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Controllability and observability of linear impulsive differential algebraic system with Caputo fractional derivative

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Abstract

Linear impulsive fractional differential-algebraic systems (LIFDAS) in a finite dimensional space are studied. We obtain the solution of LIFDAS. Using Gramian matrices, necessary and sufficient conditions for controllability and observability of time-varying LIFDAS are established. We acquired the criterion for time-invariant LIFDAS in the form of rank conditions. The results are more generalized than the results that are obtained for various differential-algebraic systems without impulses.

Keywords. Controllability; Observability; Drazin inverse; Caputo fractional derivative.
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1. Introduction

Fractional calculus has gained importance during the past four decades in various domains like fluid mechanics, physics, economics, medicine, phototonic and engineering [2, 6, 25, 26, 36]. Mathematical modeling in these fields is commonly led by algebraic systems, which may be the continuous-time linear differential-algebraic systems and discrete time linear differential-algebraic systems [14–16]. In the domain of control theory [23, 24] the differential-algebraic systems are normally combined together with the classical order derivative. But the advancement of research turns the trend towards the stability analysis of linear differential-algebraic systems which consist of fractional order derivative i.e. the generalization of the classical order derivative to an arbitrary order (non-integer). Since controllability and observability are the two main factors in the stability analysis; so, an efficient criterion is required to achieve the stability of fractional type systems in an adequate manner [3, 10–13]. Recently, Kaczorek has done a lot of work on various control theory problems both of classical and fractional order derivatives along with their applications in electrical engineering, for example [18–21]. Advancing in differential algebraic systems, an interesting phenomenon is the involvement of impulsive conditions. Many real life problems like medical injections, lasers and billiards are mathematically modeled by the impulsive differential algebraic systems. Following this trend, a detailed discussion regarding the controllability and observability on fractional continuous-time linear impulsive systems has been done by Feckan [8] and Guo [13]. Also, some results on controllability and observability have been obtained for fractional continuous-time linear systems by Younus et al. [39]. While impulsive the fractional time invariant system with the delay has been discussed by Zhou [40]. Slynko and Tunc have also discussed the delay in periodic impulsive delay in [29] and by Boyadzhiev see [4] . A detailed effort on differential systems with impulsive conditions can be seen in [28, 32] in which they have discussed not only the various types of impulsive conditions(both instantaneous and non-instantaneous) but also they have discussed the various approaches for the solution of such systems. This becomes a motivation for us to find some new results of fractional order differential-algebraic systems with impulsive conditions.

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We have considered a fractional(Caputo type derivative) differential-algebraic system with impulsive conditions. Here, the matrix E is not invertible, that is E does not exist. Among the various types of generalized inverses, the type of inverse we have chosen here for the matrix E is Drazin inverse, along with the help of regular pencil, see, for example, [5, 17, 39] . For some other related works, we referee the readers to [30, 31, 33–35]. Furthermore, we have discussed the solution of the fractional continuous-time linear impulsive differential-algebraic system, its controllability and its observability. These results are in the form of Gramian matrices and rank conditions. The present paper is organized as follows: After an introduction, Section 2 reviews the basic notions and results. In Section 3, by redefining well-known Gramian matrices, we have obtained the necessary and sufficient conditions for complete controllability of the solution of fractional differential-algebraic systems (DAS). The last section contains the results about complete observability for impulsive fractional DAS.

2. Preliminaries

For a function $f : [0, \infty) \to \mathbb{R}$, the fractional integral of order $\alpha > 0$, with the lower limit zero is defined as

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds \tag{2.1}$$

and the Caputo derivative of order α with the lower limit t_i, for a function $f \in \mathbb{C}^n [0, \infty)$, is defined as

$$^cD_{t_i,t}^\alpha\, f(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_{t_i}^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds,\ t_i > t, \tag{2.2}$$

where $f^{(n)}(t) = \frac{d^n f(t)}{dt^n}$ and $0 \leq n-1 < \alpha \leq n$.

Consider the following continuous-time linear impulsive fractional differential-algebraic system with fractional order $0 < \alpha \leq 1$:

$$\begin{cases} E\, {}^cD_{t_i,t}^\alpha\, x(t) = Ax(t) + Bu(t),\ t \in (t_i, t_{i+1}], \\ x(t_0) = x(0), \\ x(t_i^+) = (1 + c_i)(x(t_i)),\ \text{at } t = t_i, i = 1, 2, \cdots, k, \\ y = Cx(t) + Du(t), \end{cases} \tag{2.3}$$

where $c_i \in \mathbb{R}$ $(c_i \neq -1)$ are constants, $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} = T < \infty$, $x(\cdot) \in \mathbb{R}^n$, $u(\cdot) \in \mathbb{R}^m$, $y(\cdot) \in \mathbb{R}^p$ are the state, input and output vectors, respectively. Also $E, A \in \mathbb{R}^{n\times n}$ with $\det E = 0$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$.

Throughout this paper, let $\mathbb{K}[c]$ denotes the ring of polynomials over the field $\mathbb{K}$ with variable c and $\mathbb{K}^{n\times m}[c]$ denotes the ring of matrices of dimension $n \times m$ with entries from $\mathbb{K}[c]$.

Definition 2.1. [7] The matrix pair (E, A) is called the matrix pencil if $(Ec - A) \in \mathbb{K}^{n\times m}[c]$ for any $c \in \mathbb{K}$, where $E,\ A \in \mathbb{K}^{n\times n}$ and $\mathbb{K}$ is $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.2. [7] The matrix pencil $(Ec - A) \in \mathbb{K}^{n\times m}[c]$ is called regular if $n = m$ and the $\det(Ec - A)$ is not identically equal to zero. Otherwise, $(Ec - A)$ is called singular.

(A) We assume that the matrix pencil $(E,\ A)$ of system (2.3) is regular. For a matrix $X \in \mathbb{R}^{n\times n}$

$$Ind\,(X) := \min\left\{q \in \mathbb{Z} : q \geq 0 \text{ and } rankX^q = rankX^{q+1}\right\}.$$

A matrix $E^D \in \mathbb{R}^{n\times n}$ is called the Drazin inverse [20] of a matrix $E \in \mathbb{R}^{n\times n}$, if it satisfies the following conditions:

$$EE^D = E^DE,\ E^DEE^D = E^D \text{ and } E^DE^{q+1} = E^q, \tag{2.4}$$

where $q = ind\,(E)$.

To obtain $E^D \in \mathbb{R}^{n\times n}$ of any $E \in \mathbb{R}^{n\times n}$, following algorithm should be adopted:

(1) Write $E = VW$, where $V \in \mathbb{R}^{n\times r}$, $W \in \mathbb{R}^{r\times n}$ and $rankV = rankW = rankE = r$.
(2) Compute $WEV \in \mathbb{R}^{r\times r}$.
(3) The Drazin inverse of a matrix E is $E^D = V\,(WEV)^{-1}\,W$.

Example 2.3. Consider a matrix

$$E = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}.$$

Here the matrix E is not invertible, so we can write E as

$$E = VW = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix}.$$

Clearly $\det E = 0$ and $rank(E) = rank(V) = rank(W) = 1$. Moreover

$$E^2 = \begin{bmatrix} 4 & 12 \\ 4 & 12 \end{bmatrix}.$$

So, $rank(E^2) = rank(E)$. Hence $ind(E) = q = 1$.

Following the procedure of Drazin inverse, it yields

$$E^D = \begin{bmatrix} 0.0625 & 0.1875 \\ 0.0625 & 0.1875 \end{bmatrix}.$$

If the $ind(A) = 1$, then A^D becomes the group inverse and is denoted by $A^\natural$[3, p. 118]. It is well known that:

$$A = S \begin{pmatrix} J & 0 \\ 0 & N \end{pmatrix} S^{-1},\ A^D = S \begin{pmatrix} J^{-1} & 0 \\ 0 & 0 \end{pmatrix} S^{-1},$$

where J contains the Jordan blocks corresponding to nonzero eigenvalues and N is nilpotent with $N^k = 0$ and $N^{k-1} \neq 0$ [38]. Moreover,

$$\mathcal{R}(A^D) = \mathcal{R}(A^q),\ \mathcal{N}(A^D) = \mathcal{N}(A^q);\ \mathbb{R}^n = \mathcal{R}(A^D) \oplus \mathcal{N}(A^D). \tag{2.5}$$

In [5, Corollary 2], Campbell has proved that if the matrices E and A are commutative then the system will have a unique solution. For non-commutative case, to overcome this assumption we need to transform the system (2.3) into an equivalent system, which satisfies the commutative condition and hence guarantees the unique solution of the system (2.3).

By using (A), let us define the following matrices:

$$\bar{E} = E(Ec - A)^{-1},\ \bar{A} = A(Ec - A)^{-1} \text{ and } \bar{B} = (Ec - A)^{-1}B.$$

Lemma 2.4. *[5, 19] For the matrices $\bar{E}$ and $\bar{A}$:*

1. $\bar{E}\ \bar{A} = \bar{A}\ \bar{E}$, $\bar{A}^D\bar{E} = (\overline{EA})^D$, $\bar{E}^D\bar{A} = (\bar{A}\ \bar{E})^D$, $\bar{A}^D\bar{E}^D = \bar{E}^D\bar{A}^D$ *and* $\bar{E}^D\ \bar{E}\ \bar{E}^D = \bar{E}^D$ *with* $\overline{EA} = (Ec - A)^{-1}(EA)$;
2. $\mathcal{N}(\bar{A}) \cap \mathcal{N}(\bar{E}) = \{0\}$;
3. $\bar{E} = T \begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} T^{-1}$, $\bar{E}^D = T \begin{pmatrix} J^{-1} & 0 \\ 0 & 0 \end{pmatrix} T^{-1}$, $\det T \neq 0$, $J \in \mathbb{R}^{n_1 \times n_1}$, *is nonsingular,* $N \in \mathbb{R}^{n_2 \times n_2}$ *is nilpotent,* $n_1 + n_2 = n$;
4. $(\mathbb{I} - \bar{E}\bar{E}^D)\bar{A}\bar{A}^D = \mathbb{I} - \bar{E}\bar{E}^D$, $\overline{E}^D(\mathbb{I} - \bar{E}\bar{E}^D) = 0$ *and* $(\mathbb{I} - \bar{E}\bar{E}^D)(\bar{E}\bar{A}^D)^q = 0$, *where* $\mathbb{I}$ *is the identity matrix of order* $n \times n$ *and* $q = Ind(E)$.

Lemma 2.5. *Let $q_1 = Ind(\bar{A})$ and $q_2 = Ind(\bar{A})$. Then the matrices $\bar{A}$ and $\bar{E}$ satisfy the following condition:*

$$\mathcal{N}(\bar{A}^{q_1}) \cap \mathcal{N}(\bar{E}^{q_2}) = \{0\} \text{ and } \mathcal{N}(\bar{A}^D) \cap \mathcal{N}(\bar{E}^D) = \{0\}. \tag{2.6}$$

Proof. Let $x \in \mathcal{N}(\bar{A}^D) \cap \mathcal{N}(\bar{E}^D)$. Then, we have

$$\bar{A}^D x = 0 \tag{2.7}$$

and

$$\bar{E}^D x = 0. \tag{2.8}$$

From Lemma 2.4, we have

$$\left(\mathbb{I}-\bar{E}\bar{E}^{D}\right)\bar{A}\bar{A}^{D}=\left(\mathbb{I}-\bar{E}\bar{E}^{D}\right). \tag{2.9}$$

Pre-multiplying Eq. (2.7) by $\left(\mathbb{I}-\bar{E}\bar{E}^{D}\right)\bar{A}$, we have

$$\left(\mathbb{I}-\bar{E}\bar{E}^{D}\right)\bar{A}\bar{A}^{D}x=0, \tag{2.10}$$

where $\mathbb{I}$ is identity matrix.

From (2.9), it follows that

$$\left(\mathbb{I}-\bar{E}\bar{E}^{D}\right)x=0$$

or

$$\mathbb{I}x=\left(\bar{E}\bar{E}^{D}\right)x. \tag{2.11}$$

From (2.8) and (2.11), we have $\mathbb{I}x=0$, which yields that $x=0$ and $\mathcal{N}\left(\bar{A}^{D}\right)\cap\mathcal{N}\left(\bar{E}^{D}\right)=\{0\}$. Moreover, form (2.5), it implies that $\mathcal{N}\left(\bar{A}^{q_1}\right)\cap\mathcal{N}\left(\bar{E}^{q_2}\right)=\{0\}$. □

Consider the fractional differential-algebraic continuous-time linear (time invariant) equation described as

$$\bar{E}\,{}^{c}D_{t_0,t}^{\alpha}\,x(t)=\bar{A}x(t)+\bar{B}u(t),\ t\in[t_0,T]. \tag{2.12}$$

In [19, Theorem 1], the solution to the state equation (2.12) by the use of Drazin inverse method is as follows:

$$\begin{aligned}x(t)&=\Phi_0(t-t_0)\bar{E}\bar{E}^{D}v+\bar{E}^{D}\int_{t_0}^{t}\Phi(t-\tau)Bu(\tau)d\tau\\&\quad+(\bar{E}^{D}\bar{E}-\mathbb{I})\sum_{k=0}^{q-1}(\bar{E}\ \bar{A}^{D})^{k}\bar{A}^{D}\bar{B}u^{(k\alpha)}(t),\end{aligned} \tag{2.13}$$

where

$$\Phi_0(t-t_0)=\sum_{k=0}^{\infty}\frac{(\bar{E}^{D}\bar{A})^{k}\,(t-t_0)^{k\alpha}}{\Gamma(k\alpha+1)},\ \Phi(t-\tau)=\sum_{k=0}^{\infty}\frac{(\bar{E}^{D}\bar{A})^{k}\,(t-\tau)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]},$$
$$u^{(k\alpha)}(\cdot)={}^{c}D_{t_0,t}^{k\alpha}\,u(\cdot).$$

From (2.13) for $t=t_0$ we have

$$x(t_0)=x_0=\bar{E}\bar{E}^{D}v+(\bar{E}^{D}\bar{E}-\mathbb{I})\sum_{k=0}^{q-1}(\bar{E}\ \bar{A}^{D})^{k}\bar{A}^{D}\bar{B}u^{(k\alpha)}(t_0). \tag{2.14}$$

Therefore, for given admissible $u(\cdot)$, the consistent initial conditions should satisfy the equality (2.14).

Pre-multiplying (2.3) by $(Ec-A)^{-1}$, we have an equivalent system

$$\begin{cases}\bar{E}\ {}^{c}D_{t_i,\ t}^{\alpha}x(t)=\bar{A}x(t)+\bar{B}u(t),\ t\in(t_i,t_{i+1}],\\x(t_0)=x_0\\x(t_i^{+})=(1+c_i)\,x(t_i),\ \text{at } t=t_i,\\i=1,2,\cdots,k.\end{cases} \tag{2.15}$$

For impulsive case, we look at the concept of a solution. There are two main viewpoints (see for example [1]):

(V1) Using the classical Caputo derivative and working in each subinterval, determined by the impulses.

(V2) Keeping the lower limit t_0 of the Caputo derivative for all $t\geq t_0$ but considering different initial conditions on each interval (t_i,t_{i+1}).

In this article we use approach (V1). Note if for some natural i, a component of the function $\Phi_i:\mathbb{R}^n\to\mathbb{R}^n$ such that $\Phi_i=(1+c_i)\,x\,(t_i)$ satisfies the equality $\Phi_{i,j}\,(x)=x_j$, where $x\in\mathbb{R}^n$, then there will be no impulse at the point t_i. To avoid this confusing situation in the application of approach (V1), mentioned above we will assume [1]:

(H1) If $x\neq 0$, then $\Phi_{i,j}\,(x)\neq x_j$ for all $j=1,2,\cdots n$ and $i=1,2,3,\cdots$.

Theorem 2.6. *For given admissible input u and consistent initial and impulsive conditions*

$$x(t_0) = x_0 = \bar{E}\bar{E}^D v + (\bar{E}^D\bar{E} - \mathbb{I})\sum_{k=0}^{q-1}(\bar{E}\ \bar{A}^D)^k \bar{A}^D \bar{B} u^{(k\alpha)}(t_0) \text{ and}$$

$$x\left(t_k^+\right) = (1+c_i)\, x(t_i) = \bar{E}\bar{E}^D v + (\bar{E}^D\bar{E} - \mathbb{I})\sum_{k=0}^{q-1}(\bar{E}\ \bar{A}^D)^k \bar{A}^D \bar{B} u^{(k\alpha)}(t_k),$$

the solution of (2.3) (and (2.15)) is given by

$$x(t) = \begin{cases} \Phi_0(t-t_0)x_0 + \int\limits_{t_0}^{t} \Phi(t-\tau)\bar{E}^D\bar{B}u(\tau)d\tau \\ +(\bar{E}^D\bar{E}-\mathbb{I})\sum\limits_{k=0}^{q-1}(\bar{E}\ \bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t),\ \ t\in[t_0,t_1]; \\ \Phi_0(t-t_i)\left(x(t_i)\left(1+c_i\right)\right) + \int\limits_{t_i}^{t}\Phi(t-\tau)\bar{E}^D\bar{B}u(\tau)d\tau \\ +(\bar{E}^D\bar{E}-\mathbb{I})\sum\limits_{k=0}^{q-1}(\bar{E}\ \bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t), t\in(t_i,t_{i+1}] \end{cases} \tag{2.16}$$

for each $i = 1, 2, 3, \cdots, k$, where $\mathbb{I}$ is the identity matrix of order n.

3. Controllability

Definition 3.1. The system (2.3) (and (2.15)) is said to be completely controllable on the interval $J = [t_0, T]$ if for any $t > 0$, $(t \in [t_0, T])$, and $z \in \mathbb{R}^n$ there exists an admissible control input $u(t)$ such that the state variable $x(t)$ of the system (2.3) (and (2.15)) satisfies $x(t_i) = z$.

Theorem 3.2. *Let (A) hold. Then the following propositions are equivalent.*
(i) The system (2.3) (and (2.15)) is controllable on $[t_0, T]$.
(ii) Gramian matrices

$$W_c[t_i, t_{i+1}] := \int\limits_{t_i}^{t_{i+1}} \left(\Phi(t_{_{i+1}}-\tau)\bar{E}^D\bar{B}\right)\left(\Phi(t_{_{i+1}}-\tau)\bar{E}^D\bar{B}\right)^* d\tau, \tag{3.1}$$

are non-singular for $i = 0, 1, 2, \cdots, k$.

Proof. $(ii) \Rightarrow (i)$: Consider that $W_c[t_i,\ t_{i+1}]$ are non-singular for each $i = 0, 1, 2, \cdots, k$, then $W_c^{-1}[t_i,\ t_{i+1}]$ are well defined. From Eq. (2.16), for $t \in [t_0,\ t_1]$, we have

$$x(t) = \Phi_0(t-t_0)x_0 + \int\limits_{t_0}^{t}\Phi(t-\tau)\bar{E}^D\bar{B}u(\tau)d\tau \\ +(\bar{E}^D\bar{E}-\mathbb{I})\sum_{k=0}^{q-1}(\bar{E}\ \bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t). \tag{3.2}$$

For $x_0 \in \mathbb{R}^n$, we choose $u(t)$ of the form:

$$u(t) = \left(\Phi(t-\tau)\bar{E}^D\bar{B}\right)^* W_c^{-1}[t_0,t_1]\left(x_0 - \Phi_0(t-t_0)x_0\right). \tag{3.3}$$

Substituting (3.3) into (3.2), it follows that

$$\begin{aligned} x(t_1) \ &= \Phi_0(t_1-t_0)x_0 + \int\limits_{0}^{t_1}\Phi(t_1-\tau)\bar{E}^D\bar{B}\left[\left(\Phi(t_1-\tau)\bar{E}^D\bar{B}\right)^*\right. \\ &\left.\times W_c^{-1}[t_0,t_1]\left(x_0-\Phi_0(t_1-t_0)x_0\right)\right]d\tau \\ &+(\bar{E}^D\bar{E}-\mathbb{I})\sum_{k=0}^{q-1}(\bar{E}\ \bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t_1). \end{aligned} \tag{3.4}$$

Multiplying both sides of Eq. (3.4) with $\bar{E}^D$, we have

$$\begin{aligned} \bar{E}^D x(t_1) &= \bar{E}^D \Big[\Phi_0(t_1-t_0)x_0 \\ &+ \int_0^{t_1} \Phi(t_1-\tau)\bar{E}^D\bar{B}\left(\Phi(t_1-\tau)\bar{E}^D\bar{B}\right)^* W_c^{-1}[t_0,t_1] \\ &\times (x_0-\Phi_0(t_1-t_0)x_0)\,d\tau \\ &+(\bar{E}^D\bar{E}-\mathbb{I})\sum_{k=0}^{q-1}(\bar{E}\ \bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t_1)\Big]. \end{aligned} \tag{3.5}$$

With the help of properties of Drazin inverse, (3.5) has the following form:

$$\begin{aligned} \bar{E}^D x(t_1) &= \bar{E}^D \big[\Phi_0(t_1-t_0)x_0 + W_c[t_0,t_1] \\ &\times W_c^{-1}[t_0,t_1]\,(x_0-\Phi_0(t_1-t_0)x_0)\big]\big] = \bar{E}^D x_0. \end{aligned} \tag{3.6}$$

From Eq. (3.6), we have

$$\bar{E}^D x(t_1) = \bar{E}^D x_0$$

or

$$\bar{E}^D\left[x(t_1)-x_0\right] = 0. \tag{3.7}$$

Pre-multiplying Eq. (3.7), with $\bar{A}^D$, we have

$$\bar{A}^D\bar{E}^D\left[x(t_1)-x_0\right] = 0.$$

Since $\bar{A}^D\bar{E}^D = \bar{E}^D\bar{A}^D$, which implies that $\bar{E}^D\left[x(t_1)-x_0\right] \in \ker(\bar{A}^D)$ and $\bar{A}^D\left[x(t_1)-x_0\right] \in \ker(\bar{E}^D)$. Finally, from Lemma 2.5 we obtain

$$x(t_1)-x_0 = 0,$$

which implies that $x(t_1)=x_0$. Hence, the system is controllable on $[t_0,t_1]$.

Now, for $t\in(t_i,t_{i+1}]$, Eq. (2.16) gives

$$\begin{aligned} x(t) &= \Phi_0(t-t_i)x(t_i)\,(1+c_i) + \int_{t_i}^{t}\Phi(t-\tau)\bar{E}^D\bar{B}u(\tau)d\tau \\ &+(\bar{E}^D\bar{E}-\mathbb{I})\sum_{k=0}^{q-1}(\bar{E}\ \bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t). \end{aligned} \tag{3.8}$$

Multiply $\bar{E}^D$ on both sides of Eq. (3.8) and substitute $t=t_{i+1}$, then it follows that

$$\begin{aligned} \bar{E}^D x(t_{i+1}) &= \bar{E}^D \Big[\Phi_0(t_{i+1}-t_i)x(t_i)\,(1+c_i) \\ &+ \int_{t_i}^{t_{i+1}}\Phi(t_{i+1}-\tau)\bar{E}^D\bar{B}u(\tau)d\tau\Big]. \end{aligned} \tag{3.9}$$

For $x_i\in\mathbb{R}^n$, we choose $u(t)$ of the form:

$$\begin{aligned} u(t) &= \left(\Phi(t-\tau)\bar{E}^D\bar{B}\right)^* W_c^{-1}[t_i,t_{i+1}] \\ &\times (x_i-\Phi_0(t_{i+1}-t_i)x(t_i)\,(1+c_i)). \end{aligned} \tag{3.10}$$

Substituting (3.10) in (3.9), we can obtain

$$\begin{aligned} \bar{E}^D x(t_{i+1}) &= \bar{E}^D \big[\Phi_0(t_{i+1}-t_i)\,(x(t_i)\,(1+c_i)) \\ &+ \int_{t_i}^{t_{i+1}}\Phi(t_{i+1}-\tau)\bar{E}^D\bar{B}\left(\Phi(t_{i+1}-\tau)\bar{E}^D\bar{B}\right)^* W_c^{-1}[t_i,t_{i+1}] \\ &\times (x_i-\Phi_0(t_{i+1}-t_i)x(t_i)\,(1+c_i))\,d\tau\big] \end{aligned} \tag{3.11}$$

or

$$\begin{aligned} \bar{E}^D x(t_{i+1}) &= \bar{E}^D \big[\Phi_0(t_{i+1}-t_i)\,(x(t_i)\,(1+c_i)) \\ &+W_c[t_i,t_{i+1}]W_c^{-1}[t_i,t_{i+1}]\,(x_i-\Phi_0(t_{i+1}-t_i)) \\ &\times\ x(t_i)\,(1+c_i)\big], \end{aligned}$$

which results as

$$\bar{E}^D x(t_{i+1}) = \bar{E}^D x_i. \tag{3.12}$$

From (3.12), we can write

$$\bar{E}^D \left[x(t_{i+1}) - x_i\right] = 0. \tag{3.13}$$

Pre-multiplying Eq. (3.13) with $\bar{A}^D$, we have

$$\bar{A}^D \bar{E}^D \left[x(t_{i+1}) - x_i\right] = 0.$$

Since $\bar{A}^D \bar{E}^D = \bar{E}^D \bar{A}^D$, which implies that $\bar{E}^D \left[x(t_{i+1}) - x_i\right] \in \ker(\bar{A}^D)$ and $\bar{A}^D \left[x(t_{i+1}) - x_i\right] \in \ker(\bar{E}^D)$. Finally, from Lemma 2.5, we obtain the following expression

$$x(t_{i+1}) - x_i = 0,$$

which implies that $x(t_{i+1}) = x_i$. Hence the system is completely controllable on $J = [t_0, T]$.

$(i) \Rightarrow (ii)$: We assume that the system (2.3) (and (2.15)) is completely controllable. Suppose contrary that the Gramian matrix $W_c[t_0, t_1]$ is singular. Then, there exists a vector $z_0 \neq 0$ such that

$$z_0^* W_c[t_0, t_1] z_0 = 0, \tag{3.14}$$

that is,

$$z_0^* \int_{t_0}^{t_1} \left(\Phi(t_1 - \tau)\bar{E}^D \bar{B}\right) \left(\Phi(t_1 - \tau)\bar{E}^D \bar{B}\right)^* d\tau z_0 = 0. \tag{3.15}$$

The former equality implies that

$$z_0^* \left(\Phi(t_1 - t)\bar{E}^D \bar{B}\right) = 0, \ t \in \left[t_0, t_1\right]. \tag{3.16}$$

By the assumption that the system (2.3) (and (2.15)) is completely controllable on $[t_0, T]$, then for $t \in [t_0, t_1]$, we have

$$\begin{aligned} x(t) &= \Phi_0(t - t_0)x_0 + \int_{t_0}^{t} \Phi(t - \tau)\bar{E}^D \bar{B} u(\tau) d\tau \\ &+ (\bar{E}^D \bar{E} - \mathbb{I}) \sum_{k=0}^{q-1} (\bar{E}\ \bar{A}^D)^k \bar{A}^D \bar{B} u^{(k\alpha)}(t) \end{aligned} \tag{3.17}$$

and at $t = t_1$, we get

$$\begin{aligned} x(t_1) &= \Phi_0(t_1 - t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1 - \tau)\bar{E}^D \bar{B} u(\tau) d\tau \\ &+ (\bar{E}^D \bar{E} - \mathbb{I}) \sum_{k=0}^{q-1} (\bar{E}\ \bar{A}^D)^k \bar{A}^D \bar{B} u^{(k\alpha)}(t_1). \end{aligned} \tag{3.18}$$

Multiplying both sides of (3.18) with $\bar{E}^D$, we get

$$\bar{E}^D x(t_1) = \bar{E}^D \left[\Phi_0(t_1 - t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1 - \tau)\bar{E}^D \bar{B} u(\tau) d\tau\right]. \tag{3.19}$$

Since the system is controllable, then there exists control inputs $u_0(t)$ and $\tilde{u}_0(t)$ such that

$$\begin{aligned} \bar{E}^D x(t_1) &= \bar{E}^D \left[\Phi_0(t_1 - t_0)x_0 + \int_{0}^{t_1} \Phi(t_1 - \tau)\bar{E}^D \bar{B} u_0(\tau) d\tau\right] \\ &= 0. \end{aligned}$$

Hence, we derive that

$$\bar{E}^D \Phi_0(t_1 - t_0)x_0 + \bar{E}^D \int_{t_0}^{t_1} \Phi(t_1 - \tau)\bar{E}^D \bar{B} u_0(\tau) d\tau = 0 \tag{3.20}$$

and for $\tilde{u}_0(t)$, we have

$$\bar{E}^D x(t_1) = \bar{E}^D \left[\Phi_0(t_1 - t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1 - \tau)\bar{E}^D \bar{B}\ \tilde{u}_0(\tau)d\tau \right]$$
$$= \bar{E}^D z_0.$$

Then, it follows that

$$\bar{E}^D \Phi_0(t_1 - t_0)x_0 + \bar{E}^D \int_{t_0}^{t_1} \Phi(t_1 - \tau)\bar{E}^D \bar{B}\ \tilde{u}_0(\tau)d\tau = \bar{E}^D z_0.$$

It is obvious that

$$\bar{E}^D z_0 - \bar{E}^D \int_{t_0}^{t_1} \Phi(t_1 - \tau)\bar{E}^D \bar{B}\ \tilde{u}_0(\tau)d\tau = \bar{E}^D \Phi_0(t_1 - t_0)x_0. \tag{3.21}$$

From Eqs. (3.20) and (3.21), we have

$$\bar{E}^D z_0 - \bar{E}^D \int_{t_0}^{t_1} \Phi(t_1 - \tau)\bar{E}^D \bar{B}\ \tilde{u}_0(\tau)d\tau$$
$$+\bar{E}^D \int_{t_0}^{t_1} \Phi(t_1 - \tau)\bar{E}^D \bar{B} u_0(\tau)d\tau = 0.$$

More explicitly, we have

$$\bar{E}^D z_0 + \bar{E}^D \int_{t_0}^{t_1} \Phi(t_1 - \tau)\bar{E}^D \bar{B}\ \left(u_0(\tau) - \tilde{u}_0(\tau)\right) d\tau = 0. \tag{3.22}$$

Pre-multiplying both sides of (3.22) with z_0^*, we find

$$\bar{E}^D z_0^* z_0 + \bar{E}^D \int_{t_0}^{t_1} z_0^* \Phi(t_1 - \tau)\bar{E}^D \bar{B}\ \left(u_0(\tau) - \tilde{u}_0(\tau)\right) d\tau = 0.$$

From Eq. (3.16), it yields

$$\bar{E}^D z_0^* z_0 = 0. \tag{3.23}$$

Multiplying (3.23) with $\bar{A}^D$, it follows that

$$\bar{A}^D \bar{E}^D z_0^* z_0 = 0.$$

Since $\bar{A}^D \bar{E}^D = \bar{E}^D \bar{A}^D$, which implies that $\bar{E}^D\,[z_0^* z_0] \in \ker(\bar{A}^D)$ and $\bar{A}^D\,[z_0^* z_0] \in \ker(\bar{E}^D)$. Finally, by using Lemma 2.5, it implies that

$$z_0^* z_0 = 0,$$

that is, $z_0 = 0$, which leads to a contradiction.

Similarly, one can easily show that the Gramian matrices $W_c[t_i, t_{i+1}]$ are nonsingular for $t \in (t_i, t_{i+1}]$. Which completes our proof. □

Theorem 3.3. *Let (A) hold. Then, the following propositions are equivalent.*

(i) System (2.3) (and (2.15)) are completely controllable on $t \in [t_0, T]$.

(ii) For given $\bar{E}, \bar{A}$, *and* $\bar{B}$

$$rank[\bar{E}^D \bar{B}\ (\bar{E}^D \bar{A})\bar{E}^D \bar{B} \cdots (\bar{E}^D \bar{A})^{n-1}\bar{E}^D \bar{B}] = n. \tag{3.24}$$

Proof. Suppose that the system (2.3) (and (2.15)) is completely controllable on $t \in [t_0, t_1]$. From Cayley-Hamilton theorem [19] (and see [2.15]), we can write

$$\Phi(\bar{E}^D\bar{A}, t_i - \tau)\bar{E}^D\bar{B} = \sum_{i=0}^{n-1} \gamma_i(t_i - \tau)(\bar{E}^D\bar{A})^i\bar{E}^D\bar{B}. \tag{3.25}$$

We assume contrary that the rank condition (3.24) does not holds. Then, there exists a $z \neq 0 \in \mathbb{R}^n$ such that

$$z^*(\bar{E}^D\bar{A})^j\bar{E}^D\bar{B} = 0, \ j = 0, 1, \cdots, n-1,$$

which gives

$$\begin{aligned} z^*W_c[t_i, t_{i+1}] &= z^* \int_{t_i}^{t_{i+1}} \left(\Phi(t_{i+1} - \tau)\bar{E}^D\bar{B}\right)(\Phi(t_{i+1} - \tau)\bar{E}^D\bar{B})^* d\tau \\ &= z^* \int_{t_i}^{t_{i+1}} \sum_{m=0}^{n-1} \gamma_i(t_i - \tau)(\bar{E}^D\bar{A})^m\bar{E}^D\bar{B}(\Phi(t_{i+1} - \tau)\bar{E}^D\bar{B})^* d\tau = 0. \end{aligned}$$

and it implies that

$$rank[\bar{E}^D\bar{B}(\bar{E}^D\bar{A})\bar{E}^D\bar{B} \cdots (\bar{E}^D\bar{A})^{n-1}\bar{E}^D\bar{B}] < n.$$

This is a contradiction, which yields

$$rank[\bar{E}^D\bar{B}\ (\bar{E}^D\bar{A})\bar{E}^D\bar{B} \cdots (\bar{E}^D\bar{A})^{n-1}\bar{E}^D\bar{B}] = n.$$

Conversely, suppose that

$$rank[\bar{E}^D\bar{B}\ (\bar{E}^D\bar{A})\bar{E}^D\bar{B} \cdots (\bar{E}^D\bar{A})^{n-1}\bar{E}^D\bar{B}] = n.$$

But, the system (2.3) (and (2.15)) is not controllable on $t \in [t_0, t_1]$ and $t \in (t_i, t_{i+1}]$ for $i = 1, 2, \cdots, k$. Then, from Theorem 3.2, there exists a vector $z \neq 0 \in \mathbb{R}^n$, such that

$$z_0^*\left(\Phi(t_{i+1} - t)\bar{E}^D\bar{B}\right) = 0, \ t \in (t_i, t_{i+1}]. \tag{3.26}$$

In particular, at $t = t_{i+1}$, we have $z_0^*\bar{E}^D\bar{B} = 0$. Differentiating (3.25) with respect to t, we have

$$z_0^*\bar{E}^D\bar{A}\Phi(t_{i+1} - t)\bar{E}^D\bar{B} = 0.$$

For $t = t_{i+1}$,we have

$$z_0^*\bar{E}^D\bar{A}\left(\bar{E}^D\bar{B}\right) = 0.$$

Repeating this argument $(n-1)$ times, we have

$$z_0^*\left(\bar{E}^D\bar{A}\right)^j\left(\bar{E}^D\bar{B}\right) = 0, \ j = 0, 1, 2, \cdots, n-1. \tag{3.27}$$

Thus,

$$z_0^*\left[\left(\bar{E}^D\bar{B}\right)\ \bar{E}^D\bar{A}\left(\bar{E}^D\bar{B}\right)\ \cdots\ \left(\bar{E}^D\bar{A}\right)^j\left(\bar{E}^D\bar{B}\right)\right] = 0,$$

which implies that the rank condition does not holds. This contradiction proves that the system (2.3) (and (2.15)) is controllable on $t \in [t_0, T]$. Hence, the proof completes. □

Example 3.4. Consider the following linear impulsive differential-algebraic system with fractional order:

$$\begin{cases} E\,{}^cD^\alpha_{t_i,t}\, x(t) = Ax(t) + Bu(t), \ t \neq t_i, \\ x(t_i^+) = (\frac{2}{3})x(t_i), \ t = t_i, \ t_i = \frac{(i+3)}{2}, i = 1, 2, \cdots, k, \\ x(t_0) = 1, \\ y(t) = Cx(t), \end{cases} \tag{3.28}$$

where the matrices E, A, B and C are defined as

$$E = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \ A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} \ B = \begin{pmatrix} -1 \\ 6 \end{pmatrix} \text{ and}$$
$$C = \begin{pmatrix} 2 & 5 \end{pmatrix}.$$

It is clear that E^{-1} does not exists and $rank(E) = 1$. So, for system (3.28) we have obtained a regular matrix pencil (E, A) for $c = 1$, which is of the following form:

$$(Ec - A)^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{2} & \frac{1}{6} \end{pmatrix}. \tag{3.29}$$

With the help of regular pencil (3.29), we can write an equivalent system

$$\bar{E}^c D^{\alpha}_{t_i,t}\, x(t) = \bar{A}x(t) + \bar{B}u(t),$$

where the matrices $\bar{E}$, $\bar{A}$ and $\bar{B}$ are given by

$$\bar{E} = \begin{pmatrix} 0 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}, \; \bar{A} = \begin{pmatrix} -1 & -\frac{2}{3} \\ 0 & -\frac{2}{3} \end{pmatrix} \text{ and}$$
$$\bar{B} = \begin{pmatrix} -\frac{5}{2} \\ \frac{3}{2} \end{pmatrix}.$$

Moreover the obtained Drazin inverse $\bar{E}^D$ of matrix E is as follows

$$\bar{E}^D = \begin{pmatrix} 0 & -1 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

For the controllability of system (3.28), we can write the rank condition from Theorem 3.3 as

$$rank[\bar{E}^D\bar{B} \mid (\bar{E}^D\bar{A})\bar{E}^D\bar{B}] = rank \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix} = 2.$$

Hence, the system (3.28) is controllable.

4. Observability

Definition 4.1. System (2.3) (and (2.15)) are observable on the interval $[t_0, T]$ if each initial value $x(t_i) = x_i \in \mathbb{R}^n$ $(i = 1, 2, \cdots, k)$ is uniquely determined by the corresponding system input $u(t)$ and system output $y(t)$, for $t \in [t_0, T]$.

Theorem 4.2. *Let (A) hold and $c_i \neq -1$ for each $i = 0, 1, 2, \cdots, k$,, then the following propositions are equivalent.*
(i) System (2.3) (and (2.15)) are observable on $t \in [t_0, T]$.
(ii) Gramian matrices $M_i[t_i, t_{i+1}]$ are invertible for each $i = 0, 1, 2, \cdots, k$., where

$$M_i[t_i, t_{i+1}] = \int_{t_i}^{t_{i+1}} \Phi_0^*(t - t_i)C^*C\Phi_0(t - t_i)dt. \tag{4.1}$$

Proof. From system (2.3), we can write

$$y(t) = Cx(t) + Du(t). \tag{4.2}$$

Then, (2.16) becomes

$$y(t) = \begin{cases} C\Phi_0(t - t_0)x_0 + C\int_{t_0}^{t} \Phi(t - \tau)\bar{E}^D\bar{B}u(\tau)d\tau \\ +C(\bar{E}^D\bar{E} - \mathbb{I})\sum_{k=0}^{q-1}(\bar{E}\ \bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t) \\ +Du(t),\ t \in [t_0, t_1]; \\ C\Phi_0(t - t_i)\left(x(t_i)\left(1 + c_i\right)\right) + C\int_{t_i}^{t} \Phi(t - \tau)\bar{E}^D\bar{B}u(\tau)d\tau \\ +C(\bar{E}^D\bar{E} - \mathbb{I})\sum_{k=0}^{q-1}(\bar{E}\ \bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t) \\ +Du(t), t \in (t_i, t_{i+1}] \text{ for } i = 1, 2, \cdots, k. \end{cases} \tag{4.3}$$

From the definition of observability, it is equivalent to the observability of

$$y(t) = \begin{cases} C\Phi_0(t-t_0)x_0, \ t \in [t_0, t_1], \\ C\Phi_0(t-t_i)\left(x(t_i)\left(1+c_i\right)\right), \ t \in (t_i, t_{i+1}] \end{cases} \tag{4.4}$$

as $u(t) = 0$.

Multiplying (4.4) with $\Phi_0^*(t-t_0)C^*$ for $t \in [t_0, t_1]$ and $\Phi_0^*(t-t_i)C^*$ for $t \in (t_i, t_{i+1}]$ on both sides and integrating with respect to t, we have

$$\int_{t_0}^{t_1} \Phi_0^*(t-t_0)C^*y(t) = \int_{t_0}^{t_1} \Phi_0^*(t-t_0)C^*C\Phi_0(t-t_0)x_0 dt \tag{4.5}$$

and

$$\begin{aligned} \int_{t_i}^{t_{i+1}} \Phi_0^*(t-t_i)C^*y(t) = \int_{t_i}^{t_{i+1}} \Phi_0^*(t-t_i)C^* \\ \times C\Phi_0(t-t_i)\left(x(t_i)\left(1+c_i\right)\right)dt, \end{aligned} \tag{4.6}$$

which yields

$$\int_{t_0}^{t_1} \Phi_0^*(t-t_0)C^*y(t) = M_0[t_0, t_1]x_0 \tag{4.7}$$

and

$$\int_{t_i}^{t_{i+1}} \Phi_0^*(t-t_i)C^*y(t) = M_i[t_i, t_{i+1}]\left(1+c_i\right)x(t_i). \tag{4.8}$$

Obviously, left hand sides of Eq. (4.7) and (4.8) depend on $y(t)$ and $M_i[t_i, t_{i+1}]$ are invertible. So, from (4.7) and (4.8,) we note that all x_0 and x_i can be uniquely determined, respectively, by the corresponding system output $y(t)$ and proves that the system is observable on $[t_0, T]$.

For the converse part, we consider that the observability Gramians $M_i[t_i, t_{i+1}]$ are not invertible. Then, there exist nonzero vectors z and $z_i \in \mathbb{R}^n$ such that

$$z^*M_0[t_0, t_1]z = 0 \tag{4.9}$$

and

$$z_i^*M_i[t_i, t_{i+1}]z_i = 0. \tag{4.10}$$

Since $c_i \neq -1$, then all $M_i\left[t_i, t_{i+1}\right]$ are positive semi-definite. Thus, we consider $z = x_0, \cdots, z_i = x(t_i)$. From (4.4), (4.9) and (4.10), we have

$$\int_{t_0}^{t_1} y^*(\tau)y(\tau)d\tau = x_0^* \int_{t_0}^{t_1} (C\Phi_0(\tau-t_0))^* C\Phi_0(\tau-t_0)x_0 d\tau \tag{4.11}$$

and

$$\begin{aligned} \int_{t_i}^{t_{i+1}} y^*(\tau)y(\tau)d\tau = \left(x^*(t_i)\left(1+c_i\right)^2\right) \int_{t_i}^{t_{i+1}} \left(C\Phi_0(\tau-t_i)\right)^* \\ \times C\Phi_0(\tau-t_i)x(t_i)d\tau. \end{aligned} \tag{4.12}$$

So, we can write

$$\int_{t_0}^{t_1} y^*(\tau)y(\tau)d\tau = x_0^*M_0[t_0, t_1]x_0 = 0, \tag{4.13}$$

$$\int_{t_i}^{t_{i+1}} y^*(\tau)y(\tau)d\tau = \left(1+c_i\right)^2 x^*(t_i)M_i[t_i, t_{i+1}]x(t_i) = 0, \tag{4.14}$$

which implies that

$$\int_{t_0}^{t_f} \|y(\tau)\|^2 d\tau = 0,\ t_f \in [t_0, T].$$

Then, it follows that

$$0 = y(t) = \begin{cases} C\Phi_0(t-t_0)x_0,\ t \in [t_0, t_1], \\ C\Phi_0(t-t_i)\left(x(t_i)\left(1+c_i\right)\right),\ t \in (t_i, t_{i+1}], \end{cases}$$

which indicates that the system is not observable on $[t_0, T]$, which is a contradiction. Which completes the proof. □

Theorem 4.3. *Let (A) hold and $c_i \neq -1$ for each $i = 0, 1, 2, \cdots, k$. Then, the following propositions are equivalent.*
(i) System (2.3) (and (2.15)) are observable on an interval $[t_0, T]$.
(ii) For given $\bar{E}, \bar{A}$, and C

$$rank(O_b) = rank \begin{pmatrix} C \\ C(\bar{E}\ \bar{A}^D) \\ \vdots \\ C(\bar{E}\ \bar{A}^D)^{n-1} \end{pmatrix} = n. \tag{4.15}$$

Proof. Consider that the $rank(O_b) = n$ and we will show that the system (2.3) (and (2.15)) is observable. We assume contrary here that the system is not observable and the Gramian matrices $M_i[t_i, t_{i+1}]$, $i = 0, 1, 2, \cdots, k$, are not invertible. Then, from Theorem 4.2, there exists $z_i \neq 0$ such that

$$z_i^* M_i[t_i, t_{i+1}] z_i = \int_{t_i}^{t_{i+1}} z^* \Phi_0^*(t-t_i) C^* C \Phi_0(t-t_i) z dt = 0. \tag{4.16}$$

From (4.16), it implies that

$$C\Phi_0(t-t_i)z_i = 0. \tag{4.17}$$

At $t = t_i$, (4.17) yields $C\ z_i = 0$. Differentiating Eq. (4.17), $(n-1)$ times and at $t = t_i$, we obtain

$$C(\bar{E}\ \bar{A}^D)^j z_i = 0,\ j = 1, 2, \cdots, n-1.$$

Since $z_i \neq 0$, this implies that $rank(O_b) < n$, which leads to a contradiction for our assumption that $rank(O_b) = n$.

Now for the converse part of this theorem, we assume that $rank(O_b) < n$. Then, there exists a vector $z \neq 0$, $z_i \neq 0 \in \mathbb{R}^n$ such that $O_b z = 0$ and $O_b z_i = 0$. That is,

$$C(\bar{E}\ \bar{A}^D)^j z = 0,\ j = 1, 2, \cdots, n-1. \tag{4.18}$$

From the Cayley-Hamilton theorem and (4.1), we can write

$$M_0[t_0, t_1]z = \int_{t_0}^{t_1} \sum_{j=0}^{n-1} \beta_j(\tau - t_0) \Phi_0^*(t-t_0) C^* C(\bar{E}\ \bar{A}^D)^j z d\tau = 0$$

and

$$M_i[t_i, t_{i+1}]z_i = \int_{t_i}^{t_{i+1}} \sum_{j=0}^{n-1} \beta_j(\tau - t_i) \Phi_0^*(t-t_i) C^* C(\bar{E}\ \bar{A}^D)^j z_i d\tau = 0.$$

From (4.18), we conclude that $M_0[t_0, t_1]z = 0$ and $M_i[t_i, t_{i+1}]z_i = 0$. But we have assumed that both z and $z_i \neq 0$, which is a contradiction and completes our proof. □

Example 4.4. Discuss the observability of the following linear impulsive differential algebraic system with fractional order:

$$\begin{cases} E\,{}^cD^{\alpha}_{t_i,t}\, x(t) = Ax(t) + Bu(t),\ t \neq t_i, \\ x(t_i^+) = (\tfrac{2}{3})x(t_i),\ t = t_i,\ t_i = \tfrac{(i+3)}{2}, i = 1, 2, \cdots, k, \\ x(t_0) = 1, \\ y(t) = Cx(t) \end{cases} \tag{4.19}$$

with

$$E = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \; A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} \; B = \begin{pmatrix} -1 \\ 6 \end{pmatrix} \text{ and}$$
$$C = \begin{pmatrix} 2 & 5 \end{pmatrix}.$$

Clearly, E is not invertible and $rank(E) = 1$. Also, the pencil of matrices (E, A) is regular for $c = 1$, that is

$$(Ec - A)^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{2} & \frac{1}{6} \end{pmatrix}.$$

With the help of regular pencil, we can write an equivalent system $\bar{E}^c D^{\alpha}_{t_i,t}\, x(t) = \bar{A}x(t) + \bar{B}u(t)$, where

$$\bar{E} = \begin{pmatrix} 0 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}, \; \bar{A} = \begin{pmatrix} -1 & -\frac{2}{3} \\ 0 & -\frac{2}{3} \end{pmatrix} \text{ and}$$
$$\bar{B} = \begin{pmatrix} -\frac{5}{2} \\ \frac{3}{2} \end{pmatrix}.$$

The Drazin inverse $\bar{E}^D$ of matrix E is of the form:

$$\bar{E}^D = \begin{pmatrix} 0 & -1 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

For the observability of system (4.19), we can write the rank condition from Theorem 4.3 as

$$rank(O_b) = rank \begin{pmatrix} C \\ C(\bar{E}\ \bar{A}^D) \end{pmatrix}$$
$$= rank \begin{pmatrix} 2 & 5 \\ 0 & \frac{-1}{3} \end{pmatrix} = 2.$$

This fact implies that the system (4.19) is observable.

Conclusion

In this paper, controllability and observability problems have been studied, based on regular matrix pencil condition for singular impulsive fractional-order control systems with the order $0 < \alpha \leq 1$. Necessary and sufficient conditions have been presented. The results are more generalized and can be verified for the results that are obtained for various differential algebraic systems without impulses earlier for $0 < \alpha \leq 1$.

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