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Qualitative analysis of fractional differential equations with ψ-Hilfer fractional derivative

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Abstract

In this paper, we investigate the solutions of a class of ψ-Hilfer fractional differential equations with the initial values in the sense of ψ-fractional integral by using the successive approximation techniques. Next, the continuous dependence of a solution for the given Cauchy-type problem is presented.

Keywords. ψ-Hilfer fractional derivative, Cauchy-type problem, Continuous dependence.
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1. Preface

This paper tries to investigate some existence results for the following ψ-Hilfer fractional differential equation involving an initial constraint of the type ψ-Hilfer integral:

$$\begin{cases} \mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{u}(t) = \mathfrak{g}(t,\mathfrak{u}(t)), \quad t \in J := (a,b], \\ \mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(a) = \mathfrak{u}_a, \end{cases} \tag{1.1}$$

where $\mathfrak{D}^{\alpha,\beta;\psi}$ denotes the ψ-Hilfer fractional derivative of order α $(0 < \alpha < 1)$ and type β $(0 \leq \beta \leq 1)$, and the ψ-fractional integral of order $1-\gamma$ $(\gamma = \alpha + \beta - \alpha\beta)$ is denoted by $\mathfrak{I}^{1-\gamma;\psi}$. Moreover, $\mathfrak{g} : J \times \mathbb{R} \to \mathbb{R}$ is a given nonlinear continuous function.

The idea of fractional calculus and fractional order differential equations and inclusions has been a subject of interest not only among mathematicians but also among physicists and engineers. Indeed, we can find numerous applications in rheology, porous media, viscoelasticity, electrochemistry, electromagnetism, signal processing, optics, geology, viscoelastic materials, biosciences, statistical physics, thermodynamics, neural networks, etc. In recent years, there has been a significant development in fractional calculus techniques in ordinary and partial differential equations, difference differential equations, and inclusions, some recent contributions can be seen in, [2, 3, 8, 11, 13–15]. In the meantime, a real generalization of the famous Riemann-Liouville, and Caputo fractional operators [10, 12] is Hilfer fractional derivatives which has attracted much attention among various scientific disciplines. Some information about properties and applications of Hilfer derivative can be found in [7]. Existence and uniqueness results for differential equations involving Hilfer fractional operators can be seen in [1, 5, 6, 9, 17, 19–21] and references therein.

Indicating the interval of existence of solution is an essential appearance in practical use which cannot be done by fixed point techniques. This defect of fixed point method is removed by using Picard's iterative technique and existence of a solution is investigated by Dhaigude, see [4].

The outline of the article is as follows. In section 2, we declare the weighted spaces, definitions and results. In section 3, we present our existence of solution using Picard's iterative technique. In section 4, we discuss continuous

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dependence of the Cauchy type problem for their neighbouring orders and initial conditions. Finally, in the last section we provide and an example.

2. Auxiliary results

Let $C(J,\mathbb{R})$ be the space of continuous functions from the finite interval $J := [a,b]$, $(0 \leq a < b < \infty)$ into $\mathbb{R}$. A related function space is the weighted space $C_{\gamma,\psi}(J,\mathbb{R})$, containing all functions $\mathfrak{g}$ which have the following property

$$C_{\gamma,\psi}(J,\mathbb{R}) = \left\{\mathfrak{g} : J \to \mathbb{R} : (\psi(t)-\psi(a))^{\gamma}\, \mathfrak{g}(t) \in C(J,\mathbb{R})\right\}, 0 \leq \gamma < 1.$$

Easily, the proper norm for the space $C_{\gamma,\psi}(J,\mathbb{R})$, can be defined as follows:

$$\|\mathfrak{g}\|_{C_{\gamma,\psi}} = \|(\psi(t)-\psi(a))^{\gamma}\, \mathfrak{g}(t)\|_{C(J,\mathbb{R})} = \max_{t\in J} \left|(\psi(t)-\psi(a))^{\gamma}\, \mathfrak{g}(t)\right|.$$

The weighted space $C^n_{\gamma,\psi}(J,\mathbb{R})$ for $n \geq 1$ of the functions $\mathfrak{g}$ on $(J,\mathbb{R})$ is defined by

$$C^n_{\gamma,\psi}(J,\mathbb{R}) = \left\{\mathfrak{g} : J \to \mathbb{R} : \mathfrak{g}(t) \in C^{n-1}(J,\mathbb{R}); \mathfrak{g}^{(n)}(t) \in C_{\gamma,\psi}(J,\mathbb{R})\right\}, 0 \leq \gamma < 1,$$

with the norm

$$\|\mathfrak{g}\|_{C^n_{\gamma,\psi}(J,\mathbb{R})} = \sum_{k=0}^{n-1} \left\|\mathfrak{g}^{(k)}\right\|_{C(J,\mathbb{R})} + \left\|\mathfrak{g}^{(n)}\right\|_{C_{\gamma,\psi}(J,\mathbb{R})}.$$

For $n = 0$, we have, $C^0_{\gamma,\psi}(J,\mathbb{R}) = C_{\gamma,\psi}(J,\mathbb{R})$.
Here we present the following weighted space for our problem as follows

$$C^{\alpha,\beta}_{1-\gamma;\psi}(J,\mathbb{R}) = \left\{\mathfrak{g} : J \to \mathbb{R} : \mathfrak{g} \in C_{1-\gamma;\psi}(J,\mathbb{R}), \mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{g} \in C_{1-\gamma;\psi}(J,\mathbb{R})\right\},$$

and

$$C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R}) = \left\{\mathfrak{g} : J \to \mathbb{R} : \mathfrak{g} \in C_{1-\gamma;\psi}(J,\mathbb{R}), \mathfrak{D}^{\gamma;\psi}\mathfrak{g} \in C_{1-\gamma;\psi}(J,\mathbb{R})\right\}.$$

It is obvious that

$$C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R}) \subset C^{\alpha,\beta}_{1-\gamma;\psi}(J,\mathbb{R}).$$

Definition 2.1. [16] Let $0 \leq \alpha \leq 1$, and ψ be an increasing continuously differentiable function on the finite or infinite interval $J := [a,b]$ $(-\infty \leq a < t < b \leq \infty)$ such that $\psi'(t) \neq 0$ for any $t \in J$. The left-sided ψ-fractional integral of a given integrable function $\mathfrak{g} : J \to \mathbb{R}$ is defined as:

$$\left(\mathfrak{I}^{\alpha;\psi}\right)\mathfrak{g}(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \mathbf{L}^{\alpha;\psi}(t,s)\mathfrak{g}(s)ds,\ t > a. \tag{2.1}$$

where $\mathbf{L}^{\alpha;\psi}(t,s) := \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}$. In this sense, it can be defined the Riemann-Liouville fractional derivative of the function $\mathfrak{g}$ with respect to ψ of order α as follows:

$$\begin{aligned}\left(\mathfrak{D}^{\alpha;\psi}\mathfrak{g}\right)(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\mathfrak{I}^{1-\alpha;\psi}\mathfrak{g}(t)\\ &= \frac{1}{\Gamma(1-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\int_a^t \mathbf{L}^{1-\alpha;\psi}(t,s)\mathfrak{g}(s)ds.\end{aligned}$$

Inspired by the definition of (2.1), we regard an improved version of the classical Caputo derivative, so-called ψ-Caputo derivative, as follows:

$$\left({}^{C}\mathfrak{D}^{\alpha;\psi}\mathfrak{g}\right)(t) = \mathfrak{I}^{1-\alpha;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\mathfrak{g}(t). \tag{2.2}$$

Recently, Sousa and Oliveira have introduced a general definition of fractional derivatives with respect to the increasing function ψ involving two parameters $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, i.e., ψ-Hilfer fractional derivative in [16] as

$$\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{g}(t) = \mathfrak{I}^{\beta(1-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\mathfrak{I}^{(1-\beta)(1-\alpha);\psi}\mathfrak{g}(t). \tag{2.3}$$

The following formula delineates a basic relation between ψ-Hilfer fractional derivative and Riemann-Liouville fractional operators:

$$\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{g}(t)=\mathfrak{I}^{\gamma-\alpha;\psi}\mathfrak{D}^{\gamma;\psi}\mathfrak{g}(t).$$

Also, the following essential properties hold:

$$(\mathfrak{I}^{\alpha;\psi}\mathfrak{I}^{\beta;\psi}\mathfrak{g})(t)=(\mathfrak{I}^{\alpha+\beta;\psi}\mathfrak{g})(t),\ \alpha,\beta>0,$$
$$\mathfrak{I}^{\alpha;\psi}\Big((\psi(t)-\psi(a))^{\delta-1}\Big)=\frac{\Gamma(\delta)}{\Gamma(\alpha+\delta)}(\psi(t)-\psi(a))^{\alpha+\delta-1},\ \alpha,\delta>0,$$
$$\mathfrak{D}^{\alpha,\beta;\psi}\Big((\psi(t)-\psi(a))^{\delta-1}\Big)=\frac{\Gamma(\delta)}{\Gamma(\delta-\alpha)}(\psi(t)-\psi(a))^{\delta-\alpha-1},\ 0<\alpha<1,\ 0\leq\beta\leq1,\ \delta>0,$$
$$(\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{I}^{\alpha;\psi}\mathfrak{g})(t)=\mathfrak{g}(t),\ 0<\alpha<1,\ 0\leq\beta\leq1,\ \mathfrak{g}\in C^1(J,\mathbb{R}),$$
$$\mathfrak{I}^{\alpha;\psi}\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{g}(t)=\mathfrak{g}(t)-\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{g}(a),\ 0<\alpha<1,\ 0\leq\beta\leq1,\ \mathfrak{g}\in C^1(J,\mathbb{R}),$$

where $\gamma=\alpha+\beta-\alpha\beta$. On the other hand, for $0<\alpha<1$ and $0\leq\beta\leq1$, the operator $\mathfrak{D}^{\alpha,\beta;\psi}$ is bounded from $C^1_{\gamma;\psi}(J,\mathbb{R})$ into $C_{\gamma;\psi}(J,\mathbb{R})$, in which $\gamma=\alpha+\beta-\alpha\beta$.

Lemma 2.2. *If $\alpha>0$ and $0\leq\mu<1$, then $\mathfrak{I}^{\alpha;\psi}$ is bounded from $C_{\mu;\psi}(J,\mathbb{R})$ into $C_{\mu;\psi}(J,\mathbb{R})$. In addition, if $\mu\leq\alpha$, then $\mathfrak{I}^{\alpha;\psi}$ is bounded from $C_{\mu;\psi}(J,\mathbb{R})$ into $C(J,\mathbb{R})$.*

Lemma 2.3. *[18] Let $\alpha>0$, $g(t)$ be a nonnegative, nondecreasing continuous function defined on $a\leq t<b$, with $g(t)\leq K$ for some constant K, and $a(t)$ be a nonnegative function locally integrable on $a\leq t<b$ ($b\leq\infty$). Further let $\mathfrak{u}(t)$ be a nonnegative locally integrable on $a\leq t<b$ function satisfying*

$$|\mathfrak{u}(t)|\leq a(t)+g(t)\int_a^t\psi^{'}(s)\,(\psi(t)-\psi(s))^{\alpha-1}\,\mathfrak{u}(s)ds,\quad t\in J,$$

with some $\alpha>0$. Then

$$|\mathfrak{u}(t)|\leq a(t)+\int_a^t\left[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\psi^{'}(s)\,(\psi(t)-\psi(s))^{n\alpha-1}\right]a(s)ds,\quad a\leq t<b.$$

3. Existence results

In this part, we prove some results in existence and uniqueness of proposed Cauchy type problem (1.1). Before starting and proving these results, we list the following condition:

(H1) Lipschitz condition:
There exist a constant $\ell>0$ such that

$$|\mathfrak{g}(t,\mathfrak{u})-\mathfrak{g}(t,\overline{\mathfrak{u}})|\leq\ell\,|\mathfrak{u}-\overline{\mathfrak{u}}|\,,$$

for any $\mathfrak{u},\overline{\mathfrak{u}}\in\mathbb{R}$, and $t\in J$.

Lemma 3.1. *Let us consider $\gamma=\alpha+\beta-\alpha\beta$, where $0<\alpha<1$ and $0\leq\beta\leq1$. Suppose that $\mathfrak{g}:J\times\mathbb{R}\to\mathbb{R}$ be a function such that $\mathfrak{g}(\cdot,\mathfrak{u}(\cdot))\in C_{1-\gamma;\psi}(J,\mathbb{R})$ with $\mathfrak{u}\in C_{1-\gamma;\psi}(J,\mathbb{R})$. If $\mathfrak{u}\in C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R})$, then $\mathfrak{u}$ satisfies (1.1), if and only if $\mathfrak{u}$ satisfies the following Volterra integral equation of the second kind*

$$\mathfrak{u}(t)=\frac{\mathfrak{u}_a}{\Gamma(\gamma)}\,(\psi(t)-\psi(a))^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_a^t\psi^{'}(s)\,(\psi(t)-\psi(s))^{\alpha-1}\,\mathfrak{g}(s,\mathfrak{u}(s))ds. \tag{3.1}$$

Lemma 3.2. *Consider $\gamma=\alpha+\beta-\alpha\beta$, with $0<\alpha<1$ and $0\leq\beta\leq1$, then the ψ-Riemann-Liouville fractional integral operator $\mathfrak{I}^{\alpha;\psi}$ is bounded from $C_{1-\gamma;\psi}(J,\mathbb{R})$ to $C_{1-\gamma;\psi}(J,\mathbb{R})$:*

$$\left\|\mathfrak{I}^{\alpha;\psi}\mathfrak{g}\right\|_{C_{1-\gamma;\psi}}\leq M\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\,(\psi(t)-\psi(a))^{\alpha}\,, \tag{3.2}$$

where, M is the bound of bounded function $\mathfrak{g}$.

Proof. According to Lemma 2.2, the result follows. Now we prove the estimate (3.2), we have

$$\left\|\mathfrak{I}^{\alpha;\psi}\mathfrak{g}\right\|_{C_{1-\gamma;\psi}} = \left\|(\psi(t)-\psi(a))^{1-\gamma}\,\mathfrak{I}^{\alpha;\psi}\mathfrak{g}\right\|_{C} \leq \|\mathfrak{g}\|_{C_{1-\gamma;\psi}} \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(t)-\psi(a)\right)^{\alpha},$$

therefore, we get,

$$\left\|\mathfrak{I}^{\alpha;\psi}\mathfrak{g}\right\|_{C_{1-\gamma;\psi}} \leq M\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(t)-\psi(a)\right)^{\alpha}.$$

□

Theorem 3.3. *Suppose that $\gamma=\alpha+\beta-\alpha\beta$, with $0<\alpha<1$ and $0\leq\beta\leq1$. If $\mathfrak{g}:J\times\mathbb{R}\to\mathbb{R}$ be a function such that $\mathfrak{g}(\cdot,\mathfrak{u}(\cdot))\in C_{1-\gamma;\psi}(J,\mathbb{R})$ for each $\mathfrak{u}\in C_{1-\gamma;\psi}(J,\mathbb{R})$ satisfying the condition [H1], then there exists a unique solution $\mathfrak{u}$ for the Cauchy-type problem (1.1) in $C^{\alpha,\beta}_{1-\gamma;\psi}(J,\mathbb{R})$.*

Proof. Since the integral equation (3.1) holds for any subinterval $[a,t_1]$ of $[a,b]$, we choose t_1 such that satisfies

$$\ell\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(t_1)-\psi(a)\right)^{\alpha}<1. \tag{3.3}$$

First, the proof of the existence of unique solution $\mathfrak{u}\in C_{1-\gamma;\psi}([a,t_1],\mathbb{R})$ is done for this subinterval. We proceed as follows. Set Picard's sequence functions

$$\mathfrak{u}_0(t)=\frac{\mathfrak{u}_a}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}, \tag{3.4}$$

$$\mathfrak{u}_m(t)=\mathfrak{u}_0(t)+\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}_{m-1}(s))ds, \quad m\in\mathbb{N}. \tag{3.5}$$

We now show that $\mathfrak{u}_m(t)\in C_{1-\gamma;\psi}(J,\mathbb{R})$. From equation (3.4), it follows that $\mathfrak{u}_0(t)\in C_{1-\gamma;\psi}(J,\mathbb{R})$. By Lemma 3.2, $\mathfrak{I}^{\alpha;\psi}$ is a bounded operator from $C_{1-\gamma;\psi}(J,\mathbb{R})$ into itself, which gives $\mathfrak{u}_m(t)\in C_{1-\gamma;\psi}(J,\mathbb{R})$, $m\in\mathbb{N}$. By equations (3.4) and (3.5), we have

$$\|\mathfrak{u}_1(t)-\mathfrak{u}_0(t)\|_{C_{1-\gamma;\psi}([a,t_1],\mathbb{R})}=\left\|\mathfrak{I}^{\alpha;\psi}\mathfrak{g}(t,\mathfrak{u}_0(t))\right\|_{C_{1-\gamma;\psi}([a,t_1],\mathbb{R})}.$$

Now by using Lemma 3.2, we get

$$\|\mathfrak{u}_1(t)-\mathfrak{u}_0(t)\|_{C_{1-\gamma;\psi}([a,t_1],\mathbb{R})}\leq M\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(t_1)-\psi(a)\right)^{\alpha}. \tag{3.6}$$

Furthermore, we obtain

$$\begin{aligned}&\|\mathfrak{u}_2(t)-\mathfrak{u}_1(t)\|_{C_{1-\gamma;\psi}([a,t_1],\mathbb{R})}\\ &\leq M\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(t_1)-\psi(a)\right)^{\alpha}\left(\ell\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(t_1)-\psi(a)\right)^{\alpha}\right).\end{aligned} \tag{3.7}$$

Repeating this process leads to

$$\begin{aligned}&\|\mathfrak{u}_m(t)-\mathfrak{u}_{m-1}(t)\|_{C_{1-\gamma;\psi}([a,t_1],\mathbb{R})}\\ &\leq M\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(t_1)-\psi(a)\right)^{\alpha}\left(\ell\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(t_1)-\psi(a)\right)^{\alpha}\right)^{m-1}.\end{aligned} \tag{3.8}$$

By equation (3.2), we get

$$\|\mathfrak{u}_m(t)-\mathfrak{u}_{m-1}(t)\|_{C_{1-\gamma;\psi}([a,t_1],\mathbb{R})}\to0, \ \textit{as } m\to+\infty. \tag{3.9}$$

Applying Lemma 3.2, it can be shown that

$$\begin{aligned}&\left\|\mathfrak{I}^{\alpha;\psi}\mathfrak{g}(t,\mathfrak{u}_m(t))-\mathfrak{I}^{\alpha;\psi}\mathfrak{g}(t,\mathfrak{u}(t))\right\|_{C_{1-\gamma;\psi}([a,t_1],\mathbb{R})}\\&\leq \ell\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(t_1)-\psi(a)\right)^{\alpha}\left\|\mathfrak{u}_m(t)-\mathfrak{u}(t)\right\|_{C_{1-\gamma;\psi}([a,t_1],\mathbb{R})},\end{aligned}$$

and hence by Lemma 3.2,

$$\left\|\mathfrak{I}^{\alpha;\psi}\mathfrak{g}(t,\mathfrak{u}_m(t))-\mathfrak{I}^{\alpha;\psi}\mathfrak{g}(t,\mathfrak{u}(t))\right\|_{C_{1-\gamma;\psi}([a,t_1],\mathbb{R})}\to 0,\ as\ m\to+\infty. \tag{3.10}$$

From equations (3.9) and (3.10), it can be verified that $\mathfrak{u}(t)$ is the solution of (3.1) in $C_{1-\gamma;\psi}([a,t_1],\mathbb{R})$.

Now for the uniqueness of the solution $\mathfrak{u}(t)$, let there exists two different solutions $\mathfrak{u}(t)$ and $\mathfrak{v}(t)$ for (3.1) on $[a,t_1]$. Substituting them into (3.1) and applying Lemma 2.2 and considering the condition [H1], we have

$$\begin{aligned}\left\|\mathfrak{u}(t)-\mathfrak{v}(t)\right\|_{C_{1-\gamma;\psi}([a,t_1],\mathbb{R})}&\leq\left\|\mathfrak{I}^{\alpha;\psi}\mathfrak{g}(t,\mathfrak{u}(t))-\mathfrak{I}^{\alpha;\psi}\mathfrak{g}(t,\mathfrak{v}(t))\right\|_{C_{1-\gamma;\psi}([a,t_1],\mathbb{R})}\\&\leq\ell\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(t_1)-\psi(a)\right)^{\alpha}\left\|\mathfrak{u}(t)-\mathfrak{v}(t)\right\|_{C_{1-\gamma;\psi}([a,t_1],\mathbb{R})}.\end{aligned} \tag{3.11}$$

This yields $\ell\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(t_1)-\psi(a)\right)^{\alpha}\geq 1$, which violates the condition (3.3). Thus there is a unique solution $\mathfrak{u}(t)=\mathfrak{u}_1(t)\in C_{1-\gamma;\psi}([a,t_1],\mathbb{R})$ on the subinterval $[a,t_1]$.

Next, we consider the interval $[t_1,t_2]$, where $t_2=t_1+h_1$, $h_1>0$ such that $t_2<b$. Now the integral equation (3.1) takes the form

$$\begin{aligned}\mathfrak{u}(t)=&\frac{\mathfrak{u}_a}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s))ds\\&+\frac{1}{\Gamma(\alpha)}\int_{a}^{t_1}\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s))ds,\quad t\in[t_1,t_2].\end{aligned} \tag{3.12}$$

Since $\mathfrak{u}(t)$ is the unique function defined on $[a,t_1]$, the last integral is the known function and therefore the integral equation (3.12) is rewritten in the following form

$$\mathfrak{u}(t)=\mathfrak{u}^*(t)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s))ds, \tag{3.13}$$

where

$$\mathfrak{u}^*(t)=\frac{\mathfrak{u}_a}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{a}^{t_1}\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s))ds, \tag{3.14}$$

is the known function. By a similar argument, we can deduce that there is a unique solution $\mathfrak{u}(t)=\mathfrak{u}_2(t)\in C_{1-\gamma;\psi}([t_1,t_2],\mathbb{R})$ on $[t_1,t_2]$. Taking the interval $[t_2,t_3]$, in which $t_3=t_2+h_2$, $h_2>a$ with the constraint $t_3<b$, and repeating the above process, one can found a unique solution $\mathfrak{u}(t)\in C_{1-\gamma;\psi}(J,\mathbb{R})$ of the integral equation (3.1) such that $\mathfrak{u}(t)=\mathfrak{u}_j(t)\in C_{1-\gamma;\psi}([t_{j-1},t_j],\mathbb{R})$ for $j=1,2,...,l$, and $a=\mathfrak{u}_0<\mathfrak{u}_1<...<\mathfrak{u}_l=b$. Applying the differential equation (1.1) and the Lipschitz hypothesis [H1], we can derive

$$\begin{aligned}\left\|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{u}_m(t)-\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{u}(t)\right\|_{C_{1-\gamma;\psi}}&=\left\|\mathfrak{g}(t,\mathfrak{u}_m(t))-\mathfrak{g}(t,\mathfrak{u}(t))\right\|_{C_{1-\gamma;\psi}}\\&\leq\ell\left\|\mathfrak{u}_m(t)-\mathfrak{u}(t)\right\|_{C_{1-\gamma;\psi}}.\end{aligned} \tag{3.15}$$

Clearly, (3.9) and (3.15) implies that $\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{u}(t)\in C_{1-\gamma;\psi}(J,\mathbb{R})$.

Thus, the proof of the theorem is complete. □

4. Continuous Dependence

In this part, we first investigate the continuous dependence of solution of the ψ-Hilfer fractional differential equation via generalized Gronwall's inequality as a handy tool. Consider the Eq. (1.1). To present the dependence of solution on the order, let us consider the solutions of two equations with the neighbouring orders. Before we present the continuous dependence of the Cauchy-type problem (1.1), we will study some results about the Cauchy-type problem involving ψ-fractional derivative of the form

$$\begin{cases} \mathfrak{D}^{\alpha;\psi}\mathfrak{u}(t) = \mathfrak{g}(t,\mathfrak{u}(t)), \ t \in J, \\ \mathfrak{I}^{1-\alpha;\psi}\mathfrak{u}(a) = \mathfrak{u}_a. \end{cases} \tag{4.1}$$

It was shown that the above problem is equivalent to the following integral equation

$$\mathfrak{u}(t) = \frac{\mathfrak{u}_a}{\Gamma(\alpha)} \left(\psi(t) - \psi(a)\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} \mathfrak{g}(s, \mathfrak{u}(s))ds. \tag{4.2}$$

First we present the continuous dependence of the solution of the Cauchy-type problem involving ψ-fractional differential equation

Theorem 4.1. *Suppose that $\alpha > 0$, $\nu > 0$ with $0 < \alpha - \nu \leq 1$. Let $\mathfrak{u}$ be a continuous function satisfying Lipschitz condition [H1] in $\mathbb{R}$. For $a \leq t < h < b$, assume that $\mathfrak{u}$ is the solution of Eq. (1.1) and $\overline{\mathfrak{u}}$ is the solution of equation*

$$\begin{cases} \mathfrak{D}^{\alpha-\nu;\psi}\overline{\mathfrak{u}}(t) = \mathfrak{g}(t,\overline{\mathfrak{u}}(t)), \\ \mathfrak{I}^{1-(\alpha-\nu);\psi}\overline{\mathfrak{u}}(a) = \overline{\mathfrak{u}}_a. \end{cases} \tag{4.3}$$

Then, for $a < t \leq h$, the following estimate holds

$$|\overline{\mathfrak{u}}(t) - \mathfrak{u}(t)| \leq K_1(t) + \int_a^t \left[\sum_{k=1}^{\infty} \left(\frac{\ell\Gamma(\alpha-\nu)}{\Gamma(\alpha)} \right)^k \frac{\psi^{'}(s) \left(\psi(t) - \psi(s)\right)^{k(\alpha-\nu)-1}}{\Gamma(k(\alpha-\nu))} K_1(s) \right] ds,$$

where

$$\begin{aligned} K_1(t) = &\left| \frac{\overline{\mathfrak{u}}_a}{\Gamma(\alpha-\nu)} \left(\psi(t) - \psi(a)\right)^{\alpha-\nu-1} - \frac{\mathfrak{u}_a}{\Gamma(\alpha)} \left(\psi(t) - \psi(a)\right)^{\alpha-1} \right| \\ &+ \|\mathfrak{g}\| \left| \frac{\left(\psi(t) - \psi(a)\right)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} - \frac{\left(\psi(t) - \psi(a)\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)} \right| \\ &+ \|\mathfrak{g}\| \left| \frac{\left(\psi(t) - \psi(a)\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)} - \frac{\left(\psi(t) - \psi(a)\right)^{\alpha}}{\Gamma(\alpha+1)} \right|. \end{aligned}$$

Proof. Solutions of the problems (4.1) and (4.3) are given by

$$\mathfrak{u}(t) = \frac{\mathfrak{u}_a}{\Gamma(\alpha)} \left(\psi(t) - \psi(a)\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} \mathfrak{g}(s, \mathfrak{u}(s))ds, \tag{4.4}$$

and

$$\overline{\mathfrak{u}}(t) = \frac{\overline{\mathfrak{u}}_a}{\Gamma(\alpha-\nu)} \left(\psi(t) - \psi(a)\right)^{\alpha-\nu-1} + \frac{1}{\Gamma(\alpha-\nu)} \int_a^t \psi^{'}(s) \left(\psi(t) - \psi(s)\right)^{\alpha-\nu-1} \mathfrak{g}(s, \overline{\mathfrak{u}}(s))ds, \tag{4.5}$$

respectively. Therefore, it follows that

$$\begin{aligned}
|\overline{\mathfrak{u}}(t)-\mathfrak{u}(t)| &= \Bigg|\frac{\overline{\mathfrak{u}}_a}{\Gamma(\alpha-\nu)}(\psi(t)-\psi(a))^{\alpha-\nu-1}-\frac{\mathfrak{u}_a}{\Gamma(\alpha)}(\psi(t)-\psi(a))^{\alpha-1}\\
&\quad+\frac{1}{\Gamma(\alpha-\nu)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\nu-1}\mathfrak{g}(s,\overline{\mathfrak{u}}(s))ds\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s))ds\Bigg|\\
&\le \left|\frac{\overline{\mathfrak{u}}_a}{\Gamma(\alpha-\nu)}(\psi(t)-\psi(a))^{\alpha-\nu-1}-\frac{\mathfrak{u}_a}{\Gamma(\alpha)}(\psi(t)-\psi(a))^{\alpha-1}\right|\\
&\quad+\left|\int_a^t \psi'(s)\left(\frac{(\psi(t)-\psi(s))^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}-\frac{(\psi(t)-\psi(s))^{\alpha-\nu-1}}{\Gamma(\alpha)}\right)\mathfrak{g}(s,\overline{\mathfrak{u}}(s))ds\right|\\
&\quad+\left|\int_a^t \frac{\psi'(s)}{\Gamma(\alpha)}\left((\psi(t)-\psi(s))^{\alpha-\nu-1}-(\psi(t)-\psi(s))^{\alpha-1}\right)\mathfrak{g}(s,\mathfrak{u}(s))ds\right|\\
&\quad+\left|\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-\nu-1}}{\Gamma(\alpha)}\left(\mathfrak{g}(s,\overline{\mathfrak{u}}(s))-\mathfrak{g}(s,\mathfrak{u}(s))\right)ds\right|\\
&\le \left|\frac{\overline{\mathfrak{u}}_a}{\Gamma(\alpha-\nu)}(\psi(t)-\psi(a))^{\alpha-\nu-1}-\frac{\mathfrak{u}_a}{\Gamma(\alpha)}(\psi(t)-\psi(a))^{\alpha-1}\right|\\
&\quad+\|\mathfrak{g}\|\left|\frac{(\psi(t)-\psi(a))^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}-\frac{(\psi(t)-\psi(a))^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)}\right|\\
&\quad+\|\mathfrak{g}\|\left|\frac{(\psi(t)-\psi(a))^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)}-\frac{(\psi(t)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right|\\
&\quad+\frac{\ell}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\nu-1}|\overline{\mathfrak{u}}(s)-\mathfrak{u}(s)|\,ds.
\end{aligned}$$

Then, by Gronwall's Lemma 2.3, we have,

$$|\overline{\mathfrak{u}}(t)-\mathfrak{u}(t)|\le K_1(t)+\int_a^t\left[\sum_{k=1}^{\infty}\left(\frac{\ell\Gamma(\alpha-\nu)}{\Gamma(\alpha)}\right)^k\frac{\psi'(s)(\psi(t)-\psi(s))^{k(\alpha-\nu)-1}}{\Gamma(k(\alpha-\nu))}K_1(s)\right]ds.$$

This completes the proof of the claim given above. □

Next, we study the continuous dependence of the solution on the order of the Cauchy-type problem (1.1) involving ψ-Hilfer fractional differential equation using Gronwall's Lemma, for this we consider the initial condition that given in (1.1), and the solutions of two initial value problems with a neighbouring orders and a neighbouring initial values.

Theorem 4.2. *Consider $\alpha>0$, $\nu>0$ with $0<\alpha-\nu\le 1$. Let $\mathfrak{u}$ be a continuous function satisfying Lipschitz condition [H1] in $\mathbb{R}$. For $a\le t<h<b$, assume that $\mathfrak{u}$ is the solution of Eq. (1.1) and $\overline{\mathfrak{u}}$ is the solution of equation*

$$\begin{cases}\mathfrak{D}^{\alpha-\nu,\beta;\psi}\overline{\mathfrak{u}}(t)=\mathfrak{g}(t,\overline{\mathfrak{u}}(t)),\\ \mathfrak{I}^{1-\gamma-\nu(\beta-1);\psi}\overline{\mathfrak{u}}(a)=\overline{\mathfrak{u}}_a.\end{cases}\tag{4.6}$$

Then, for $a<t\le h$, the following inequality holds

$$|\overline{\mathfrak{u}}(t)-\mathfrak{u}(t)|\le K_2(t)+\int_a^t\left[\sum_{k=1}^{\infty}\left(\frac{\ell\Gamma(\alpha-\nu)}{\Gamma(\alpha)}\right)^k\frac{\psi'(s)(\psi(t)-\psi(s))^{k(\alpha-\nu)-1}}{\Gamma(k(\alpha-\nu))}K_2(s)\right]ds,$$

in which

$$\begin{aligned} K_2(t) =& \left| \frac{\overline{\mathfrak{u}}_a}{\Gamma(\gamma+\nu(\beta-1))} \left(\psi(t)-\psi(a)\right)^{\gamma+\nu(\beta-1)} - \frac{\mathfrak{u}_a}{\Gamma(\gamma)} \left(\psi(t)-\psi(a)\right)^{\gamma-1} \right| \\ &+ \|\mathfrak{g}\| \left| \frac{\left(\psi(t)-\psi(a)\right)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} - \frac{\left(\psi(t)-\psi(a)\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)} \right| \\ &+ \|\mathfrak{g}\| \left| \frac{\left(\psi(t)-\psi(a)\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)} - \frac{\left(\psi(t)-\psi(a)\right)^{\alpha}}{\Gamma(\alpha+1)} \right|. \end{aligned}$$

Proof. Solutions of the problems (1.1) and (4.6) are given by

$$\mathfrak{u}(t) = \frac{\mathfrak{u}_a}{\Gamma(\alpha)} \left(\psi(t)-\psi(a)\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) \left(\psi(t)-\psi(s)\right)^{\alpha-1} \mathfrak{g}(s,\mathfrak{u}(s))ds, \tag{4.7}$$

and

$$\overline{\mathfrak{u}}(t) = \frac{\overline{\mathfrak{u}}_a}{\Gamma(\gamma+\nu(\beta-1))} \left(\psi(t)-\psi(a)\right)^{\gamma+\nu(\beta-1)} + \frac{1}{\Gamma(\alpha-\nu)} \int_a^t \psi^{'}(s) \left(\psi(t)-\psi(s)\right)^{\alpha-\nu-1} \mathfrak{g}(s,\overline{\mathfrak{u}}(s))ds. \tag{4.8}$$

From these statements we deduce that

$$\begin{aligned} |\overline{\mathfrak{u}}(t)-\mathfrak{u}(t)| =& \Bigg| \frac{\overline{\mathfrak{u}}_a}{\Gamma(\gamma+\nu(\beta-1))} \left(\psi(t)-\psi(a)\right)^{\gamma+\nu(\beta-1)-1} - \frac{\mathfrak{u}_a}{\Gamma(\alpha)} \left(\psi(t)-\psi(a)\right)^{\alpha-1} \\ &+ \frac{1}{\Gamma(\alpha-\nu)} \int_a^t \psi^{'}(s) \left(\psi(t)-\psi(s)\right)^{\alpha-\nu-1} \mathfrak{g}(s,\overline{\mathfrak{u}}(s))ds \\ &- \frac{1}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) \left(\psi(t)-\psi(s)\right)^{\alpha-1} \mathfrak{g}(s,\mathfrak{u}(s))ds \Bigg| \\ \leq& \left| \frac{\overline{\mathfrak{u}}_a}{\Gamma(\gamma+\nu(\beta-1))} \left(\psi(t)-\psi(a)\right)^{\gamma+\nu(\beta-1)-1} - \frac{\mathfrak{u}_a}{\Gamma(\alpha)} \left(\psi(t)-\psi(a)\right)^{\alpha-1} \right| \\ &+ \left| \int_a^t \psi^{'}(s) \left(\frac{\left(\psi(t)-\psi(s)\right)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)} - \frac{\left(\psi(t)-\psi(s)\right)^{\alpha-\nu-1}}{\Gamma(\alpha)} \right) \mathfrak{g}(s,\overline{\mathfrak{u}}(s))ds \right| \\ &+ \left| \int_a^t \frac{\psi^{'}(s)}{\Gamma(\alpha)} \left(\left(\psi(t)-\psi(s)\right)^{\alpha-\nu-1} - \left(\psi(t)-\psi(s)\right)^{\alpha-1} \right) \mathfrak{g}(s,\mathfrak{u}(s))ds \right| \\ &+ \left| \int_a^t \frac{\psi^{'}(s) \left(\psi(t)-\psi(s)\right)^{\alpha-\nu-1}}{\Gamma(\alpha)} \left(\mathfrak{g}(s,\overline{\mathfrak{u}}(s)) - \mathfrak{g}(s,\mathfrak{u}(s)) \right) ds \right| \\ \leq& \left| \frac{\overline{\mathfrak{u}}_a}{\Gamma(\gamma+\nu(\beta-1))} \left(\psi(t)-\psi(a)\right)^{\gamma+\nu(\beta-1)-1} - \frac{\mathfrak{u}_a}{\Gamma(\alpha)} \left(\psi(t)-\psi(a)\right)^{\alpha-1} \right| \\ &+ \|\mathfrak{g}\| \left| \frac{\left(\psi(t)-\psi(a)\right)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)} - \frac{\left(\psi(t)-\psi(a)\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)} \right| \\ &+ \|\mathfrak{g}\| \left| \frac{\left(\psi(t)-\psi(a)\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)} - \frac{\left(\psi(t)-\psi(a)\right)^{\alpha}}{\Gamma(\alpha+1)} \right| \\ &+ \frac{\ell}{\Gamma(\alpha)} \int_a^t \psi^{'}(s) \left(\psi(t)-\psi(s)\right)^{\alpha-\nu-1} \left|\overline{\mathfrak{u}}(s)-\mathfrak{u}(s)\right| ds. \end{aligned}$$

Then, we have by Gronwall's Lemma 2.3,

$$|\overline{\mathfrak{u}}(t)-\mathfrak{u}(t)| \leq K_2(t)+\int_a^t\left[\sum_{k=1}^{\infty}\left(\frac{\ell\Gamma(\alpha-\nu)}{\Gamma(\alpha)}\right)^k \frac{\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{k(\alpha-\nu)-1}}{\Gamma(k(\alpha-\nu))}K_2(s)\right]ds.$$

This completes the proof. □

In the next theorem, we shall make a small change of the initial condition that is given in (1.1), as follows

$$\mathfrak{J}^{1-\gamma;\psi}\overline{\mathfrak{u}}(a)=\overline{\mathfrak{u}}_a+\epsilon, \tag{4.9}$$

where ϵ is an arbitrary constant.

We state and prove the result as follows:

Theorem 4.3. *Let $\gamma=\alpha+\beta-\alpha\beta$, where $0<\alpha<1$ and $0\leq\beta\leq1$. Let $\mathfrak{g}:J\times\mathbb{R}\to\mathbb{R}$ be a function such that $\mathfrak{g}(\cdot,\mathfrak{u}(\cdot))\in C_{1-\gamma;\psi}(J,\mathbb{R})$ for any $\mathfrak{u}\in C_{1-\gamma;\psi}(J,R)$, and satisfies the condition [H1]. For $a\leq t<h<b$, assume that $\mathfrak{u}$ is the solution of Eq. (1.1) and $\overline{\mathfrak{u}}$ is the solution of equation*

$$\begin{cases} \mathfrak{D}^{\alpha,\beta;\psi}\overline{\mathfrak{u}}(t)=\mathfrak{g}(t,\overline{\mathfrak{u}}(t)), t\in J,\\ \mathfrak{J}^{1-\gamma;\psi}\overline{\mathfrak{u}}(a)=\overline{\mathfrak{u}}_a+\epsilon, \end{cases} \tag{4.10}$$

Then,

$$|\mathfrak{u}(t)-\overline{\mathfrak{u}}(t)|\leq\frac{\epsilon}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}E_{\alpha,\gamma}(\ell\left(\psi(t)-\psi(a)\right)^{\alpha})$$

holds, where $E_{\alpha,\gamma}=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\gamma)}$ is Mittag-Leffler function.

Proof. According to Theorem 3.3 we have $\mathfrak{u}(t)=\lim_{n\to\infty}\mathfrak{u}_m(t)$ in which $\mathfrak{u}_0(t)$ and $\mathfrak{u}_m(t)$ are as defined in equations (3.4) and (3.5). Clearly, it can be seen that $\overline{\mathfrak{u}}(t)=\lim_{n\to\infty}\overline{\mathfrak{u}}_m(t)$, and

$$\overline{\mathfrak{u}}_0(t)=\frac{\overline{\mathfrak{u}}_a+\epsilon}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}, \tag{4.11}$$

$$\overline{\mathfrak{u}}_m(t)=\overline{\mathfrak{u}}_0(t)+\frac{1}{\Gamma(\alpha)}\int_a^t\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\overline{\mathfrak{u}}_{m-1}(s))ds. \tag{4.12}$$

It follows from (3.4) and (4.11) that

$$\begin{aligned} |\mathfrak{u}_0(t)-\overline{\mathfrak{u}}_0(t)| &= \left|\frac{\mathfrak{u}_a}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}-\frac{\overline{\mathfrak{u}}_a+\epsilon}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}\right|\\ &\leq\frac{\epsilon}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}. \end{aligned} \tag{4.13}$$

Now, by using equations (3.5) and (4.12) and applying the Lipschitz condition [H1], we get

$$\begin{aligned} |\mathfrak{u}_1(t)-\overline{\mathfrak{u}}_1(t)| &= \Bigg|\frac{\epsilon}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_a^t\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\left(\mathfrak{g}(s,\mathfrak{u}_0(s))-\mathfrak{g}(s,\overline{\mathfrak{u}}_0(s))\right)ds\Bigg|\\ &\leq\frac{\epsilon}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}\\ &\quad+\frac{\ell}{\Gamma(\alpha)}\int_a^t\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}|\mathfrak{u}_0(s)-\overline{\mathfrak{u}}_0(s)|\,ds\\ &\leq\frac{\epsilon}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}\left[1+\frac{\ell}{\Gamma(\alpha+\gamma)}\left(\psi(t)-\psi(a)\right)^{\alpha}\right]. \end{aligned}$$

Then, we have

$$|\mathfrak{u}_1(t)-\bar{\mathfrak{u}}_1(t)| \leq \frac{\epsilon}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}\left[1+\frac{\ell}{\Gamma(\alpha+\gamma)}\left(\psi(t)-\psi(a)\right)^{\alpha}\right]. \tag{4.14}$$

Similarly,

$$|\mathfrak{u}_2(t)-\bar{\mathfrak{u}}_2(t)| \leq \frac{\epsilon}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}\sum_{i=0}^{2}\left[\frac{\ell^i}{\Gamma(\alpha i+\gamma)}\left(\psi(t)-\psi(a)\right)^{\alpha i}\right]. \tag{4.15}$$

By induction we can show

$$|\mathfrak{u}_m(t)-\bar{\mathfrak{u}}_m(t)| \leq \frac{\epsilon}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}\sum_{i=0}^{m}\left[\frac{\ell^i}{\Gamma(\alpha i+\gamma)}\left(\psi(t)-\psi(a)\right)^{\alpha i}\right]. \tag{4.16}$$

Taking limit as $m\to\infty$, we have

$$\begin{aligned}|\mathfrak{u}(t)-\bar{\mathfrak{u}}(t)| \leq &\frac{\epsilon}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}\sum_{i=0}^{\infty}\left[\frac{\ell^i}{\Gamma(\alpha i+\gamma)}\left(\psi(t)-\psi(a)\right)^{\alpha i}\right]\\ =&\frac{\epsilon}{\Gamma(\gamma)}\left(\psi(t)-\psi(a)\right)^{\gamma-1}E_{\alpha,\gamma}(\ell\left(\psi(t)-\psi(a)\right)^{\alpha}),\end{aligned} \tag{4.17}$$

which completes the proof. □

5. An Example

We consider the problem as a special case of ψ-Hilfer fractional derivative by assuming $\psi(t)=t$. Therefore, the problem with Hilfer fractional derivative is given by

$$\begin{cases}\mathfrak{D}^{\alpha,\beta;t}\mathfrak{u}(t)=\mathfrak{g}(t,\mathfrak{u}(t)), \quad t\in[0,1],\\ \mathfrak{I}^{1-\gamma;t}\mathfrak{u}(a)=\mathfrak{u}_a,\end{cases} \tag{5.1}$$

Now, set $\alpha=\frac{2}{3}$, $\beta=\frac{1}{2}$ and we deduce $\gamma=\frac{5}{6}$. Consider $\mathfrak{g}(t,\mathfrak{u}(t))=\frac{e^{-t-10}}{1+|\mathfrak{u}(t)|}$. Moreover, $\mathfrak{g}$ satisfies the condition (A1) with $\ell=\frac{1}{e^{10}}$, i.e.,

$$|\mathfrak{g}(s,\mathfrak{u}(s))-\mathfrak{g}(s,\mathfrak{y}(s))| \leq \frac{1}{e^{10}}\left|\mathfrak{u}-\mathfrak{y}\right|.$$

On the other hand we obtain

$$\frac{\frac{1}{e^{10}}}{\Gamma(\frac{2}{3})}B\left(\frac{5}{6},\frac{2}{3}\right)\approx 0.00005.$$

This problem satisfies all the assumptions provided above, so the problem (5.1) has a unique solution.

6. Conclusions

This paper attempts to obtain some existence and uniqueness results for a ψ-Hilfer fractional differential equation involving an initial constraint of the type ψ-Hilfer integral. Next, we investigate the continuous dependence of solutions of the proposed equation via generalized Gronwall's inequality. Finally, a simple example of the Cauchy type problem by assuming $\psi(t)=t$ is offered.

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