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Existence of solution for nonlinear integral inclusions

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Abstract

In this paper, we prove the existence of solution of two nonlinear integral inclusions by using generalization of Krasnoselskii fixed point theorem for set-valued mappings. As an application we prove the existence of solution of the boundary valued problem of ordinary differential inclusion.

Keywords. Krasnoselskii fixed point theorem, Caratheodory map, Nonlinear integral inclusions.

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1. Introduction

A fundamental tool for studying problems $x = A(x) + B(x)$ is the well-known Krasnoselskii fixed point theorem [22]. This result combined the topological Schauder fixed point theorem with the geometrical Banach fixed point theorem. Because of its importance for mathematical theory, it has been extended in various directions, see [2, 4, 5, 14–17] and the references therein.

Krasnoselskii fixed point theorem plays a key role in the study of the existence of continuous solutions for perturbed nonlinear differential and the mixed type of integral equations, see [7, 15, 17, 18].

The fixed point theory for set-valued mappings is an important topic of set-valued analysis. Several well-known fixed point theorems for single-valued mappings such as Banach contraction and Schauder fixed point theorems have been extended to set-valued mappings in Banach spaces. Recently, set-valued version of Krasnoselskii's fixed point theorem has been stated by Boriceanu [8] and Petruşel [25] and subsequent extensions have been derived by many authors, see [1, 6, 9–11, 19, 23].

Graef et al. [19] presented a Krasnoselskii-type fixed point theorem for set-valued mappings by using the measure of noncompactness combined with an approximation method. Also, Basoc and Cardinali [6] obtained two Krasnoselskii-Sadowskii-type fixed point theorems for set-valued mappings. They used the classic Covitz-Nadler fixed point theorem and a fixed point result for condensing set-valued mappings.

Set-valued version of Krasnoselskiis fixed point theorem has nice applications to perturbed nonlinear differential inclusions and integral inclusion see, [12, 24]. Dhage [11] proved the existence results for a certain functional integral inclusion under mixed Lipschitz and Caratheodory conditions on $C([0, 1]; \mathbb{R})$. In this paper, we prove the existence of solution for two nonlinear integral inclusions on $C([0, b]; \mathbb{R}^n)$ by using two fixed point theorems have been presented by Graef et al. and Basoc and Cardinali.

As an application of the integral inclusion, we prove existence of solution for the nonlinear functional point boundary valued problem of ordinary differential inclusion.

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2. Preliminaries

In this section, we introduce some definitions and facts which will be used in the sequel.
Let X be a Banach space we denote by $\mathcal{P}(X)$ and $\mathcal{P}_b(X)$ the family of all nonempty subsets of X and the family of all nonempty bounded subsets of X, respectively. Also, we denote by $\mathcal{P}_{cp}(X)$ and $\mathcal{P}_{cv}(X)$ the family of all nonempty compact subsets of X and the family of all nonempty convex subsets of X, respectively.

Definition 2.1. Let X and Y be two Banach spaces. A set-valued map $F : X \to \mathcal{P}(Y)$ is said to be:

(i) upper semi-continuous, if for each closed set $B \subseteq Y$, $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is closed in X.
(ii) lower semi-continuous if for each open set $V \subseteq Y$, $F^-(V)$ is an open set in X.
(iii) continuous if it is both upper and lower semi-continuous.
(iv) weakly lower semi-continuous at $x_0 \in X$ (see [26]) if for every $\varepsilon > 0$ and for every neighborhood V of x_0 there is a point $x_1 \in V$ such that for every $z \in F(x_1)$ there is a neighborhood U_z of x_0 such that
$$z \in \cap\{F(x) + \varepsilon B(0,1) : x \in U_z\}$$
where $B(0,1) = \{x \in X : \|x\| < 1\}$.

Notice that every lower semi-continuous map is weakly lower semi-continuous.
Let $A \in \mathcal{P}_b(X)$ then the Kuratowski measure of non-compactness of $A \in \mathcal{P}_b(X)$ is defined by $\gamma(A) = \inf\{\varepsilon > 0 : \ A = \cup_{i=1}^n A_i, \mathrm{diam(A_i)} \leq \varepsilon\}$, where $\mathrm{diam(A)} = \sup\{\|x-y\| : \ x, y \in A\}$.

Definition 2.2. [20] Let X be a Banach space. A bounded set-valued map $F : X \to \mathcal{P}(X)$ is said to be

(i) compact if $\overline{F(A)}$ is compact for each $A \in \mathcal{P}_b(X)$.
(ii) k-set contraction if there exists $k \in [0,1)$ such that $\gamma(F(A)) \leq k\gamma(A)$ for each $A \in \mathcal{P}_b(X)$.
(iii) condensing if $\gamma(F(A)) < \gamma(A)$ for each $A \in \mathcal{P}_b(X)$ with $\gamma(A) > 0$.

Let (Ω, Σ, μ) be a measurable space and X a separable Banach space. Let $F : \Omega \to \mathcal{P}(X)$ be a set-valued map. We introduce the set
$$S_F = \{f(.) \in L^1(\Omega, X) : \ f(\omega) \in F(\omega) \ \mu - a.e.\}.$$
If $Gr(F) = \{(\omega, x) \in \Omega \times X : \ x \in F(\omega)\} \in \Sigma \times B(X)$, the set S_F is nonempty if and only if $\inf_{x \in F(\omega)} \|x\| \in L^1(\Omega, \mathbb{R})$, where $B(X)$ denotes the family of all Borel subsets of X. Having this set, we can now define an integral for the set-valued map $F(.)$. So we set
$$\int_\Omega F(\omega) d\mu = \{\int_\Omega f(\omega) d\mu : \ f(.) \in S_F\},$$
where $\int_\Omega f(\omega) d\mu$ is understood as a Bochner integral. This set-valued integral was introduced by Aumann [3] as the generalization of the integral of a point valued function and of the Minkowski sum of sets.

Definition 2.3. [21] Let X be a Banach space, $J = [0, b] \subset \mathbb{R}$ and A be a subset of $J \times X$. A is called $L \otimes B$ measurable if A belongs to the σ-algebra generated by all sets of the form $I \times D$ where I is Lebesgue measurable in J and D is Borel measurable in X.

Definition 2.4. [21] A map $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be integrably bounded if there exists an integrable function $m \in L^1(J; \mathbb{R}^+)$ such that $\|y\| \leq m(t)$ for every $x \in \mathbb{R}^n$, $t \in J$ and $y \in F(t, x)$.

Definition 2.5. [21] A set-valued map $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be Caratheodory map provided

(i) $F(t, .) : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is upper semi-continuous for a.e. $t \in J$; and
(ii) $F(., x) : J \to \mathcal{P}(\mathbb{R}^n)$ is measurable for every $x \in \mathbb{R}^n$.

Let $C(J; \mathbb{R}^n)$ denotes the Banach space of all continuous functions $y : J \to \mathbb{R}^n$ equipped with a standard norm $\|y\|_\infty := \sup_{t \in J} \|y(t)\|$. Let $L^1(J; \mathbb{R}^n)$ be the Banach space of measurable functions $y : J \to \mathbb{R}^n$ which are Lebesgue integrable and normed by $\|y\|_{L^1} = \int_0^b \|y(t)\| dt$.

Lemma 2.6. *[13] Let X be a Banach space, $F : J \to \mathcal{P}_{cp,cv}(X)$ be an Caratheodory set-valued map with $S_{F,y} \neq \emptyset$ and Γ be a linear continuous mapping from $L^1(J; X)$ into $C(J; X)$. Then the operator*

$$\Gamma \circ S_F : C(J; X) \longrightarrow \mathcal{P}_{cp,cv}(C(J; X))$$

$$y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(J; X) \times C(J; X)$

3. Main results

In this section, first we present as an application of Theorem 7.3 of [19], the study of the following nonlinear integral inclusion:

$$x(t) \in g(t, x(t)) + \int_0^t F(s, x(s))ds$$

We begin by recalling the following definition.

Definition 3.1. [17] A mapping $A : D(A) \subseteq X \to X$ is said to be ϕ-expansive if there exists a function $\phi : [0, \infty) \to [0, \infty)$ such that $\|Ax - Ay\| \geq \phi(\|x - y\|)$ for all $x, y \in D(A)$, and ϕ satisfies the following conditions:

(i) $\phi(0) = 0$;
(ii) $\phi(r) > 0$ for $r > 0$;
(iii) either ϕ is continuous or ϕ is nondecreasing.

In what follows, for any map B, we denote the image of B by $R(B)$. We also need the following lemma.

Lemma 3.2. *[17] Let M be a nonempty bounded closed subset of a Banach space X and let $B : M \to X$ be a ϕ-expansive mapping. Then B is injective and the mapping $B^{-1} : R(B) \to M$ is uniformly continuous.*

In [19], Graef et al. proved the following theorem:

Theorem 3.3. *[19] Let X be a Banach space, M be a closed bounded convex subset of X. Suppose that $A : M \to \mathcal{P}_{cp,cv}(X)$ is an upper semi-continuous set-valued mapping, and let $B : M \to X$ is a continuous mapping such that*

(i) *A is compact;*
(ii) *B is k-set contractive;*
(iii) *$(I - B)^{-1} : R(I - B) \to M$ exists and is uniformly continuous;*
(iv) *$A(M) + B(M) \subset M$.*

Then there exists $x \in M$ such that $x \in A(x) + B(x)$.

We will now study the existence of solutions for the nonlinear integral inclusion

$$x(t) \in g(t, x(t)) + \int_0^t F(s, x(s))ds \tag{3.1}$$

on $C(J; \mathbb{R}^n)$, where the maps F and g satisfy the following conditions:

(E_1) The map $g : J \times \mathbb{R}^n \to \mathbb{R}^n$ is uniformly continuous on the bounded subsets of $J \times X$, $g(t, .)$ is a k-set-contraction mapping and let $M_r := \max\{\|g(t, x)\| : \|x\| \leq r\ ,\ t \in J\}$,
(E_2) $I - g(t, .) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is ϕ-expansive,
(E_3) The set-valued map $F : J \times \mathbb{R}^n \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is a Caratheodory and there exist functions $m \in L^1(J; \mathbb{R}^+)$ and an increasing function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\|F(t, x)\| \leq m(t)\psi(\|x\|)$.
(E_4) $\lim_{r\to\infty} \dfrac{\|m\|_{L^1}\psi(r) + M_r}{r} < 1$.

Theorem 3.4. *Equation (3.1) has a solution in $C(J; \mathbb{R}^n)$ whenever the conditions (E_1)- (E_4) are satisfied.*

Proof. We define

$$A : C(J;\mathbb{R}^n) \to \mathcal{P}(C(J;\mathbb{R}^n)),$$
$$x \mapsto A(x)(t) = \{h(t) \in C(J;\mathbb{R}^n) : \ h(t) = \int_0^t \upsilon(s)ds\}$$

where $\upsilon \in S_{F,x} = \{\upsilon \in L^1(J;\mathbb{R}^n) : \ \upsilon(t) \in F(t,x(t)) \ a.e \ t \in J\}$. Since F is a Caratheodory map, the set $S_{F,x}$ is nonempty, and

$$B : C(J;\mathbb{R}^n) \to C(J;\mathbb{R}^n)$$
$$x \mapsto B(x)(t) = g(t,x(t)).$$

Our task consists into see that $A+B$ has a fixed point. First, notice that $A(x)$ is convex for each $x \in C(J;\mathbb{R}^n)$. This follows from the convexity of $S_{F,x}$ since F has convex values.

Step 1. We prove that A has a closed graph. Let $h_n \in A(x_n)$ be such that $x_n \to x^*$ and $h_n \to h^*$. We prove that $h^* \in A(x^*)$. Since $h_n \in A(x_n)$ so there exists $\upsilon_n \in S_{F,x_n}$ such that

$$h_n(t) = \int_0^t \upsilon_n(s)ds.$$

Consider the linear continuous operator

$$\Gamma : L^1(J;\mathbb{R}^n) \to C(J;\mathbb{R}^n)$$
$$\upsilon \longmapsto \Gamma(\upsilon)(t) = \int_0^t \upsilon(s)ds.$$

By Lemma 6.155 of [13], $\Gamma \circ S_F$ has a closed graph, Since

$$h_n(.) \in \Gamma(S_{F,x_n}),$$

then $(x^*, h^*) \in Gr(\Gamma \circ S_F)$. Hence, there exists $\upsilon^* \in S_{F,x^*}$ such that

$$h^*(t) = \int_0^t \upsilon^*(s)ds \ t \in J,$$

for some $\upsilon^* \in S_{F,x^*}$. Therefore, A has a closed graph and so A is closed valued.

Step 2. A maps bounded sets into bounded sets in $C(J;\mathbb{R}^n)$. Indeed, it is enough to show that there exists a positive constant l such that for each $h \in A(x), x \in B_q = \{y \in C(J;\mathbb{R}^n) : \ \|y\| \le q\}$ one has $\|h\| \le l$. If $h \in A(x)$, then there exists $\upsilon \in S_{F,x}$ such that for each $t \in J$; we have

$$h(t) = \int_0^t \upsilon(s)ds.$$

By (E_3), we have for each $t \in J$

$$\|h(t)\| \le \int_0^t \|\upsilon(s)\|ds \le \psi(q) \int_0^t m(s)ds = l.$$

Then for each $h \in A(B_q)$; we have $\|h\|_\infty \le l$.

Step 3. A maps bounded sets into equicontinuous sets of $C(J;\mathbb{R}^n)$. Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ and $B_q = \{y \in C(J;\mathbb{R}^n) : \ \|y\| \le q\}$ be a bounded set of $C(J;\mathbb{R}^n)$. For each $x \in B_q$ and $h \in A(x)$, there exists $\upsilon \in S_{F,x}$ such that

$$h(t) = \int_0^t \upsilon(s)ds \ t \in J.$$

Thus,

$$\|h(\tau_2) - h(\tau_1)\| \le \int_{\tau_1}^{\tau_2} \|\upsilon(s)\|ds \le \psi(q) \int_{\tau_1}^{\tau_2} m(s)ds.$$

As $|\tau_2 - \tau_1| \to 0$, the right-hand side of the above inequality tends to zero. As a consequence of step 1, 2 and 3 together with the Ascoli-Arzela theorem, we conclude that $A : C(J;\mathbb{R}^n) \to \mathcal{P}_{cp,cv}(C(J;\mathbb{R}^n))$ is a compact set-valued map and upper semi-continuous map.

On the other hand, since g is uniformly continuous on the bounded subsets of $J \times \mathbb{R}^n$ and $g(t,.)$ is k-set-contraction, we easily obtain that B is a continuous k-set-contraction mapping on $J \times \mathbb{R}^n$. Also, by Theorem 4.4 of [17], we see that $I - B$ is a ϕ-expansive operator on $C(J;\mathbb{R}^n)$.

Finally, we show that there exists $r_0 > 0$ such that $A(B_{r_0}(0)) + B(B_{r_0}(0)) \subseteq B_{r_0}(0)$. Otherwise, for every $r > 0$ we can find $u_r, v_r \in B_{r_0}(0)$ and $h \in A(u_r)$ such that $\|h + Bv_r\|_\infty > r$. This means that

$$\frac{1}{r}\|h + Bv_r\|_\infty > 1.$$

Also there exists $\upsilon \in S_{F,u_r}$ such that

$$h(t) = \int_0^t \upsilon(s)ds.$$

Then we may assume that

$$\begin{aligned} |h(t) + Bv_r(t)| &\leq M_r + \int_0^t \|\upsilon(s)\|ds \\ &\leq M_r + \int_0^b m_1(s)\psi(r)ds \leq M_r + \|m\|_{L^1}\psi(r). \end{aligned}$$

Consequently,

$$\liminf_{r\to\infty} \frac{1}{r}\|h + Bv_r\|_\infty \leq \liminf_{r\to\infty} \frac{M_r + \|m\|_{L^1}\psi(r)}{r} < 1.$$

This is a contradiction. Thus, by Theorem 3.3, there exists $x \in C(J,\mathbb{R}^n)$ such that $x \in A(x) + B(x)$. Therefore integral inclusion (3.1) has a solution. □

For the rest of this section, we prove the existence solution for the following nonlinear integral inclusion:

$$x(t) \in p(t) + \int_0^{\alpha(t)} k_1(t,s)F(s, x(\eta(s)))ds + \int_0^{\beta(t)} k_2(t,s)G(s, x(\theta(s)))ds.$$

For this purpose, we need the following theorem proved by Basoc and Cardinali [6].

Theorem 3.5. *Let X be a Banach space and M be a closed bounded convex subset of X. Suppose that $A, B : M \to \mathcal{P}_{cp,cv}(X)$ are set-valued mappings such that*

(i) *A is weakly lower semi-continuous;*
(ii) *A maps bounded sets into relatively compact sets;*
(iii) *B is condensing;*
(iv) *B has closed graph;*
(v) *$B(M) + A(M) \subset M$.*

Then there exists $x \in M$ such that $x \in A(x) + B(x)$.

We consider the following nonlinear integral inclusion:

$$x(t) \in p(t) + \int_0^{\alpha(t)} k_1(t,s)F(s, x(\eta(s)))ds + \int_0^{\beta(t)} k_2(t,s)G(s, x(\theta(s)))ds. \tag{3.2}$$

where $p : J \to \mathbb{R}^n, k_1, k_2 : J \times J \to \mathbb{R}, F, G : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ and $\alpha, \beta, \eta, \theta : J \to J$.
We consider the following hypotheses in the sequel.

(H_1) The functions $\alpha, \beta, \eta, \theta : J \to J$ are continuous.
(H_2) The function $p : J \to \mathbb{R}^n$ is continuous.
(H_3) The functions k_1, k_2 are continuous on $J \times J$ with $K_1 = \max_{t,s\in J} |k_1(t,s)|$ and $K_2 = \max_{t,s\in J} |k_2(t,s)|$.
(H_4) the set-valued map $F : J \times \mathbb{R}^n \to \mathcal{P}_{cp}(\mathbb{R}^n)$ satisfying
 (i) F is an integrably bounded. i.e. there exist functions $m_1 \in L^1(J;\mathbb{R}^+)$ such that $\|y\| \leq m_1(t)$ for every $x \in \mathbb{R}^n$, $t \in J$ and $y \in F(t,x)$.
 (ii) the mapping $(t,x) \to F(t,x)$ is $L \otimes B$ measurable;
 (iii) the mapping $x \to F(t,x)$ is lower semi-continuous for a.e. $t \in J$.

(H_5) The set-valued map $G : J \times \mathbb{R}^n \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is a Caratheodory and there exist two functions $m_2, \psi_2 \in L^1(J;\mathbb{R}^+)$ and an increasing function $\psi_2 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$\|G(t,x)\| \le m_2(t)\psi_2(\|x\|) \ a.e\ t \in J,$$

for all $x \in \mathbb{R}^n$.

(H_6) $\lim_{r\to\infty} \dfrac{\|p\| + K_1\|m_1\|_{L^1} + K_2\|m_2\|_{L^1}\psi_2(r)}{r} < 1.$

Notice that Dhage in [11] proved the existence results for above nonlinear integral inclusion under mixed Lipschitz and Caratheodory conditions.

Theorem 3.6. *Equation (3.2) has a solution in $C(J;\mathbb{R}^n)$ whenever the conditions (H_1)- (H_6) are satisfied.*

Proof. First, condition (H_4) and (H_5) imply by Lemma 6.143 and Lemma 6.138 of [13], there exists a continuous map $f : C(J;\mathbb{R}^n) \to L^1(J;\mathbb{R}^n)$ such that $(fx)(t) \in F(t, x(\eta(t)))$ for all $t \in J$ and $x \in C(J;\mathbb{R}^n)$. Now, we define

$$A : C(J;\mathbb{R}^n) \to C(J;\mathbb{R}^n)$$

$$x \mapsto A(x)(t) = \int_0^{\alpha(t)} k_1(t,s)(fx)(s)ds.$$

and

$$B : C(J;\mathbb{R}^n) \to \mathcal{P}(C(J;\mathbb{R}^n))$$

$$x \mapsto B(x)(t) = \{h(t) \in C(J;\mathbb{R}^n) : \ h(t) = p(t) + \int_0^{\beta(t)} k_2(t,s)v(s)ds, \ v \in S_{G,x}\},$$

where $S_{G,x} = \{v \in L^1(J;\mathbb{R}^n) : \ v(t) \in G(t, x(\theta(t))) \ a.e\ t \in J\}$. Since G is a Caratheodory map, the set $S_{G,x}$ is nonempty. Then solution of equation (3.2) is equivalent to the operator inclusion $x(t) \in Ax(t) + Bx(t)$ for each $t \in J$. We will show that the set-valued operators A and B satisfy all the conditions of Theorem 3.5.

First, notice that $B(x)$ is convex for each $x \in C(J;\mathbb{R}^n)$. This follows from the convexity of $S_{G,x}$, since G has convex values.

Step 1. We prove that B has a closed graph. Let $h_n \in B(x_n)$ be such that $x_n \to x^*$ and $h_n \to h^*$. We shall prove that $h^* \in B(x^*)$. Since $h_n \in B(x_n)$ so there exists $v_n \in S_{G,x_n}$ such that

$$h_n(t) = p(t) + \int_0^{\beta(t)} k_2(t,s)v_n(s)ds.$$

Consider the linear continuous operator

$$\Gamma : L^1(J;\mathbb{R}^n) \to C(J;\mathbb{R}^n)$$

$$v \longmapsto \Gamma(v)(t) = \int_0^{\beta(t)} k_2(t,s)v(s)ds.$$

Since

$$h_n(.) - p(.) \in \Gamma(S_{G,x_n}),$$

by Lemma 6.155 of [13], $\Gamma \circ S_G$ has a closed graph, then $(x^*, h^* - p) \in Gr(\Gamma \circ S_G)$. Hence, there exists $v^* \in S_{G,x^*}$ such that

$$h^*(t) = p(t) + \int_0^{\beta(t)} k_2(t,s)v^*(s)ds \ t \in J$$

for some $v^* \in S_{G,x^*}$. Therefore, B has a closed graph and so B is closed valued.

Step 2. B maps bounded sets into bounded sets in $C(J;\mathbb{R}^n)$. Indeed, it is enough to show that there exists a positive constant l such that for each $h \in B(x), x \in B_q = \{x \in C(J;\mathbb{R}^n) : \ \|x\| \le q\}$ one has $\|h\| \le l$. If $h \in B(x)$, then there exists $v \in S_{G,x}$ such that for each $t \in J$, we have

$$h(t) = p(t) + \int_0^{\beta(t)} k_2(t,s)v(s)ds.$$

Since p is continuous then there exists ρ such that $\|p\| \leq \rho$. By (H_3), we have for each $t \in J$

$$\|h(t)\| \leq \rho + \int_0^{\beta(t)} \|\upsilon(s)\| ds \leq \rho + \psi_2(q) \int_0^b K_2 m_2(s) ds = l.$$

Then for each $h \in B(B_q)$, we have $\|h\|_\infty \leq l$.
Step 3. B maps bounded sets into equicontinuous sets of $C(J;\mathbb{R}^n)$. Let $\tau_1, \tau_2 \in J$; $\tau_1 < \tau_2$ and $B_q = \{y \in C(J;\mathbb{R}^n) : \|y\| \leq q\}$ be a bounded set of $C(J;\mathbb{R}^n)$. For each $x \in B_q$ and $h \in B(x)$, there exists $\upsilon \in S_{G,x}$ such that

$$h(t) = \int_0^{\beta(t)} k_2(t,s)\upsilon(s) ds \; t \in J.$$

Thus,

$$\begin{aligned}
\|h(\tau_2) - h(\tau_1)\| &= \|p(\tau_2) + \int_0^{\beta(\tau_2)} k_2(\tau_2,s)\upsilon(s)ds - p(\tau_1) - \int_0^{\beta(\tau_1)} k_2(\tau_1,s)\upsilon(s)ds\| \\
&\leq \|p(\tau_2) - p(\tau_1)\| + \| \int_0^{\beta(\tau_2)} k_2(\tau_2,s)\upsilon(s)ds - \int_0^{\beta(\tau_2)} k_2(\tau_1,s)\upsilon(s)ds\| \\
&+ \| \int_0^{\beta(\tau_2)} k_2(\tau_1,s)\upsilon(s)ds - \int_0^{\beta(\tau_1)} k_2(\tau_1,s)\upsilon(s)ds\| \\
&\leq \|p(\tau_2) - p(\tau_1)\| + \int_0^{\beta(\tau_2)} |k_2(\tau_2,s) - k_2(\tau_1,s)| \|\upsilon(s)\| ds \\
&+ \int_{\beta(\tau_1)}^{\beta(\tau_2)} |k_2(\tau_1,s)| \|\upsilon(s)\| ds \\
&\leq \|p(\tau_2) - p(\tau_1)\| + \psi_2(q) \int_0^1 |k_2(\tau_2,s) - k_2(\tau_1,s)| m_2(s) ds \\
&+ \psi_2(q) \int_{\beta(\tau_1)}^{\beta(\tau_2)} K_2 m_2(s) ds.
\end{aligned}$$

As $|\tau_2 - \tau_1| \to 0$, the right-hand side of the above inequality tends to zero. As a consequence of step 1, 2 and 3 together with the Ascoli-Arzela theorem, we conclude that $B : C(J;\mathbb{R}^n) \to \mathcal{P}_{cp,cv}(C(J;\mathbb{R}^n))$ is compact and upper semi-continuous map.
On the other hand we prove that A is compact and continuous map. by a similar proof as that of step 2 and 3 for B, we conclude that A is compact map. Now we prove that A is a continuous map. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C(J, \mathbb{R}^n)$. Then

$$\begin{aligned}
\|(Ax_n)(t) - (Ax)(t)\| &= \| \int_0^{\alpha(t)} k_1(t,s)(fx_n)(s) - \int_0^{\alpha(t)} k_1(t,s)(fx)(s)\| \\
&\leq \int_0^{\alpha(t)} |k_1(t,s)| \|(fx_n)(s) - (fx)(s)\| \\
&\leq K_1 \int_0^b \|(fx_n)(s) - (fx)(s)\|.
\end{aligned}$$

Since f is continuous then

$$\|Ax_n - Ax\|_\infty \leq K_1 \|fx_n - fx\|_{L^1} \to 0,$$

as $n \to \infty$. So A is continuous.
Finally, let us show that there exists $r_0 > 0$ such that $A(B_{r_0}(0)) + B(B_{r_0}(0)) \subseteq B_{r_0}(0)$. Otherwise, for every $r > 0$ we can find $u_r, v_r \in B_{r_0}(0)$ and $h \in B(v_r)$ such that $\|A(u_r) + h\|_\infty > r$. This means that

$$\frac{1}{r} \|A(u_r) + h\|_\infty > 1.$$

Also there exists $\upsilon \in S_{G,v_r}$ such that

$$h(t) = p(t) + \int_0^{\beta(t)} k_2(t,s)\upsilon(s)ds.$$

Then we may assume that

$$\begin{aligned} \|(Au_r)(t) + h(t)\| &\leq \int_0^{\alpha(t)} |k_1(t,s)| \|(fu_r)(s)\| ds + p(t) + \int_0^t |k_2(t,s)| \|\upsilon(s)\| ds \\ &\leq K_1 \int_0^b m_1(s) ds + \|p\| + K_2 \int_0^b m_2(s)\psi_2(r) ds \\ &\leq K_1 \|m_1\|_{L^1} + \|p\| + K_2 \|m_2\|_{L^1} \psi_2(r). \end{aligned}$$

Consequently,

$$\liminf_{r\to\infty} \frac{1}{r} \|A(u_r) + h\|_\infty \quad \leq \quad \liminf_{r\to\infty} \frac{\|p\| + K_1\|m_1\|_{L^1} + K_2\|m_2\|_{L^1}\psi_2(r)}{r} < 1.$$

This is a contradiction. Therefore $A + B$ satisfies to condition Theorem 3.5, then there exists $x \in C(J, \mathbb{R}^n)$ such that $x \in A(x) + B(x)$. So integral inclusion (3.2) has a solution.

□

As an application of the integral inclusion (3.2), we consider the following the boundary valued problem of ordinary differential inclusion,

$$\begin{cases} x''(t) \in F(t, x(\eta(t))) + G(t, x(\theta(t))) \qquad a.e\ t \in J, \\ \\ x(0) = x(1) = \overbrace{(0, ..., 0)}^{n} \end{cases} \tag{3.3}$$

where $F, G : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ and $\eta, \theta : J \to J$ are continuous.

Theorem 3.7. *Assume that the hypotheses $(H4)$ and (H_5) hold. Also suppose*

$$\lim_{r\to\infty} \frac{\|m_1\|_{L^1} + \|m_2\|_{L^1}\psi_2(r)}{r} < 1.$$

Then problem (3.3) has a solution on J.

Proof. First, similar proof of Theorem 5.1 of [17], problem (3.3) is reformulated as a nonlinear integral inclusion

$$x(t) \in \int_0^1 k(t,s)F(s, x(\eta(s)))ds + \int_0^1 k(t,s)G(s, x(\theta(s)))ds, \tag{3.4}$$

where $k(t,s)$ is the Greens function of (3.3) that is the solution to the linear homogeneous equation with boundary condition

$$\begin{cases} g''(t) = 0 \qquad\qquad a.e\ t \in J, \\ \\ g(0) = g(1) = 0, \end{cases} \tag{3.5}$$

where $g \in C(J, \mathbb{R})$. In this case, we have $|k(t,s)| \leq \frac{1}{4}$. Therefore, S satisfies the condition Theorem 3.6 with $k_1(t,s) = k_2(t,s) = k(t,s)$, $\alpha(t) = \beta(t) = 1$ and $p(t) = 0$ so nonlinear integral inclusion (3.4) has a solution. Then problem (3.3) has a solution on J.

□

Conclusion

In this work, we successfully apply two fixed point theorems have been presented by Graef et al. and Basoc and Cardinali to obtain the solution of two nonlinear integral inclusions on $C([0,b];\mathbb{R}^n)$. As an application of these integral inclusions, the existence of solution of the boundary valued problem of ordinary differential inclusion is proved. Notice that, the existence of solution of the two nonlinear integral inclusions on $C([0,b];X)$, where X is Banach spaces, could be studied in the future researches

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