6}$$

where $A_k(0 \leq k \leq m)$ and $A_{-k} = B_k(1 \leq k \leq m)$ are constants to be determined, such that $A_m \neq 0, B_m \neq 0$ and $\phi = \phi(\xi)$ satisfies the following ordinary differential equation:

$$\phi'(\xi) = a \sinh(\phi(\xi)) + b \cosh(\phi(\xi)) + c. \tag{1.7}$$

We will consider the following special solutions of equation (1.7):
Family 1: **When** $a^2 + c^2 - b^2 < 0$ **and** $b - c \neq 0$**,** then

$$\phi(\xi) = 2\tanh^{-1}\left[-\frac{a}{b-c} + \frac{\sqrt{b^2-a^2-c^2}}{b-c}\tan\left(\frac{\sqrt{b^2-a^2-c^2}}{2}\bar{\xi}\right)\right].$$

Family 2: **When** $a^2 + c^2 - b^2 > 0$ **and** $b - c \neq 0$**,** then

$$\phi(\xi) = 2\tanh^{-1}\left[-\frac{a}{b-c} - \frac{\sqrt{a^2+c^2-b^2}}{b-c}\tanh\left(\frac{\sqrt{a^2+c^2-b^2}}{2}\bar{\xi}\right)\right].$$

Family 3: **When** $a^2 + c^2 - b^2 < 0$**,** $b \neq 0$ **and** $c = 0$, then

$$\phi(\xi) = 2\tanh^{-1}\left[-\frac{a}{b} + \frac{\sqrt{b^2-a^2}}{b}\tan\left(\frac{\sqrt{b^2-a^2}}{2}\bar{\xi}\right)\right].$$

Family 4: **When** $\mathrm{a}^2 + \mathrm{c}^2 - \mathrm{b}^2 > 0$**,** $c \neq 0$ **and** $b = 0$, then

$$\phi(\xi) = 2\tanh^{-1}\left[\frac{a}{c} + \frac{\sqrt{a^2+c^2}}{c}\tanh\left(\frac{\sqrt{a^2+c^2}}{2}\bar{\xi}\right)\right].$$

Family 5: **When** $a^2 + c^2 - b^2 < 0$**,** $\mathrm{b} - \mathrm{c} \neq 0$ **and** $\mathrm{a} = 0$, then

$$\phi(\xi) = 2\tanh^{-1}\left[\sqrt{\frac{b+c}{b-c}}\tan\left(\frac{\sqrt{b^2-c^2}}{2}\bar{\xi}\right)\right].$$

Family 6: **When** $a = 0$ **and** $c = 0$, then

$$\phi(\xi) = \ln\left[\tan\left(\frac{b}{2}\bar{\xi}\right)\right].$$

Family 7: **When** $b = 0$ **and** $c = 0$, then

$$\phi(\xi) = \ln\left[-\tanh\left(\frac{a}{2}\bar{\xi}\right)\right].$$

Family 8: **When** $a^2 + b^2 = c^2$**,** then

$$\phi(\xi) = 2\tanh^{-1}\left[\frac{a}{-b+\sqrt{a^2+b^2}} + \frac{\sqrt{2}a}{-b+\sqrt{a^2+b^2}}\tanh\left(\frac{\sqrt{2}a}{2}\bar{\xi}\right)\right].$$

Family 9: **When** $a = b = c = ka$**,** then

$$\phi(\xi) = 2\tanh^{-1}\left[e^{ka\bar{\xi}} - 1\right].$$

Family 10: **When** $a = c = ka$ **and** $b = -ka$**,** then

$$\phi(\xi) = 2\tanh^{-1}\left[\frac{e^{ka\bar{\xi}}}{-1+e^{ka\bar{\xi}}}\right].$$

Family 11: **When** b = a**,** then

$$\phi(\xi) = -2\tanh^{-1}\left[\frac{(a+c)e^{c\bar{\xi}} - 1}{(a-c)e^{c\bar{\xi}} - 1}\right].$$

Family 12: **When** $b = c$**,** then

$$\phi(\xi) = 2\tanh^{-1}\left[\frac{e^{a\bar{\xi}} - c}{a}\right].$$

Family 13: **When** $a = -c$ and $b = c$, then

$$\phi(\xi) = 2\tanh^{-1}\left[1 + e^{-c\bar{\xi}}\right].$$

Family 14: **When** $b = -b$ and $c = -b$, then

$$\phi(\xi) = 2\tanh^{-1}\left[\frac{b + e^{a\bar{\xi}}}{a}\right].$$

Family 15: **When** $b = -b$**,** $a = -b$**,** and $c = b$, then

$$\phi(\xi) = 2\tanh^{-1}\left[\frac{1}{e^{b\bar{\xi}} - 1}\right].$$

Family 16: **When** $b = -c$**,** then

$$\phi(\xi) = 2\tanh^{-1}\left[\frac{ae^{a\bar{\xi}}}{ce^{a\bar{\xi}} - 1}\right].$$

Family 17: **When** $a = 0$ **and** $b = c$**,** then

$$\phi(\xi) = 2\tanh^{-1}\left[c\bar{\xi}\right].$$

Family 18: **When** $a = 0$ **and** $b = -c$**,** then

$$\phi(\xi) = 2\tanh^{-1}\left[\frac{1}{c\bar{\xi}}\right].$$

Family 19: **When** $b = 0$ **and** $a = c$**,** then

$$\phi(\xi) = 2\tanh^{-1}\left[1 + \sqrt{2}\tanh\left(\frac{\sqrt{2}c}{2}\bar{\xi}\right)\right].$$

Family 20: **When** $a = 0$ **and** $b = 0$**,** then $\phi(\xi) = c\xi + C$,
where $\overline{\xi} = \xi + C, A_k, B_k(k = 1, 2, ..., m), a, b$ and c are constants to be determined later. But, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (1.7). Putting (1.6) into Eq. (1.5), we can get a set of nonlinear equations including $A_0, A_k, B_k(k = 1, 2, ..., m)$ a, b, c, then solve the algebraic equations.

2. The extended (1 + 1)D FOIE

By make the transformation $\xi = \mu(x - \lambda t)$, then Eq. (1.1) becomes to

$$(\mu^2 - \lambda^2)\mathrm{u}' + \mu^2\mathrm{u}''' + \frac{\mu}{2}(\alpha + \beta)(\mathrm{u}')^2 = 0, \tag{2.1}$$

where obtained by twice integrating and neglecting the constant of integration. The next step is to expand the unknowns $u(\xi)$ in power series in terms of $p + \tanh(\phi/2)$,

$$u(\xi) = \sum_{k=-m}^{m} A_k \left[p + \tanh(\phi(\xi)/2)\right]^k, \tag{2.2}$$

which $A_{-k} = B_k$. In order to determine value of m, we balance the linear term of the highest order u''' with the highest order nonlinear term $(u')^2$ in Eq. (2.1) we get

$$u(\xi) = A_m \left(\tanh(\phi(\xi)/2)\right)^m + ..., \tag{2.3}$$

$$\frac{du(\xi)}{d\xi} = \frac{m(b-c)}{2} A_m \left(\tanh(\phi(\xi)/2)\right)^{m+1} + ..., \tag{2.4}$$

$$\left(\frac{du(\xi)}{d\xi}\right)^2 = \frac{m^2(b-c)^2}{4} A_m^2 \left(\tanh(\phi(\xi)/2)\right)^{2m+2} + ..., \tag{2.5}$$

$$\frac{d^2u(\xi)}{d\xi^2} = \frac{m(m+1)(b-c)^2}{2} A_m \left(\tanh(\phi(\xi)/2)\right)^{m+2} + ..., \tag{2.6}$$

$$\frac{d^3u(\xi)}{d\xi^3} = \frac{m(m+1)(m+2)(b-c)^3}{2} A_m \left(\tanh(\phi(\xi)/2)\right)^{m+3} + ... \tag{2.7}$$

By considering the homogeneous balance principle between the highest order derivatives u''' and nonlinear terms $(u')^2$, we obtain $m + 3 = 2m + 2$, then $m = 1$. Suppose that the exact solution for Eq. (2.1) will be as

$$u(\xi) = \sum_{k=-1}^{1} A_k (p + \tanh(\phi/2))^k. \tag{2.8}$$

By substituting (2.8) into Eq. (2.1) and collecting all terms with the same order of $\tanh(\phi/2)$ together, we can obtain a set of algebraic equations for $A_0, A_1, B_1, p, \mu, \lambda, a, b$ and c as follows:

Coefficients of $Y = \tanh(\phi/2)$

$$
\begin{cases}
Y^8: & \mu A_1(b-c)^2(6\mu b + A_1\beta + A_1\alpha - 6\mu c) = 0,\\
Y^7: & 4\mu(b-c)A_1(6\mu b + A_1\beta + A_1\alpha - 6\mu c)(pb - cp + a) = 0,\\
Y^6: & 2A_1\mu\beta(2A_1a^2 - A_1c^2 + A_1b^2 - B_1c^2 - B_1b^2 - 8A_1pac - 6A_1p^2bc + 8A_1pab + 2B_1bc +\\
& 3A_1p^2c^2 + 3A_1p^2b^2) + 2A_1\mu\alpha(2A_1a^2 - A_1c^2 + A_1b^2 - B_1b^2 - B_1c^2 - 6A_1p^2bc - 8A_1pac +\\
& 8A_1pab + 2B_1bc + 3A_1p^2c^2 + 3A_1p^2b^2) + 2A_1\mu^2(-54b^2cp^2 + 48ab^2p + 48ac^2p + 54bc^2p^2 -\\
& 18c^3p^2 - 4b^2c + 14a^2b + 18b^3p^2 - 14a^2c - 96abcp + 2b + 4c^3 + 4b^3 - 2c - 2b) + 4A_1\lambda^2c = 0,\\
Y^5: & ...,\\
Y^4: & ...,\\
Y^3: & ...,\\
Y^2: & ...,\\
Y^1: & \mu^2(8B_1b^2cp - 48B_1abc + 40B_1a^2cp - 8bpB_1 - 8B_1ap^2 + 8B_1b^3p - 8A_1bc^2p^3 + 16B_1ac^2p^2 -\\
& 16B_1ab^2p^2 - 16A_1ac^2p^4 + 8A_1b^2cp^3 + 16A_1a^2cp^3 + 16A_1a^2bp^3 + 8A_1a^3p^4 + 40B_1a^2bp +\\
& 16A_1ab^2p^4 - 8B_1c^2bp + 8A_1p^4a + 16A_1p^3c + 16A_1p^3b - 8B_1c^3p - 8B_1cp - 24B_1ab^2 -\\
& 24B_1ac^2 + 8A_1b^3p^3 - 8A_1c^3p^3 - 8B_1a^3p^2) + \mu\beta(4A_1^2p^4ac - 8A_1p^2B_1ab + 4A_1^2p^4ab +\\
& 4A_1^2p^3c^2 + 8A_1^2p^3bc - 4A_1pB_1b^2 + 4B_1^2ac + 4B_1^2ab + 4A_1^2p^3b^2 - 8A_1pB_1bc - 8A_1p^2B_1ac -\\
& 4A_1pB_1c^2) + \mu\alpha(-8A_1pB_1bc - 8A_1p^2B_1ac + 4A_1^2p^3c^2 + 4A_1^2p^3b^2 - 8A_1p^2B_1ab -\\
& 4A_1pB_1b^2 + 4A_1^2p^4ac - 4A_1pB_1c^2 + 4A_1^2p^4ab + 4B_1^2ac + 4B_1^2ab + 8A_1^2p^3bc) -\\
& 16\lambda^2A_1p^3c + 8\lambda^2B_1cp - 16\lambda^2A_1p^3b + 8\lambda^2B_1ap^2 + 8pb\lambda^2B_1 - 8\lambda^2A_1p^4a = 0,\\
Y^0: & (b+c)[\mu^2(2b^2A_1p^4 + 4A_1p^4 - 4B_1p^2 - 4B_1a^2p^2 - 6B_1b^2 - 6B_1c^2 + 2B_1c^2p^2 -\\
& 12B_1cb + 12B_1acp - 2A_1c^2p^4 - 2b^2B_1p^2 + 4A_1a^2p^4 + 12bB_1ap) + \mu\beta(-2A_1p^2B_1c + B_1^2b +\\
& A_1^2p^4c + B_1^2c + A_1^2p^4b - 2A_1p^2B_1b) + \mu\alpha(-2A_1p^2B_1b + B_1^2c + A_1^2p^4b + A_1^2p^4c +\\
& B_1^2b - 2A_1B_1p^2c) + 4\lambda^2B_1p^2 - 4\lambda^2A_1p^4] = 0.
\end{cases}
\tag{2.9}
$$

Solving the above algebraic equations (2.9), we have the following sets of coefficients as
Set I ((2.8) and Family 7):

$$A_1 = 0,\ B_1 = \mp\frac{12p}{\alpha+\beta}\sqrt{\lambda^2-\mu^2},\ a = \pm\frac{\sqrt{\lambda^2-\mu^2}}{\mu},\ b = c = 0, \tag{2.10}$$

$$u_1(x,t) = A_0 \mp \frac{12p\sqrt{\lambda^2-\mu^2}}{\alpha+\beta}\left\{p + \ln\left[-\tanh\left(\frac{\sqrt{\lambda^2-\mu^2}}{2}(x-\lambda t) + C\right)\right]\right\}^{-1}. \tag{2.11}$$

Set II ((2.8) and Family 6):

$$\lambda = \pm\sqrt{1-b^2}\mu,\ A_1 = 0,\ B_1 = \frac{6b\mu(p^2+1)}{\alpha+\beta},\ b = b,\ a = c = 0, \tag{2.12}$$

$$u_2(x,t) = A_0 + \frac{6b\mu(p^2+1)}{\alpha+\beta}\left\{p + \ln\left[\tan\left(\frac{b\mu}{2}(x \mp \sqrt{1-b^2}\mu t) + C\right)\right]\right\}^{-1}. \tag{2.13}$$

Set III ((2.8) and Family 20):

$$\lambda = \pm\sqrt{1+c^2}\mu,\ \ B_1 = -\frac{6c\mu(p^2-1)}{\alpha+\beta},\ A_1 = a = b = 0,\ \ c = c, \tag{2.14}$$

$$u_3(x,t) = A_0 - \frac{6c\mu(p^2-1)}{\alpha+\beta}\left[p + \tanh\left(\frac{c\mu(x \mp \sqrt{1-c^2}\mu t) + C}{2}\right)\right]^{-1}. \tag{2.15}$$

Set IV ((2.8) and Family 18):

$$a = A_1 = 0,\ B_1 = \mp\frac{12p^2}{\alpha+\beta}\sqrt{\lambda^2-\mu^2},\ b = -c = \pm\frac{\sqrt{\lambda^2-\mu^2}}{\mu}, \tag{2.16}$$

$$u_4(x,t) = A_0 \mp \frac{12p^2}{\alpha+\beta}\sqrt{\lambda^2-\mu^2}\left[p + \frac{1}{c\mu(x-\lambda t)+C}\right]^{-1}. \tag{2.17}$$

Set V ((2.8) and Family 18):

$$\lambda = \pm\frac{1}{12}\sqrt{144\mu^2+(\alpha+\beta)^2A_1^2},\ B_1 = a = 0,\ c = -b = \frac{(\alpha+\beta)A_1}{12\mu}, \tag{2.18}$$

$$u_5(x,t) = A_0 + A_1\left[p + \frac{1}{c\mu\left(x \mp \frac{1}{12}\sqrt{144\mu^2+(\alpha+\beta)^2A_1^2 t}\right)+C}\right]. \tag{2.19}$$

Set VI ((2.8) and Family 4):

$$\lambda = \pm\sqrt{1+a^2+c^2}\mu,\ b = A_1 = 0,\ B_1 = -\frac{6\mu(2ap+c(p^2-1))}{\alpha+\beta}, \tag{2.20}$$

$$u_6(x,t) = A_0 - \frac{6\mu(2ap+c(p^2-1))}{\alpha+\beta} \tag{2.21}$$

$$\left\{p + \frac{a}{c} + \frac{\sqrt{c^2+a^2}}{c}\tanh\left[\frac{\mu\sqrt{c^2+a^2}}{2}\left(x \mp \sqrt{1+a^2+c^2}\mu t\right)\right]\right\}^{-1}.$$

Set VII ((2.8) and Family 4):

$$\lambda = \pm\frac{1}{3}\sqrt{9\mu^2+(p^2+1)(\alpha+\beta)^2A_1^2},\ B_1 = (p^2+1)A_1,\ \ a = -cpA_1,\ b = 0,\ c = \frac{(\alpha+\beta)A_1}{6\mu}, \tag{2.22}$$

$$u_7(x,t) = A_0 + A_1\left\{p - pA_1 + \sqrt{1+p^2A_1^2}\tanh\left[\frac{c\mu\sqrt{1+p^2A_1^2}}{2}\xi\right]\right\} \tag{2.23}$$

$$+(p^2+1)A_1\left\{p - pA_1 + \sqrt{1+p^2A_1^2}\tanh\left[\frac{c\mu\sqrt{1+p^2A_1^2}}{2}\xi\right]\right\}^{-1},$$

$$\xi = x \mp \frac{1}{3}\sqrt{9\mu^2+(p^2+1)(\alpha+\beta)^2A_1^2}t.$$

Set VIII ((2.8) and Family 4):

$$\lambda = \pm\frac{1}{6}\sqrt{36\mu^2+(p^2+1)(\alpha+\beta)^2A_1^2},\ b = B_1 = 0,\ a = -cpA_1,\ c = \frac{(\alpha+\beta)A_1}{6\mu}, \tag{2.24}$$

$$u_8(x,t) = A_0 + A_1\left\{p - pA_1 + \sqrt{1+p^2A_1^2}\tanh\left[\frac{c\mu\sqrt{1+p^2A_1^2}}{2}\xi\right]\right\}, \tag{2.25}$$

$$\xi = x \mp \frac{1}{6}\sqrt{36\mu^2+(p^2+1)(\alpha+\beta)^2A_1^2}t.$$

Set IX ((2.8) and Family 4):

$$\lambda = \pm\frac{1}{6}\sqrt{36\mu^2(a^2+1)+(\alpha+\beta)^2A_1^2},\ b = B_1 = 0,\ \ c = \frac{(\alpha+\beta)A_1}{6\mu}, \tag{2.26}$$

$$u_9(x,t) = A_0 + A_1\left\{p + \frac{a}{c} + \frac{\sqrt{c^2+a^2}}{c}\tanh\left[\frac{\mu\sqrt{c^2+a^2}}{2}\xi + C\right]\right\}, \tag{2.27}$$

$$\xi = x \mp \frac{1}{6}\sqrt{36\mu^2(a^2+1)+(\alpha+\beta)^2A_1^2}t.$$

Set X ((2.8) and Family 3):

$$\lambda = \pm\sqrt{a^2-b^2+1}\mu,\ c = A_1 = 0,\ B_1 = -\frac{6\mu(2ap-b(p^2+1))}{\alpha+\beta}, \tag{2.28}$$

$$u_{10}(x,t) = A_0 - \frac{6\mu(2ap-b(p^2+1))}{\alpha+\beta}\left[p-\frac{a}{b}+\frac{\sqrt{b^2-a^2}}{b}\tan\left(\frac{\mu\sqrt{b^2-a^2}}{2}\right)\right]^{-1}, \tag{2.29}$$

$$\xi = x \mp \sqrt{a^2-b^2+1}\mu t.$$

Set XI ((2.8) and Family 3):

$$\lambda = \pm\frac{1}{6}\sqrt{36\mu^2(a^2+1)+A_1^2(\alpha+\beta)^2},\quad c = B_1 = 0,\quad b = -\frac{(\alpha+\beta)A_1}{6\mu}, \tag{2.30}$$

$$u_{11}(x,t) = A_0 + A_1\left\{p-\frac{a}{b}+\frac{\sqrt{b^2-a^2}}{b}\tan\left[\frac{\mu\sqrt{b^2-a^2}}{2}\right]\right\}, \tag{2.31}$$

$$\xi = x \mp \frac{1}{6}\sqrt{36\mu^2(a^2+1)+A_1^2(\alpha+\beta)^2}t.$$

Set XII ((2.8) and Family 3):

$$\lambda = \pm\frac{1}{3}\sqrt{9\mu^2+A_1^2(\alpha+\beta)^2(p^2-1)},\ B_1 = (p^2-1)A_1,\ a = pb,\ b = -\frac{(\alpha+\beta)A_1}{6\mu},\ c = 0. \tag{2.32}$$

$$u_{12}(x,t) = A_0 + A_1\sqrt{1-p^2}\tan\left[\frac{b\mu\sqrt{1-p^2}}{2}\left(x\mp\frac{1}{3}\sqrt{9\mu^2+A_1^2(\alpha+\beta)^2(p^2-1)}t+C\right)\right] - \tag{2.33}$$

$$A_1\sqrt{1-p^2}\cot\left[\frac{b\mu\sqrt{1-p^2}}{2}\left(x\mp\frac{1}{3}\sqrt{9\mu^2+A_1^2(\alpha+\beta)^2(p^2-1)}t+C\right)\right].$$

Set XIII ((2.8) and Family 3):

$$\lambda = \pm\frac{1}{6}\sqrt{36\mu^2+A_1^2(\alpha+\beta)^2(p^2-1)},\ c = B_1 = 0,\ p = p,\ a = bp,\ b = -\frac{(\alpha+\beta)A_1}{6\mu}. \tag{2.34}$$

$$u_{13}(x,t) = A_0 + A_1\sqrt{1-p^2}\tan\left(\frac{b\mu\sqrt{1-p^2}}{2}\left(x\mp\frac{1}{6}\sqrt{36\mu^2+A_1^2(\alpha+\beta)^2(p^2-1)t}\right)\right). \tag{2.35}$$

Set XIV ((2.8) and Families 2, 16, 18):

$$\lambda = \pm\sqrt{1+c^2p^2}\mu,\quad A_1 = 0,\quad B_1 = -\frac{2\mu c}{\alpha+\beta},\quad a = -pc,\quad b = -c, \tag{2.36}$$

$$u_{14}(x,t) = A_0 + \frac{4\mu c^2}{p(\alpha+\beta)}\left[1-\tanh\left(\frac{pc\mu}{2}\left(x\mp\sqrt{1+c^2p^2}\mu t\right)+C\right)\right]^{-1}, \tag{2.37}$$

$$u_{15}(x,t) = A_0 - \frac{2\mu c}{\alpha+\beta}\left[p-\frac{p}{e^{pc\mu\left(x\mp\sqrt{1+c^2p^2}\mu t\right)+C}-1}\right]^{-1}, \tag{2.38}$$

$$u_{16}(x,t) = A_0 - \frac{2\mu c^2}{\alpha+\beta}[\mu(x\mp\mu t)+C].$$

Set XV ((2.8) and Family 9):

$$\mu=-\frac{(\alpha+\beta)A_1}{2ap^3},\ \lambda=\pm\frac{(\alpha+\beta)A_1}{2ap^3}\sqrt{1+a^2-2a^2p^4},\ B_1=-p^2A_1,\ b=c=ap, \tag{2.39}$$

$$u_{17}(\xi)=A_0+A_1e^{-\frac{(\alpha+\beta)A_1}{2}\left(x\mp\frac{(\alpha+\beta)A_1}{2ap^3}\sqrt{1+a^2-2a^2p^4}t\right)+C}-\frac{p^2A_1}{e^{-\frac{(\alpha+\beta)A_1}{2}\left(x\mp\frac{(\alpha+\beta)A_1}{2ap^3}\sqrt{1+a^2-2a^2p^4}t\right)+C}}. \tag{2.40}$$

3. The extended $(2+1)$D FOIE

As, the second example, by make the transformation $\xi=\mu(x-\lambda t)$, Eq. (1.2) becomes to

$$(1-\lambda^2)u'+\mu^2u'''+\frac{\alpha\mu}{2}(u')^2=0, \tag{3.1}$$

where obtained by twice integrating and neglecting the constant of integration. Balancing the u''' and $(u')^2$, by using the homogenous principle, we obtain

$$m+3=2m+2, \qquad \Rightarrow m=1. \tag{3.2}$$

The exact solution is as

$$u(\xi)=A_0+A_1\left[p+\tanh(\phi/2)\right]+B_1\left[p+\tanh(\phi/2)\right]^{-1}. \tag{3.3}$$

Inserting (3.3) into Eq. (3.1), we can obtain the following results
Coefficients of $Y=\tanh(\phi/2)$

$$\begin{cases}
Y^8: & \mu A_1(b-c)^2(6\mu b+\alpha A_1-6\mu c)=0,\\
Y^7: & 4\mu(b-c)A_1(6\mu b+\alpha A_1-6\mu c)(pb+a-cp)=0,\\
Y^6: & 2A_1\mu\alpha(A_1b^2-B_1b^2-B_1c^2-6A_1p^2bc+2A_1a^2+8A_1pab+3A_1p^2b^2-8A_1pac+\\
& 2B_1bc+3A_1p^2c^2-A_1c^2)+2A_1\mu^2(4c^3-14a^2c-4b^2c+48ac^2p+48ab^2p+54bc^2p^2-\\
& 54b^2cp^2-96abcp+18b^3p^2+14a^2b+4b^3-18c^3p^2)+4A_1(b-c)(1-\lambda^2)=0,\\
Y^5: & ...,\\
Y^4: & ...,\\
Y^3: & ...,\\
Y^2: & ...,\\
Y^1: & \mu^2(40B_1a^2cp-16A_1ac^2p^4-8B_1c^3p-8A_1c^3p^3-16B_1ab^2p^2-8B_1a^3p^2+8B_1b^3p+\\
& 8A_1a^3p^4-24B_1ac^2-48B_1abc-8A_1bc^2p^3+16B_1ac^2p^2+16A_1a^2bp^3+16A_1ab^2p^4+\\
& 8B_1b^2cp+8A_1b^3p^3-8B_1c^2bp+16A_1a^2cp^3+8A_1b^2cp^3-24B_1ab^2+40B_1a^2bp)+\\
& \mu\alpha(-4A_1pB_1b^2+4A_1^2p^3b^2-4A_1pB_1c^2-8A_1p^2B_1ab+4A_1^2p^4ab+4A_1^2p^4ac+4B_1^2ac+4B_1^2ab\\
& +8A_1^2p^3bc-8A_1pB_1bc-8A_1p^2B_1ac+4A_1^2p^3c^2)-8bpB_1+16A_1p^3c-8B_1cp-8\lambda^2A_1p^4a+\\
& 16A_1p^3b+8\lambda^2B_1cp-16\lambda^2A_1p^3c-8B_1ap^2+8A_1p^4a+8\lambda^2B_1ap^2-16\lambda^2A_1p^3b+8\lambda^2B_1bp=0,\\
Y^0: & (b+c)[\mu^2(-2b^2B_1p^2+2b^2A_1p^4-6B_1b^2+12B_1acp+4A_1a^2p^4+12bB_1aP-12B_1cB+\\
& 2B_1c^2p^2-6B_1c^2-4B_1a^2p^2-2A_1c^2p^4)+\mu\alpha(B_1^2b+A_1^2p^4b-2A_1p^2B_1c-2A_1p^2B_1b+\\
& B_1^2c+A_1^2p^4c)-4\lambda^2A_1p^4-4B_1p^2+4\lambda^2B_1p^2+4A_1p^4]=0.
\end{cases} \tag{3.4}$$

Solving the above algebraic equations (3.4), we have the following results as
Set I ((3.3) and Family 7):

$$A_1=0,\quad B_1=\mp\frac{12p\sqrt{\lambda^2-1}}{\alpha},\quad a=\pm\frac{\sqrt{\lambda^2-1}}{\mu},\quad b=0,\quad c=0, \tag{3.5}$$

$$u_1(x,t) = A_0 \mp \frac{12p\sqrt{\lambda^2-1}}{\alpha}\left\{p + \ln\left[-\tanh\left(\pm\frac{\sqrt{\lambda^2-1}}{2}(x+y-\lambda t)+C\right)\right]\right\}^{-1}. \tag{3.6}$$

Set II ((3.3) and Family 18):

$$\lambda = \pm\sqrt{1+\mu^2c^2}, \quad A_1 = 0, \quad B_1 = -\frac{12\mu c p^2}{\alpha}, \quad a = 0, \quad b = -c, \tag{3.7}$$

$$u_2(x,t) = A_0 - \frac{12\mu c p^2}{\alpha}\left(p + \frac{1}{c\mu(x+y\mp\sqrt{1+\mu^2c^2}t)+C}\right)^{-1}. \tag{3.8}$$

Set III ((3.3) and Family 17):

$$\lambda = \pm\frac{1}{12}\sqrt{144+\alpha^2A_1^2}, \quad B_1 = 0, \quad a = 0, \quad b = -\frac{\alpha A_1}{12\mu}, \quad c = \frac{\alpha A_1}{12\mu}, \tag{3.9}$$

$$u_3(x,t) = A_0 + A_1\left[p + \frac{1}{c\mu\left(x+y\mp\frac{1}{12}\sqrt{144+\alpha^2A_1^2t}\right)+C}\right]^{-1}. \tag{3.10}$$

Set IV ((3.3) and Family 4):

$$\lambda = \pm\sqrt{1+\mu^2(a^2+c^2)}, \quad A_1 = b = 0, \quad B_1 = -\frac{6\mu(2ap+c(p^2-1))}{\alpha}, \tag{3.11}$$

$$u_4(x,t) = A_0 - \frac{6\mu(2ap+c(p^2-1))}{\alpha}\left[p + \frac{a}{c} + \frac{\sqrt{c^2+a^2}}{c}\zeta\right]^{-1}, \tag{3.12}$$

$$\zeta = \tanh\left(\frac{\mu\sqrt{c^2+a^2}}{2}(x+y\mp\sqrt{1+\mu^2(a^2+c^2)}t)\right).$$

Set V ((3.3) and Family 4):

$$\lambda = \pm\frac{1}{6}\sqrt{36+\alpha^2A_1^2(p^2+1)}, \; b = B_1 = 0, \; c = -\frac{a}{p} = \frac{\alpha p A_1}{6\mu}, \tag{3.13}$$

$$u_5(x,t) = A_0 + A_1\sqrt{1+p^2}\tanh\left[\frac{\sqrt{1+p^2}\alpha A_1}{12}\left(x+y\mp\frac{1}{6}\sqrt{36+\alpha^2A_1^2(p^2+1)t}\right)+C\right]. \tag{3.14}$$

Set VI ((3.3) and Family 4):

$$\lambda = \pm\frac{1}{3}\sqrt{9+\alpha^2A_1B_1}, \; B_1 = (p^2+1)A_1, \; c = -\frac{a}{p} = \frac{\alpha p A_1}{6\mu}, \tag{3.15}$$

$$u_6(x,t) = A_0 + A_1\sqrt{1+p^2}\left\{\tanh\left[\frac{\sqrt{1+p^2}\alpha A_1}{12}\left(x+y\mp\frac{1}{3}\sqrt{9+\alpha^2A_1^2(p^2+1)t}\right)+C\right]+\right. \tag{3.16}$$

$$\left.\coth\left[\frac{\sqrt{1+p^2}\alpha A_1}{12}\left(x+y\mp\frac{1}{3}\sqrt{9+\alpha^2A_1^2(p^2+1)t}\right)+C\right]\right\}.$$

Set VII ((3.3) and Family 4):

$$\lambda = \pm\frac{1}{6}\sqrt{36(1+a^2\mu^2)+\alpha^2A_1^2}, \; b = B_1 = 0, \; c = \frac{\alpha A_1}{6\mu}, \tag{3.17}$$

$$u_7(x,t) = A_0 + A_1 \left\{ p + \frac{a}{c} + \frac{\sqrt{c^2+a^2}}{c} \tanh\left[\frac{\mu\sqrt{c^2+a^2}}{2}\xi \right] \right\}, \tag{3.18}$$

$$\xi = x + y \mp \frac{1}{6}\sqrt{36(1+a^2\mu^2)+\alpha^2 A_1^2}t + C.$$

Set VIII ((3.3) and Family 3):

$$\lambda = \sqrt{1+\mu^2(a^2-b^2)},\ c = A_1 = 0,\ B_1 = \frac{6\mu(b(p^2+1)-2ap)}{\alpha}, \tag{3.19}$$

$$u_8(x,t) = A_0 - \frac{6\mu(2ap-b(p^2+1))}{\alpha} \left\{ p - \frac{a}{b} + \frac{\sqrt{b^2-a^2}}{b} \tan[\zeta] \right\}^{-1}, \tag{3.20}$$

$$\zeta = \frac{\mu\sqrt{b^2-a^2}}{2}\left(x + y \mp \sqrt{1+\mu^2(a^2-b^2)}t \right).$$

Set IX ((3.3) and Family 3):

$$\lambda = \pm\frac{1}{6}\sqrt{36(1+a^2\mu^2)-\alpha^2 A_1^2},\ c = B_1 = 0,\ b = -\frac{\alpha A_1}{6\mu}, \tag{3.21}$$

$$u_9(x,t) = A_0 + A_1 \left\{ p - \frac{a}{b} + \frac{\sqrt{b^2-a^2}}{b} \tan\left[\frac{\mu\sqrt{b^2-a^2}}{2}\xi \right] \right\}, \tag{3.22}$$

$$\xi = x + y \mp \frac{1}{6}\sqrt{36(1+a^2\mu^2)-\alpha^2 A_1^2}t.$$

Set X ((3.3) and Family 3):

$$\lambda = \pm\frac{1}{6}\sqrt{36+\alpha^2 A_1^2(p^2-1)},\ c = B_1 = 0,\ \frac{a}{p} = -\frac{\alpha A_1}{6\mu}, \tag{3.23}$$

$$u_{10}(x,t) = A_0 - A_1\sqrt{1-p^2}\tan\left[\frac{\alpha A_1\sqrt{1-p^2}}{12}\left(x + y \mp \frac{1}{6}\sqrt{36+\alpha^2 A_1^2(p^2-1)t} \right) + C \right]. \tag{3.24}$$

Set XI ((3.3) and Family 4):

$$\lambda = \pm\frac{1}{3}\sqrt{9+\mu^2 A_1 B_1},\ B_1 = (p^2-1)A_1,\ \frac{a}{p} = -\frac{\alpha A_1}{6\mu},\ c = 0, \tag{3.25}$$

$$u_{11}(x,t) = A_0 - A_1\sqrt{1-p^2}\left\{ \tan\left[\frac{\alpha A_1\sqrt{1-p^2}}{12}\left(x + y \mp \frac{1}{3}\sqrt{9+\mu^2 A_1^2(p^2-1)t} \right) \right] + \right. \tag{3.26}$$

$$\left. \cot\left[\frac{\alpha A_1\sqrt{1-p^2}}{12}\left(x + y \mp \frac{1}{3}\sqrt{9+\mu^2 A_1^2(p^2-1)t} \right) \right] \right\}.$$

Set XII ((3.3) and Families 2,1, 18):

$$\lambda = \pm\sqrt{1+\mu^2 c^2 p^2},\ A_1 = 0,\ B_1 = -\frac{2c\mu}{\alpha},\ a = -pc,\ b = -c, \tag{3.27}$$

$$u_{12}(x,t) = A_0 - \frac{4c\mu}{p\alpha}\left\{ 1 + \tanh\left[\frac{pc\mu}{2}\left(x + y \mp \sqrt{1+\mu^2 c^2 p^2 t} \right) \right] \right\}^{-1}, \tag{3.28}$$

$$u_{13}(x,t) = A_0 - \frac{2c\mu}{\alpha}\left[p + \frac{p}{e^{pc\mu\left(x+y\mp\sqrt{1+\mu^2c^2p^2}t\right)} - 1}\right]^{-1}, \tag{3.29}$$

$$u_{14}(x,t) = A_0 - \frac{2c^2\mu^2}{\alpha}\left(x + y \mp t + C\right).$$

Set XIII ((3.3) and Family 12):

$$\mu = -\frac{\alpha A_1}{2ap^3},\ \lambda = \pm\frac{\sqrt{4p^6 + A_1^2\alpha^2(1-2p^4)}}{2p^3},\ B_1 = -p^2A_1,\ b = c = ap, \tag{3.30}$$

$$u_{15}(x,t) = A_0 + A_1 e^{-\frac{\alpha A_1}{2p^3}\left(x+y\mp\frac{\sqrt{4p^6+A_1^2\alpha^2(1-2p^4)}}{2p^3}t+C\right)} - p^2A_1 e^{-\frac{\alpha A_1}{2p^3}\left(x+y\mp\frac{\sqrt{4p^6+A_1^2\alpha^2(1-2p^4)}}{2p^3}t+C\right)}. \tag{3.31}$$

4. Discussion and Results

Wazwaz [36] studied the exact solutions of the $(1+1)$-dimensional nonlinear fifth-order integrable equation through Hirota s direct method and found only two soliton solutions by supposing $\beta = 4$ in Eq. (1.1) as single kink solution ((70) in [36]), single singular kink solution ((72) in [36]), two kink solutions ((75) in [36]), two singular kink solutions ((77) in [36]). On the other hand, by means of the ITEM we have obtained 15 solutions for $(1+1)$-dimensional FOIE. Our solutions with IThEM are including hyperbolic, periodic, singular kink and rational solutions. Moreover, for particular values of the free parameters, some of our solutions coincide with solutions of Wazwaz [36]. It proves that the other solutions are newly derived through the IThEM. Similarly, it can be shown that Wazwaz [37] with Bäcklund transformation and the simplified Hirota's method have obtained some solutions including kink solutions for i odd, and anti-kink solutions for i even in ((16) in [37]). Also, Wazwaz [35] with Hereman-Nuseir method derive the conditions for the cases of complete integrability of this equation and found multiple soliton solutions as single soliton solution and single singular soliton solution ((11), (12), (22) and (23) in [35]) respectively, for $\alpha = \beta$ and $\alpha = 0$, and β a nonzero constant. But, with help the IThEM we have obtained 17 solutions for $(2+1)$-dimensional FOIE. Our solutions with IThEM are including hyperbolic, periodic, singular kink and rational solutions. In Figures 1-6, we plot two and three dimensional graphics of absolute values of the (1+1)-dimensional fifth-order integrable equations with appropriate parametric selections. Figs. 1-6 illustrate the effects of the constant parameters on the (1+1)-dimensional fifth-order integrable equations with selecting free parameter. Solutions u_1, u_{14} represent the soliton solution. Solutions u_2, u_{13} represent the periodic solutions. Also, solution u_4 represent the cuspon solution. Moreover, solution u_{15} represent the singular kink-type solution and u_{17} kink-type solution of (1+1)-dimensional fifth-order integrable equation.

5. Conclusion

In this paper, we applied the IThEM for deriving the exact soliton wave solutions for the nonlinear fifth-order integrable equations. As a result, we received many new exact solitary solutions for the nonlinear fifth-order integrable equations which are expressed by the hyperbolic, the trigonometric functions, the polynomial functions and the rational functions. We then think that these results will help for conducting future research in various areas of physics such as mathematical physics, nonlinear mechanics and other applied fields and so on. The method used here can be applied to other nonlinear partial differential equations.

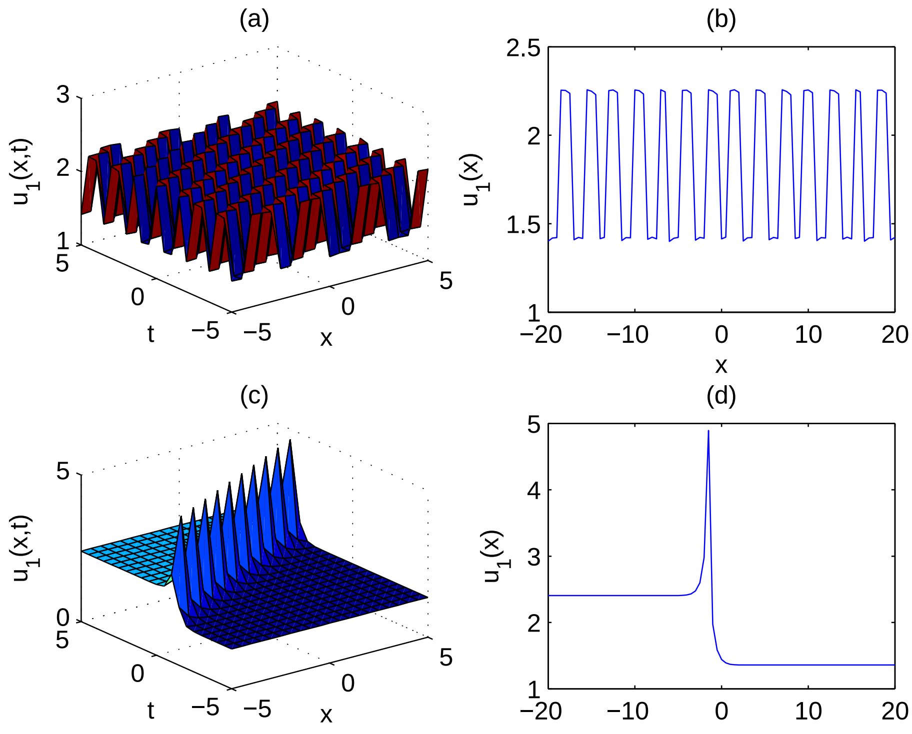

FIGURE 1. 2D and 3D plots of absolute values of (2.11) for (a) and (b) $\alpha = \beta = 4, A_0 = 0.5, \mu = 3, C = p = \lambda = 2$, and for (c) and (d) $\alpha = \beta = 4, A_0 = \mu = 0.5, C = p = \lambda = 2$,.

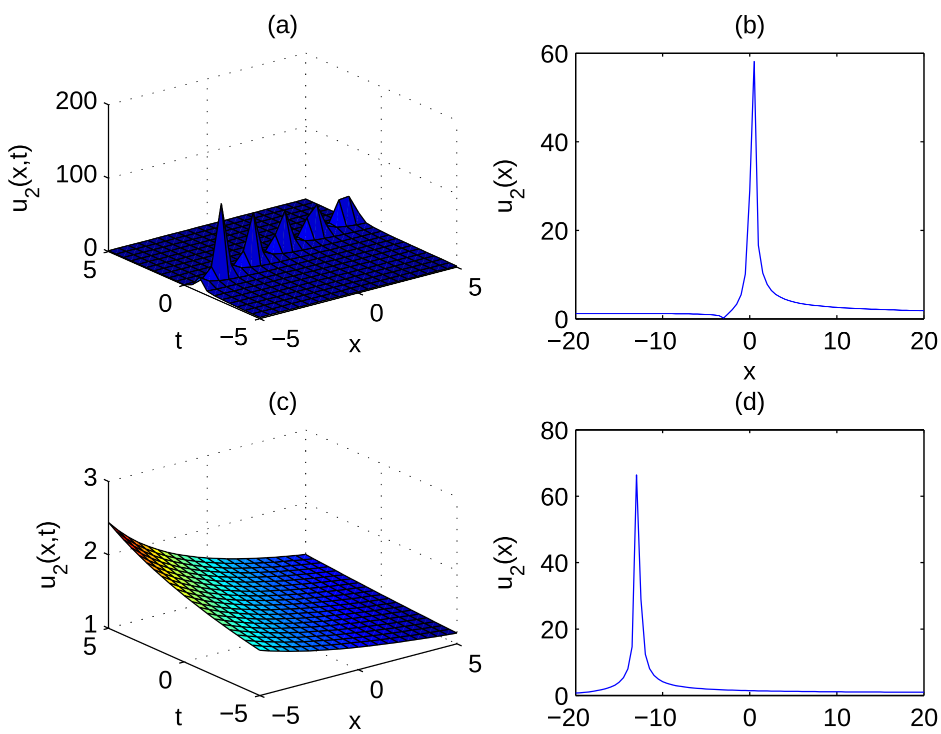

FIGURE 2. 2D and 3D plots of absolute values of (2.12) for (a) and (b) $\alpha = \beta = 4, A_0 = 0.5, b = 0.2, \mu = 4, p = C = 2$, and for (c) and (d) $\alpha = \beta = 4, b = 0.2, A_0 = mu = 0.5, p = C = 2$.

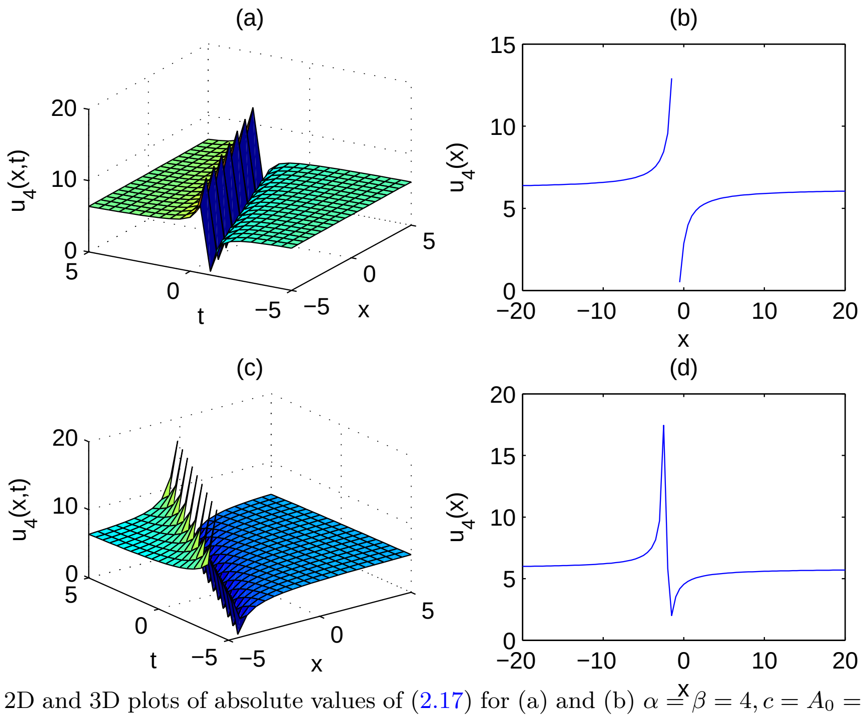

FIGURE 3. 2D and 3D plots of absolute values of (2.17) for (a) and (b) $\alpha = \beta = 4, c = A_0 = 0.5, b = 0.2, \mu = p = C = 2, \lambda = 3$, and for (c) and (d) $\alpha = \beta = 4, c = A_0 = 0.5, b = 0.2, \mu = p = C = 2, \lambda = 0.5$.

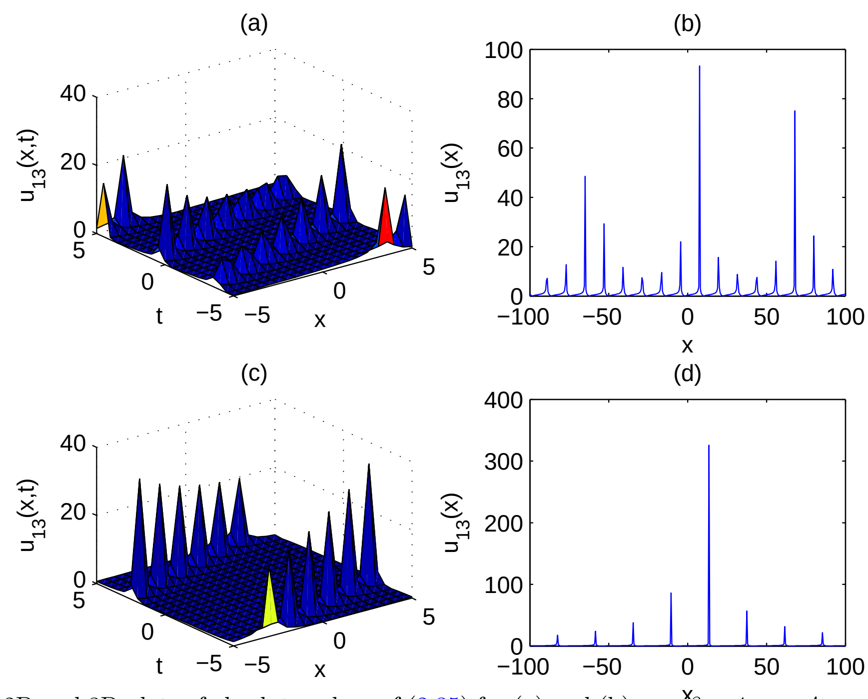

FIGURE 4. 2D and 3D plots of absolute values of (2.35) for (a) and (b) $\alpha = \beta = 4, p = A_0 = A_1 = 0.5, \mu = 3$, and for (c) and (d) $\alpha = \beta = 4, p = 0.9, A_0 = A_1 = 0.5, \mu = 3$.

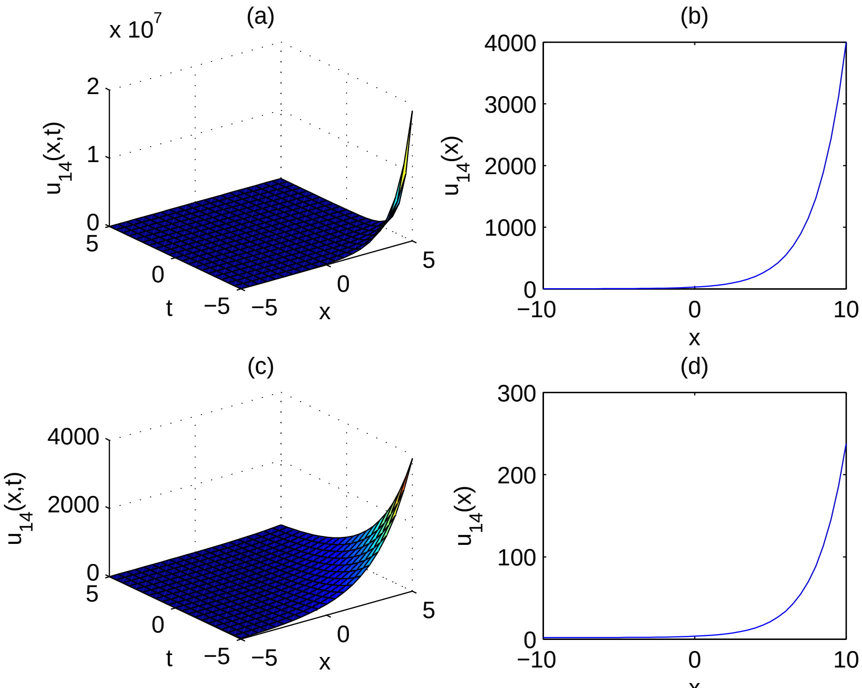

FIGURE 5. 2D and 3D plots of absolute values of (2.37) when $\alpha = \beta = 4, p = 0.5, A_0 = 1, \mu = 1, c = C = 2$ and for (b) $(t = 2)$, and for (d) $(t = 10)$.

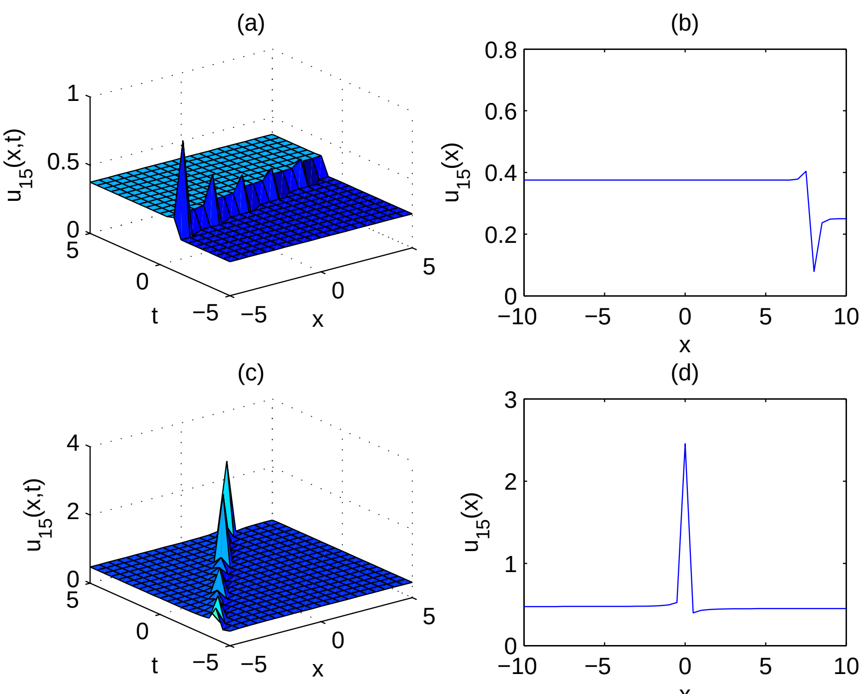

FIGURE 6. 2D and 3D plots of absolute values of (2.38) for (a) and (b) $\alpha = \beta = 4, C = p = c = 2, A_0 = 0.5, \mu = 1$, and for (c) and (d) $\alpha = \beta = 4, c = p = C = 2, A_0 = 0.5, \mu = 0.2$.

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