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Synchronization between integer and fractional chaotic systems Via tracking control and non linear control with application

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Abstract

In this paper, the synchronization between complex fractional-order chaotic systems and the integer-order hyperchaotic system has been investigated. Due to increased complexity and the presence of additional variables, it seems to be very interesting and can be associated with real-life problems. Based on the idea of tracking control and nonlinear control, we have designed the controllers to obtain the synchronization between the chaotic systems. To establish the efficacy of the methods computations have been carried out. Excellent agreement between the analytical and computational studies has been observed. The achieved synchronization is illustrated in the field of secure communication. The results have been compared with published literature.

Keywords. Chaos synchronization; Nonlinear control; Tracking control; Secure communication.

2010 Mathematics Subject Classification. 37D45, 37N10, 37E99, 37F99.

1. Introduction

Nature being nonlinear, has invoked huge interest among the researchers in the field of nonlinear sciences. In fact most systems that we encounter in this world are non linear. Some of these systems show chaos-hyper chaos even though they are completely deterministic, i.e. we can determine the state of the system at any time knowing their exact state at some initial time. Despite this property these systems remain unpredictable .In this situation, synchronizing such two systems is in itself a matter of astonishment.By synchronization, we mean that the trajectories of the coupled systems evolve with time to a common rhythm . The most significant work in this direction has been carried out by Pecora and Carrol in 1990 . Since then different types of synchronization schemes [23, 24] have been developed viz. complete synchronization, anti- synchronization, projective synchronization, lag synchronization, hybrid synchronization, combination synchronization, combination-combination synchronization, compound synchronization [14, 19, 30] and also various control design methods have been used such as active control, adaptive control, feedback control, optimal control [5, 27], back-stepping control, tracking control [13] technique and so on [9–11, 18, 21, 25, 29] . However, complete synchronization scheme [22] was first reported by Liu et al. in 2002.

These studies have continuously motivated the researcher's interests due to their applications in various systems related to physical, biological and chemical sciences. Such systems can be represented in the form of ordinary differential equations, partial differential equations, iterated maps, fractional differential equations [12, 15]. Recently, a lot of significant work has been done in the field of fractional calculus. These studies stand significant as the synchronization [20] between different structures of different dimensions finds many practical applications. Because of the complex nature of the fractional order slave system relatively fewer results on synchronization of such systems have been developed to the best of our knowledge.

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With the increased number of variables describing the system, the sensitivity to initial conditions and parameters of system have increased making the system highly unpredictable. With the increased complexity, such synchronization technique finds application in secure communication, cryptography, earthquake dynamics etc. Because of such extensive applications, chaos synchronization between complex fractional order and integer order chaotic systems has attracted the researchers all over the world.

2. Preliminaries

2.1. **Definition:** Fractional order [2, 6] differentiation and integration is a generalization of integer order.Much work involving fractional calculus [1, 26] has been done during last two decades. Major contribution in this area has been made by Liouville, Riemann and Caputo.

- THE GRUNWALD-LETNIKOV DEFINITION:

$$ {}_aD_x^{\alpha}g(x) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{n=0}^{[\frac{x-a}{h}]}(-1)^{n\alpha}C_n g(x-nh) $$

, where $a = x - nh$
i.e. $n = \frac{x-a}{h}$

- RIEMANN-LIOUVILLE DEFINITION:

$$ {}_aD_x^{\alpha}g(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_a^x \frac{g(\tau)d\tau}{(x-\tau)^{\alpha-n+1}} $$

, where n is integer
α is real number and $(n-1) \leq \alpha < n$

- CAPUTO DEFINITION:

$$ {}_aD_x^{\alpha}g(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{g^{(n)}(\tau)d\tau}{(x-\tau)^{\alpha-n+1}} $$

, where n is integer
α is real number and $(n-1) \leq \alpha < n$ and $\Gamma(.)$ is the Gamma function.
Throughout our studies Caputo version has been used. Grunwald Letnikov approximation function has been used for numerical simulations in MATLAB.

2.2. **Problem Formulation Using Tracking Controllers:** Consider the integer order chaotic master system:

$$ \dot{X} = h(X,t) $$

and a complex fractional order chaotic slave system with tracking controllers:

$$ \frac{d^qY(t)}{dt^q} = f(Y,t) + v(X(t)) + V(X(t),Y(t)), $$

where $X = [X_1, X_2, ..., X_n]^T \in R^n$, $Y = [Y_1', Y_2', ..., Y_n']^T \in C^n$ are the state vectors of the master and slave systems respectively, and $Y_i' = y_a + iy_b$ where y_a, y_b are real variables. $h : R^n \to R^n$, $f : C^n \to C^n$ are non linear functions, $v(X(t)) + V(X(t), Y(t)$ is the tracking controller. The controller is designed in such a manner that the trajectories of the complex fractional order slave system are synchronized with that of the integer order master system.
Defining the synchronization error as

$$ e(t) = Y(t) - X(t) $$

.
Dividing the error vector $e(t)$ into two vectors E_1 and E_2, the error dynamics simplifies to

$$\begin{pmatrix} \frac{d^q E_1}{dt^q} \\ \frac{d^q E_2}{dt^q} \end{pmatrix} = \begin{pmatrix} P_1 E_1(t) + F_1(X, E_1, E_2) \\ P_2 E_2(t) + F_{21}(X, E_1, E_2) + F_{22}(X, E_1, E_2) \end{pmatrix}$$

where $F_1(X, E_1, E_2) \in R^m$, $F_{21}(X, E_1, E_2) \in R^{2n-m}$, $F_{22}(X, E_1, E_2) \in R^{2n-m}$

$$\lim_{E_1 \to 0} F_{21}(X, E_1, E_2) = 0$$

and $P_1 \in R_{m \times m}$, $P_2 \in R_{2n-m \times 2n-m}$ are constant matrices.
Rewriting the controller function $V(X(t), Y(t))$ as

$$\begin{pmatrix} A_1 E_1(t) - F_1(X, E_1, E_2) \\ A_2 E_2(t) - F_{22}(X, E_1, E_2) \end{pmatrix}$$

where A_1, A_2 are matrices to be determined by the following Lemma and Theorem based on the stability theory.

Lemma 2.1. For the following fractional order chaotic system:

$$\frac{d^q Y}{dt^q} = AY; Y(0) = Y_0$$

where $q = {(q_i)_{i=1}^n}^T \in R^n$, $A \in R_{n \times n}$ and $0 < q_i < 1, \forall i = 1, 2, ...n$, the system (1) achieves stability asymptotically at equilibrium point iff $|arg\lambda| > \frac{q'\pi}{2}$ for all eigen values λ of the matrix A, where $q' = max(q_i)_{i=1}^n$.

The above methodology about the error dynamical system can be summarized in the form of the following theorem.

Theorem 2.1. *If the choice of the matrices A_1 and A_2 is made in such a way that$A_1 + B_1 < 0$ and all the eigen values of matrix $A_2 + B_2$ are in accordance with the above lemma,then the trajectories of the master and slave systems are said to be synchronized with the action of the tracking controller $v(X(t)) + V(X(t), Y(t))$.*

3. System Description:

3.1. **Master System:** Hu[7], constructed in 2009 a five dimensional hyperchaotic Lorenz system by adding one linear and one non linear feedback controller to the Lorenz system. Five Lyapunov Exponents of the above mentioned system using MATLAB,at time t=300 have been computed as

$$\lambda_1 = -12.5971, \lambda_2 = 0.0015927, \lambda_3 = 0.029795, \lambda_4 = 0.29558, \lambda_5 = 0.60563$$

With three positive Lyapunov Exponents, the system so obtained is hyperchaotic .
The 5-D system is given by:

$$\begin{aligned} \dot{x_1} &= \alpha(x_2 - x_1) + x_4 \\ \dot{x_2} &= r x_1 - x_2 - x_1 x_3 - x_5 \\ \dot{x_3} &= -\beta x_3 + x_1 x_2 \\ \dot{x_4} &= k_1 x_4 - x_1 x_3 \\ \dot{x_5} &= k_2 x_2 \end{aligned} \tag{3.1}$$

has a hyperchaotic attractor when$(\alpha, r, \beta, k_1, k_2) = (10, 28, 8/3, 2, 3)$ where $k_1, k_2 > 0$ are controlled parameters and α, β, r are three parameters.

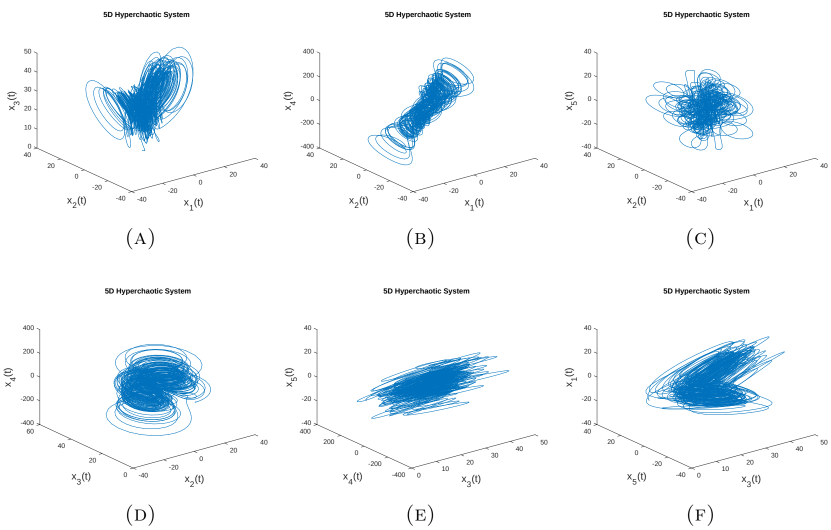


FIGURE 1. Phase portraits of the 5-D Hyper chaotic System in (a)$x_1-x_2-x_3$space (b)$x_1-x_2-x_4$space (c)$x_1 - x_2 - x_5$space (d)$x_2 - x_3 - x_4$space (e)$x_3 - x_4 - x_5$space (f)$x_3 - x_5 - x_1$space

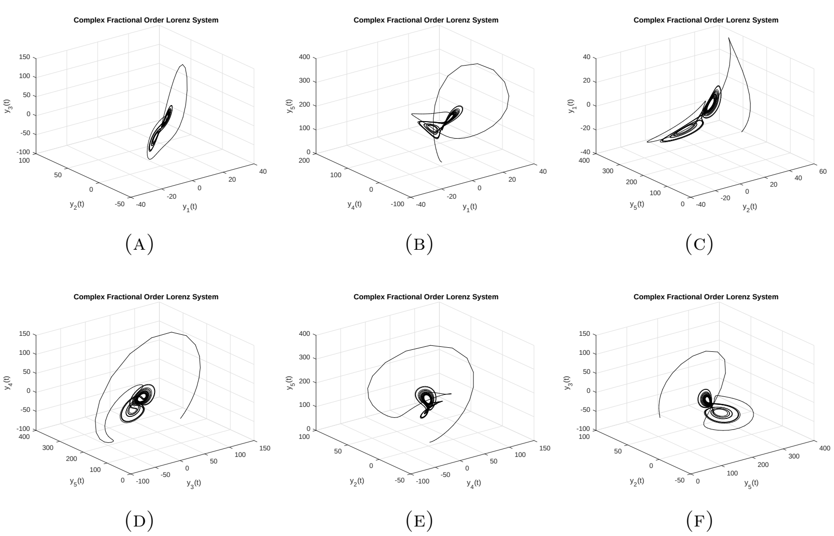


FIGURE 2. Phase portraits of the fractional order Complex Lorenz System for the order of the derivative q=.95; in (a)$y_1 - y_2 - y_3$space;(b)$y_1 - y_4 - y_5$space;(c)$y_2 - y_5 - y_1$space;(d)$y_3 - y_5 - y_4$space;(e)$y_4 - y_2 - y_5$space;(f)$y_5 - y_2 - y_3$space

Phase portraits of the hyperchaotic master system (3.1) are shown in Fig.1 for the initial condition (-1,2,1,3,-2).

3.2. **Slave System:** The complex fractional order Lorenz system is described as:

$$\frac{d^q y_1'}{dt^q} = a_1(y_2' - y_1')$$
$$\frac{d^q y_2'}{dt^q} = a_2 y_1' - y_2' - y_1' y_3'$$
$$\frac{d^q y_3'}{dt^q} = \frac{1}{2}(\bar{y}_1' y_2' + y_1' \bar{y}_2' - c y_3')$$

where $y' = [y_1', y_2', y_3']^T$ is the state variable vector.
y_1', y_2' are complex variables and y_3' is taken as real variable.
a_1, a_2, c are the three parameters.
Substituting

$$y_1' = y_1 + i y_2$$
$$y_2' = y_3 + i y_4$$
$$y_3' = y_5$$

where x_1, x_2, x_3, x_4, x_5 are real variables, we get the dynamical system as:

$$\begin{aligned}
\frac{d^q y_1}{dt^q} &= a_1(y_3 - y_1)\\
\frac{d^q y_2}{dt^q} &= a_1(y_4 - y_2)\\
\frac{d^q y_3}{dt^q} &= a_2 y_1 - y_3 - y_1 y_5\\
\frac{d^q y_4}{dt^q} &= a_2 y_2 - y_4 - y_2 y_5\\
\frac{d^q y_5}{dt^q} &= y_1 y_3 + y_2 y_4 - c y_5
\end{aligned} \tag{3.2}$$

Phase portraits of the chaotic slave system (3.2) are shown in Fig. 2 for initial condition as (2,3,5,6,9) and parameters value $a_1 = 10, a_2 = 180, c = 1$ for fractional order q=0.95.

4. Synchronization of 5-D hyperchaotic integer order system and complex fractional order Lorenz system Via. Tracking Control Method

Consider system (3.1) as master system and system (3.2) as slave system. The slave system with control functions is given as:

$$\frac{d^q y_1}{dt^q} = a_1(y_3 - y_1) + v_1(x(t)) + V_1(x(t), y(t))$$
$$\frac{d^q y_2}{dt^q} = a_1(y_4 - y_2) + v_2(x(t)) + V_2(x(t), y(t))$$
$$\frac{d^q y_3}{dt^q} = a_2 y_1 - y_3 - y_1 y_5 + v_3(x(t)) + V_3(x(t), y(t))$$
$$\frac{d^q y_4}{dt^q} = a_2 y_2 - y_4 - y_2 y_5 + v_4(x(t)) + V_4(x(t), y(t))$$
$$\frac{d^q y_5}{dt^q} = y_1 y_3 + y_2 y_4 - c y_5 + v_5(x(t)) + V_5(x(t), y(t))$$

where $v(x(t)) + V(x(t), y(t))$ is tracking controller.
Here $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)), y(t) = (y_1(t), y_2(t), y_3(t), y_4(t), y_5(t))$
We design tracking controller as

$$v(x(t)) = \frac{d^q x(t)}{dt^q} - f(x(t))$$

where

$$f(x(t)) = \begin{bmatrix} a_1(x_3 - x_1) \\ a_1(x_4 - x_2) \\ a_2x_1 - x_3 - x_1x_5 \\ a_2x_2 - x_4 - x_2x_5 \\ x_1x_3 + x_2x_4 - cx_5 \end{bmatrix}$$

$$\begin{bmatrix} \frac{d^q y_1}{dt^q} \\ \frac{d^q y_2}{dt^q} \\ \frac{d^q y_3}{dt^q} \\ \frac{d^q y_4}{dt^q} \\ \frac{d^q y_5}{dt^q} \end{bmatrix} = \begin{bmatrix} a_1(y_3 - y_1) \\ a_1(y_4 - y_2) \\ a_2y_1 - y_3 - y_1y_5 \\ a_2y_2 - y_4 - y_2y_5 \\ y_1y_3 + y_2y_4 - cy_5 \end{bmatrix} + \frac{d^q x(t)}{dt^q} - \begin{bmatrix} a_1(x_3 - x_1) \\ a_1(x_4 - x_2) \\ a_2x_1 - x_3 - x_1x_5 \\ a_2x_2 - x_4 - x_2x_5 \\ x_1x_3 + x_2x_4 - cx_5 \end{bmatrix} + V(x(t), y(t))$$

Defining $e_i = y_i - x_i, \forall i = 1, 2, 3, 4, 5$,we have

$$\begin{bmatrix} \frac{d^q e_1}{dt^q} \\ \frac{d^q e_2}{dt^q} \\ \frac{d^q e_3}{dt^q} \\ \frac{d^q e_4}{dt^q} \\ \frac{d^q e_5}{dt^q} \end{bmatrix} = \begin{bmatrix} a_1(e_3 - e_1) \\ a_1(e_4 - e_2) \\ a_2e_1 - e_3 - e_1e_5 - x_1e_5 - x_5e_1 \\ a_2e_2 - e_4 - e_2e_5 - x_5e_2 - x_2e_5 \\ e_1e_3 + x_1e_3 + x_3e_1 + e_4e_2 + x_4e_2 + x_2e_4 - ce_5 \end{bmatrix} + V(x(t), y(t)) \tag{4.1}$$

Let $e(t) = (E_1(t), E_2(t))^T$ where $E_1(t) = e_1, E_2(t) = (e_2, e_3, e_4, e_5)^T$, equation (4.1) simplifies to

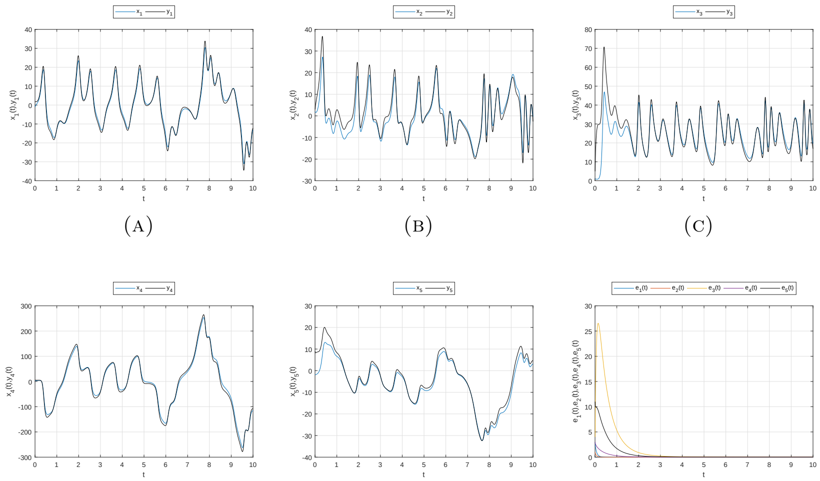
$$\begin{bmatrix} \frac{d^q E_1(t)}{dt^q} \\ \frac{d^q E_2(t)}{dt^q} \end{bmatrix} = \begin{bmatrix} P_1E_1(t) + F_1(x(t), E_1(t), E_2(t)) \\ P_2E_2(t) + F_{21}(x(t), E_1(t), E_2(t)) + F_{22}(x(t), E_1(t), E_2(t)) \end{bmatrix} + V(x(t), y(t)) \tag{4.2}$$

where $P_1 = -a_1, P_2 = \begin{bmatrix} -a_1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -c \end{bmatrix}, F_{21}(x(t), E_1(t), E_2(t)) = \begin{bmatrix} 0 \\ a_2e_1 - e_1e_5 - x_5e_1 \\ 0 \\ e_1e_3 + x_3e_1 \end{bmatrix},$

$$F_{22}(x(t), E_1(t), E_2(t)) = \begin{bmatrix} a_1e_4 \\ -x_1e_5 \\ a_2e_2 - e_2e_5 - x_2e_5 - x_5e_2 \\ x_1e_3 + e_4e_2 + x_4e_2 + x_2e_4 \end{bmatrix}$$

Clearly,

$$\lim_{E_1 \to 0} F_{21}(x(t), E_1(t), E_2(t)) = 0$$



(A) (B) (C)

(D) (E) (F)

FIGURE 3. Synchronized trajectories via tracing control method and synchronization error

Let us choose the control functions as follows:

$$V(x(t),y(t)) = \begin{bmatrix} A_1E_1(t) - F_1(x(t),E_1(t),E_2(t)) \\ A_2E_2(t) - F_{22}(x(t),E_1(t),E_2(t)) \end{bmatrix}, \tag{4.3}$$

where A_1, A_2 are matrices to be determined.Using equation (4.3) in equation (4.2), we get

$$\begin{bmatrix} \frac{d^qE_1(t)}{dt^q} \\ \frac{d^qE_2(t)}{dt^q} \end{bmatrix} = \begin{bmatrix} (P_1 + A_1)E_1(t) \\ (P_2 + A_2)E_2(t) + F_{21}(x(t),E_1(t),E_2(t)) \end{bmatrix}$$

. From Lemma 2.1 and Theorem 2.1 as stated in section 2, synchronization between the master and slave system is achieved, choosing

$$A_1 = [-1], A_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

.

5. Synchronization of 5-D hyperchaotic integer order system and complex fractional order Lorenz system Via. Non Linear Control Method

Consider system (3.1) as master system and system (3.2) as slave system. The slave system with control functions is given as:

$$\frac{d^qy_1}{dt^q} = a_1(y_3 - y_1) + u_1$$

,

$$\frac{d^qy_2}{dt^q} = a_1(y_4 - y_2) + u_2$$

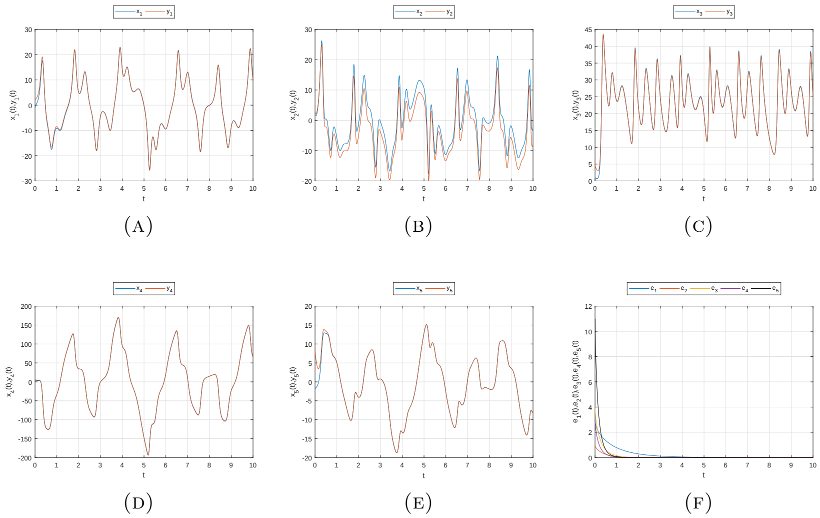


FIGURE 4. Synchronized trajectories via non linear control method and synchronization error

,

$$\frac{d^q y_3}{dt^q} = a_2 y_1 - y_3 - y_1 y_5 + u_3$$

,

$$\frac{d^q y_4}{dt^q} = a_2 y_2 - y_4 - y_2 y_5 + u_4$$

,

$$\frac{d^q y_5}{dt^q} = y_1 y_3 + y_2 y_4 - c y_5 + u_5$$

, where u_i are the non linear controllers.
The error system is given by:

$$e_i = y_i - x_i$$

, for $i = 1, 2..5$. The error dynamical is therefore:

$$d^q e_i = d^q y_i - d^q x_i \tag{5.1}$$

Choosing the non linear control functions as:

$$\begin{aligned} u_1 &= -10y_3 + 10y_1 + 10x_2 - 10x_1 + x_4 - e_1, \\ u_2 &= -10y_4 + 10y_2 + 28x_1 - x_2 - x_1x_3 - x_5 - 2e_2, \\ u_3 &= -180y_1 + y_3 + y_1y_5 - \frac{8}{3}x_3 + x_1x_2 - 3e_3, \\ u_4 &= -180y_2 + y_4 + y_2y_5 + 2x_4 - x_1x_3 - 4e_4, \\ u_5 &= -y_1y_3 - y_2y_4 + y_5 + 3x_2 - 5e_5. \end{aligned} \tag{5.2}$$

From (3.1) and (3.2), substituting (5.2) into(5.1),we get

$$d^q e_1 = -e_1$$

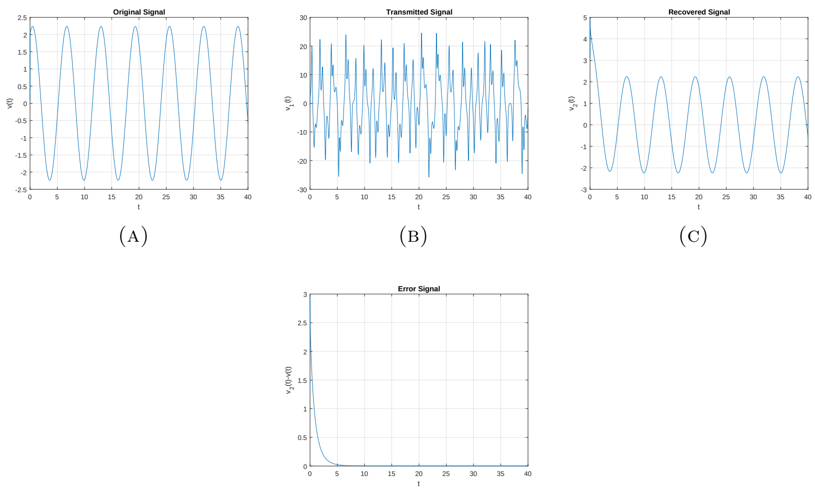


FIGURE 5. Illustration in secure communication

,

$$d^q e_2 = -2e_2$$

,

$$d^q e_3 = -3e_3$$

,

$$d^q e_4 = -4e_4$$

,

$$d^q e_5 = -5e_5$$

. Consider the Lyapunov function

$$V(e(t)) = \frac{1}{2}({e_1}^2 + {e_2}^2 + {e_3}^2 + {e_4}^2 + {e_5}^2)$$

,

$$d^q V(e(t)) \leq e_1(-e_1) + e_2(-2e_2) + e_3(-3e_3) = -{e_1}^2 - 2{e_2}^2 - 3{e_3}^2 - 4{e_4}^2 - 5{e_5}^2$$

,

$\Rightarrow d^q V(e(t))$ is negative definite.
Therefore by Stability Theory of Lyapunov, errors vanish as $t \to \infty$, implying synchronization of the systems.

6. Numerical Simulations And Illustration In Secure Communication

The choice of parameters and initial conditions for which the master system shows chaotic behavior are $(\alpha, r, \beta, k_1, k_2) = (10, 28, 8/3, 2, 3)$ and (-1,2,1,3,-2) respectively. The slave system shows chaotic behavior for the following choice of parameters and initial conditions $(a_1, a_2, c) = (10, 180, 1)$ and (2,3,5,6,9) respectively. Here we have chosen

$$A_1 = -1, A_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

to check the effectiveness of the designed tracking controller. The synchronized state trajectories of the corresponding variables of the master and slave system using tracking control method are shown in Figure.3 and using non linear control method are shown in Figure. 4. For the initial conditions$(e_1(0), e_2(0), e_3(0), e_4(0), e_5(0)) = (3, 1, 4, 3, 11)$, the error trajectories using both methods tend to zero with time.

The achieved synchronization between integer order chaotic system and complex fractional order chaotic systems is illustrated as an application in secure communication using non linear control method. based on the technique of masking the secret message with chaotic signals.[3, 4, 8] Chaotic systems being highly sensitive to initial conditions and parameter values increase the anti-attack capacity of signals, as without prior and exact knowledge of the values of the system the secret message cannot be hacked by the intruders. From the transmitting end the secret message is mixed with chaotic signals from chaotic system. At the receiving end only after performing desired synchronization applying desired controllers the secret message is retrieved. An illustration in the field of secure communication is as follows:

Let the original message to be transmitted be $sint+2cost$. It is mixed with chaotic signal $x_1(t)$ and transmitted. At the receiving end after performing synchronization on applying suitable controllers, the original message is recovered. The obtained results are shown in Fig. 5.

7. Comparison With Previous Literature

In [16] the authors have conducted double compound combination anti synchronization among hyper chaotic systems by active control method where they have the synchronization error components converge to zero at approx. 5 units. In [28] the authors achieved combination projective synchronization with error vanishing at approx. 4 units. In [31] triple compound synchronization is achieved in presence of disturbances at approx. 5 units whereas in [17] dual combination combination error components reach zero at 0.1,2,4,5,6,6 units respectively.

In our paper we have achieved synchronization between chaotic systems of different structures and orders using tracking control method, with errors e_1, e_2, e_3, e_4, e_5 converge to zero at respectively 0.3, 0.5, 2, 3, 3 units. This clearly shows the efficacy of our synchronization scheme.

8. Conclusion

In the present manuscript, synchronization of integer order chaotic system with complex fractional order chaotic system using the tracking control scheme based on stability theory and non linear control has been studied. Because of the complex fractional order systems, these studies may have applications in the real life problems. With the increase in banking transactions, telephone communications, online purchases, instances of cyber crime has also increased. Therefore, it is the need of the hour to develop more complex chaos synchronization so as to obstruct the intruders to interrupt. The synchronization is illustrated in secure communication. In secure communication, this may enhance the security of the signals. The obtained results are also compares with previous published literature. Further in this direction different synchronization techniques can be developed on these systems.

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No conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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