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Finite volume element approximation for time dependent convection diffusion reaction equations with memory

Anas Rachid*
Laboratoire LMA,ENS de Casablanca, Hassan II university of Casablanca, B.P 50069, Ghandi Casablanca, Morocco.
ENSAM de Casablanca, Hassan II university of Casablanca, Casablanca, Morocco.
E-mail: `rachid.anas@gmail.com`

Mohamed Bahaj
Faculty of Sciences and Technology, University Hassan 1^{st},Settat, Morocco.
E-mail: `mohamedbahaj@gmail.com`

Rachid Fakhar
Laboratoire LS3M, Universit Hassan 1^{st}, 25000 Khouribga, Morocco.
E-mail: `rachid.fakhar@gmail.com`

Abstract Error estimates for element schemes for time-dependent for convection-diffusion-reaction equations with memory are derived and stated. For the spatially discrete scheme, optimal order error estimates in L^2, H^1, and $W^{1,p}$ norms for $2 \leq p < \infty$, are obtained. In this paper, we also study the lumped mass modification. Based on the Crank-Nicolson method, a time discretization scheme is discussed and related error estimates are derived.

Keywords. Finite volume method, Crank-Nicolson method, Parabolic integrodifferential equation, Full discrete scheme, Error estimates.

2010 Mathematics Subject Classification. 35K10, 65N08, 65N15.

1. Introduction

The main purpose of this paper is to study semi-discrete and full-discrete finite volume element method (FVE) for convection-diffusion-reaction equations with memory of the form

$$\begin{aligned} u_t - \nabla.(A(x)\nabla u) + \nabla.(bu) + cu - \int_0^t \nabla.(B(x,t,s)\nabla u(s))ds \\ = f(x,t), \quad \text{in } \Omega \times (0,T], \end{aligned} \tag{1.1}$$

$$\begin{cases} u = 0, & \text{on } \partial\Omega \times (0,T], \\ u(\cdot,0) = u_0, & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^d, d = 2,3$, with smooth boundary $\partial\Omega$ and $T < \infty$. Here $A = A(x)$ is a symmetric and uniformly positive definite dispersion-diffusion matrix in Ω, the parameter b is the divergence free groundwater velocity and c is

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$*$ corresponding.

the constant reaction parameter and $B(t,s)$ an arbitrary second order linear partial differential operator, A, B both with coefficients depending smoothly on x. The nonhomogeneous term $f = f(x,t)$ and $u_0(x)$ are known functions, which are assumed to be smooth and satisfy certain compatibility conditions for $x \in \Omega$ and $t = 0$. Problem (1.1) occurs in nonlocal reactive flows in porous media, viscoelasticity and heat conduction through materials with memory.

The finite volume method is an important numerical tool for solving partial differential equations. It has been widely used in several engineering fields, such as fluid mechanics, heat and mass transfer and petroleum engineering. The method can be formulated in the finite difference framework or in the Petrov-Galerkin framework. Usually, the former one is called finite volume method[3], MAC (marker and cell) method [9] or cell-centered method[4], and the latter one is called finite volume element method (FVE) [6, 12, 13, 14], covolume method[8] or vertex-centered method [2, 10]. We refer to the monographs [15, 23] for the general presentation of these methods. The most important property of the FVE method is that it can preserve the conservation laws (mass, momentum, and heat flux) on each control volume. This important property, combined with adequate accuracy and ease of implementation, has attracted more people to do research in this field.

Recently Bahaj and Rachid [1] studied the FVE method for Self-Adjoint Parabolic Integrodifferential Equations and have obtained an optimal-order estimate in the L^2 and H^1 -norms. Ewing, Lin and Lin in [14] and Jianguo and Shitong in [17] elaborate the FVE method for general self adjoint elliptic problems. Ma, Shu and Zho in [19] presented and analyzed the semi-discrete and full discrete symmetric finite volume schemes for a class of parabolic problems. In [12, 13] the authors have studied the FVE method for one and two-dimensional parabolic integrodifferential equations and have obtained an optimal-order estimate in the L^2-norm. The regularity required on the exact solution u is $W^{3,p}$ for $p > 1$ which is higher when compared that for finite element methods.

The new contribution of this work is to extend the results from [1] to the finite volume discretization for time-dependent convection-diffusion-reaction equations with memory (1.1). Both spatially discrete scheme and discrete-in-time scheme are analyzed and optimal error estimates in L^2 and H^1 norms are proved using only energy method. We also explore and generalize that idea to develop the lumped mass modification and $W^{1,p}$ estimates, $2 \leq p < \infty$. Our analysis avoid the use of semigroup theory and the regularity requirement on the solution is the same as that of finite element method. Further, based on the Crank-Nicolson method the fully discrete scheme is analyzed and related optimal error estimates are established.

This paper is organized as follows. In section 2, we introduce some notations and present some preliminary materials to be used later. The Ritz-Volterra projection to finite volume element spaces is introduced and related estimates are carried out in section 3. In section 4 we estimate the error of the finite volume element approximations derived in the previous section. In section 5 the lumped mass are presented and optimal estimates in L^2 and H^1 norms are obtained Finally, The Crank-Nicolson scheme is studied in section 6.

2. Finite volume element scheme

In this section, we introduce some material which will be used repeatedly below. Throughout this paper, C (with or without index) denotes a generic positive constant which does not depend on the spatial and time discretization parameters h and k, respectively.

2.1. **Notations.** We will use $\|\cdot\|_m$ and $|\cdot|_m$ (resp.$\|\cdot\|_{m,p}$ and $|\cdot|_{m,p}$) to denote the norm and semi-norm of the Sobolev space $H^m(\Omega)$ (resp.$W^{m,p}(\Omega)$). The scalar product and norm in $L^2(\Omega)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Let $H_0^1(\Omega)$ be the standard Sobolev subspace of $H^1(\Omega)$ of functions vanishing on $\partial\Omega$.

The weak form of (1.1) is to find $u(\cdot,t) : [0,T] \to H_0^1(\Omega)$, such that

$$(u_t, v) + A(u,v) + \int_0^t B(t,s;u(s),v)\,ds = (f,v), \quad \forall v \in H_0^1(\Omega), \tag{2.1}$$
$$u(\cdot,0) = u_0,$$

where

$$\begin{aligned} A(u,v) &= \int_\Omega (A(x)\nabla u.\nabla v - bu\nabla v + cuv)\,dx, \\ B(t,s;u(s),v) &= \int_\Omega B(x,t,s)\nabla u(s).\nabla v\,dx. \end{aligned}$$

Note that the bilinear form $A(.,.)$ may not be coercive but it can be made coercive by adding a sufficiently large constant $\lambda \in \mathbb{R}$ times the L^2-inner product. That is, it satisfies Gärding's type inequality

$$A(v,v) + \lambda\|v\|^2 \geq \frac{\alpha}{2}\|v\|_1^2 \qquad \forall\, v \in H_0^1(\Omega).$$

Introducing the transformation $\overline{u} = e^{-\lambda t}u$ as a new dependent variable, we rewrite (1.1) as

$$\begin{aligned} \overline{u}_t - \nabla.(A(x)\nabla\overline{u}) + \nabla.(b\overline{u}) &- \int_0^t \nabla.(B(x,t,s)\nabla\overline{u}(s))\,ds \\ &+ (\lambda + c)\overline{u} = \overline{f}(x,t), \text{ in } \quad \Omega\times(0,T], \\ \overline{u}(0) &= u_0. \end{aligned}$$

The new bilinear form $A(.,.)$ is given

$$A(u,v) = \int_\Omega A(x,t)\nabla u.\nabla v - bu\nabla v + \int_\Omega (\lambda + c)uv\,dx.$$

Let $\mathcal{T}_h$ be a decomposition of Ω into triangles (for the 2-D case) or tetrahedral (for the 3-D case) with $h = \max h_K$, where h_K is the diameter of the element $K \in \mathcal{T}_h$.

In order to describe the FVEM for solving the problem (1.1), we shall introduce a dual partition $\mathcal{T}_h^*$ based upon the original partition $\mathcal{T}_h$ whose elements are called control volumes. We construct the control volumes in the same way as in [13, 16].

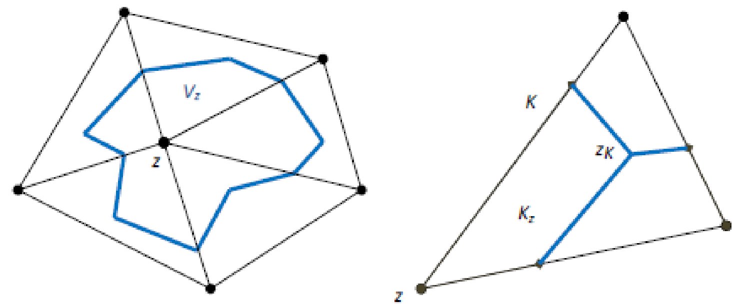

FIGURE 1. Left-hand side: A sample region with blue lines indicating the corresponding control volume V_z. Right-hand side: A triangle K partitioned into three subregions K_z

Let z_K be a point of $K \in \mathcal{T}_h$. In the 2-D case, on each edge e of K a point q_e is selected, then we connect z_K with line segments to q_e. Thus partitioning K into three quadrilaterals K_z, $z \in Z_h(K)$, where $Z_h(K)$ are the vertices of K. Then with each vertex $z \in Z_h = \cup_{K \in \mathcal{T}_h} Z_h(K)$ we associate a control volume V_z, which consists of the union of the subregions K_z, sharing the vertex z. (see Figure 1)

Similarly, in 3-D case, on each two faces S_1 and S_2 of K sharing an edge e, a point q_{S_i} is selected, then we connect q_{S_i} with an arbitrary point q_e of e and with z_K by line segments. Thus partitioning K into twelve (12) tetrahedron K_z, $z \in Z_h(K)$. (see,Figure 2). Then for $z \in Z_h$, the control volume V_z consists of the union of the subregions K_z, sharing the vertex z. Thus we finally obtain a group of control volumes covering the domain Ω, which is called the dual partition $\mathcal{T}_h^*$ of the triangulation $\mathcal{T}_h$. We denote by Z_h^0 the set of interior vertices and $N_h = \#Z_h^0$. For a vertex $z_i \in Z_h^0$, let $\Pi(i)$ be the index set of those vertices that, along with z_i; are in some element of T_h. (Figure 2)

There are various ways to introduce a regular dual partition $\mathcal{T}_h^*$. In this paper, we shall also use the construction of the control volumes in which z_K be the barycenter of $K \in \mathcal{T}_h$. In the 2-D case, we choose q_e to be the midpoint of the edge e (Figure 3).

In the 3-D case, we choose, q_e to be the midpoint of the edge e and q_{S_i} to be the medi center of the face S_i(Figure 4).

We call the partition T_h^* regular or quasi-uniform, if there exists a positive $C > 0$ such that

$$C^{-1}h^2 \leq meas\,(V_i) \leq Ch^2, \quad \forall V_i \in T_h^*.$$

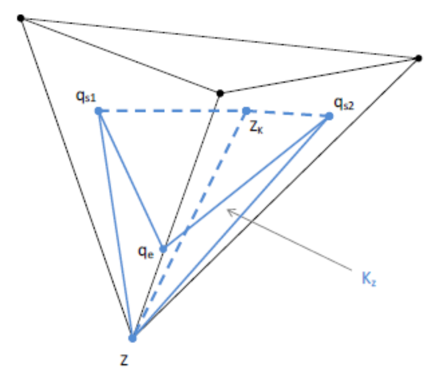

FIGURE 2. A tetrahedron K partitioned into twelve subregions K_z

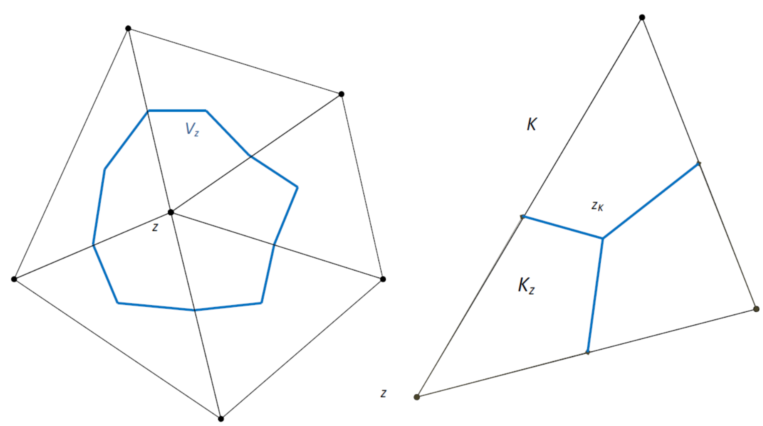

FIGURE 3. z_K is the barycenter of K, q_e to be the midpoint of the edge e

The barycenter-type dual partition can be introduced for any finite element triangulation T_h and leads to relatively simple calculations. Besides, if the finite element triangulation T_h is quasi-uniform, i.e., there exists a positive $C > 0$ such that

$$C^{-1}h^2 \leq meas\,(K) \leq Ch^2, \quad \forall K \in T_h,$$

then the dual partition T_h^* is also quasi-uniform.

Based on the triangulation T_h, let S_h be the standard conforming finite element space of piecewise linear functions, defined on the triangulation T_h,

$$S_h = \{v \in \mathcal{C}\,(\Omega) : v|_K \text{ is linear } \forall K \in T_h, \text{ and } v|_\Gamma = 0\}.$$

Let $I_h : \mathcal{C}\,(\Omega) \to S_h$ be the standard interpolation operators,

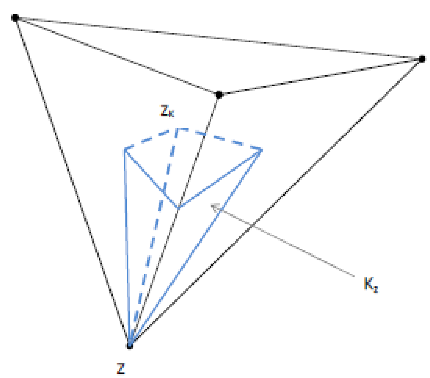

FIGURE 4. q_e is the midpoint of the edge e, q_{S_i} is the medi center of the face S_i

$$I_h u = \sum_{z \in Z_h^0} v_z(t) \varphi_z(x), \quad \forall v \in S_h,$$

where $\{\varphi_z\}_{z \in Z_h^0}$ are the standard nodal basis functions of S_h and $v_z(t) = v(t; z)$.

2.2. **Construction of the FVE scheme.** We formulate the FVE for the problem (1.1) as follows: Given a $z \in Z_h^0$ and $K \in \mathcal{T}_h$, integrating (1.1) over the associated control volume V_z and applying Green's formula, we obtain an integral conservation form

$$\int_{V_z} u_t - \int_{\partial V_z} (A(x) \nabla u - bu) .nds \quad - \quad \int_{\partial V_z} \int_0^t B(x,t,s) \nabla u.nds \\ + \int_{V_z} (\lambda + c) u = \int_{V_z} f(x,t), \quad (2.2)$$

where n denotes the unit outer normal vector to ∂V_z.

Let $I_h^* : \mathcal{C}(\Omega) \to S_h^*$ be the transfer operator defined by

$$I_h^* v = \sum_{z \in Z_h^0} v(z) \chi_z, \quad \forall v \in S_h,$$

where

$$S_h^* = \{v \in L^2(\Omega) : v_i|_{V_z} \text{ is constant}, \ \forall z \in Z_h^0\},$$

and χ_z is the characteristic function of the control volume V_z.

Now for $t > 0$ and for an arbitrary $I_h^* v$, we multiply (2.2) by $v(z)$ and sum over all $z \in Z_h^0$. Then the semi discrete FVE approximation u_h of (1.1) is a solution to the problem: find $u_h(t) \in S_h$ for $t > 0$ such that

$$\begin{aligned} &(u_{ht}, v_h) + A(u_h, v_h) + \int_0^t B(t, s; u_h(s), v_h)\, ds = (f, v_h), \quad v_h \in S_h^*, \\ &u_h(0) = u_{0h} \in S_h. \end{aligned} \tag{2.3}$$

Here the bilinear forms $A(t; u, v)$ and $B(t, s; u, v)$ are defined by

$$A(u,v) = \begin{cases} -\sum\limits_{z\in Z_h^0} v_z \int_{\partial V_z} (A(x)\nabla u - bu)\cdot n ds + v_z \int_{V_z} (\lambda + c)\, u\, dx, \\ \qquad\qquad (u,v) \in \left(\left(H_0^1 \cap H^2\right) \cup S_h\right) \times S_h^* \\ \int_\Omega A(x)\nabla u.\nabla v dx - bu\nabla v + \int_\Omega (\lambda + c)\, uv\, dx, \quad (u,v) \in H_0^1 \times H_0^1 \end{cases}$$

and

$$B(t,s;u,v) = \begin{cases} -\sum\limits_{z\in Z_h^0} v_z \int_{\partial V_z} B(x,t,s)\nabla u.n ds, \quad (u,v) \in \left(\left(H_0^1 \cap H^2\right) \cup S_h\right) \times S_h^*, \\ \int_\Omega B(x,t,s)\nabla u.\nabla v dx, \qquad (u,v) \in H_0^1 \times H_0^1. \end{cases}$$

Let

$$u_h = \sum_{j=1}^{N_h} \alpha_z(t)\varphi_z(x) \ , \ \text{ and } \quad \alpha(t) = \left(\alpha_1(t), \alpha_2(t), ..., \alpha_{N_h}(t)\right)^T.$$

Then we can rewrite the scheme (2.3) as systems of ordinary differential equations

$$M_h \alpha'(t) + A_h \alpha(t) + \int_0^t B_h(t,s)\,\alpha(s)\, ds = F_h(t). \tag{2.4}$$

Here $F_h(t) = \left(f_1(t), f_2(t), ..., f_{N_h}(t)\right)^T$, the mass matrix $M_h = \left\{M_{h_{ij}}\right\} = \{(\varphi_i, \chi_j)\}$ is tridiagonal and that both $A_h = \{A(\varphi_i, \chi_j)\}$ and $B_h(t,s) = \{B(t,s;\varphi_i,\chi_j)\}$ are positive definite.

In order to describe features of the bilinear forms defined in (2.3) we introduce some discrete norms on S_h in the same way as in [13],

$$\begin{aligned} &\|v_h\|_{0,h}^2 = (v_h, v_h)_{0,h} = (I_h^* v_h, I_h^* v_h), \\ &|v_h|_{1,h}^2 = \sum_{x_i\in Z_h^0}\sum_{x_j\in\Pi(i)} meas(V_i)\left((v_{hi} - v_{hj})/d_{ij}\right)^2, \quad \text{with } v_{hi} = v_h(x_i) \\ &\|v_h\|_{1,h}^2 = \|v_h\|_{0,h}^2 + |v_h|_{1,h}^2, \qquad |||v_h|||^2 = (v_h, I_h^* v_h), \end{aligned}$$

where $d_{ij} = d(x_i, x_j)$ is the distance between x_i and x_j. Obviously, these norms are well defined for $v_h \in S_h^*$ as well and $\|v_h\|_{0,h} = |||v_h|||$.

Below, we state the equivalence of discrete norms $\|.\|_{0,h}$ and $\|.\|_{1,h}$ with usual norms $\|.\|$ and $\|.\|_1$, respectively on S_h.

Lemma 2.1 ([13]). *There exist two positive constants C_0 and C_1 such that for all $v_h \in S_h$, we have*

$$C_0 \|v_h\|_{0,h} \leq \|v_h\| \leq C_1 \|v_h\|_{0,h}, \quad \forall \ v_h \in S_h,$$
$$C_0 |||v_h||| \leq \|v_h\| \leq C_1 |||v_h|||, \quad \forall \ v_h \in S_h,$$
$$C_0 \|v_h\|_{1,h} \leq \|v_h\|_1 \leq C_1 \|v_h\|_{1,h}, \quad \forall \ v_h \in S_h.$$

Next, we recall some properties of the bilinear forms (see [13, 20]).

Lemma 2.2 ([13]). *There exist two positive constants C and C_0 such that for all $u_h, v_h \in S_h$, we have*

$$A\left(u_h, I_h^* v_h\right) \leq C \|u_h\|_1 \|v_h\|_1, \quad \forall \ u_h, v_h \in S_h,$$
$$A\left(v_h, I_h^* v_h\right) \geq C_0 \|v_h\|_1^2, \quad \forall \ v_h \in S_h.$$

The following Lemmas is proved in [4, 13], which gives the key feature of the bilinear forms in the FVE method.

Lemma 2.3 ([4]). *Assume that $\varphi \in W_0^{1,p}$. Then we have*

$$A(\varphi, v_h) - A\left(\varphi, I_h^* v_h\right) = \sum_{K \in \tau_h} \int_{\partial K} \left(\left(A(x) \nabla \varphi - b\varphi\right).\mathbf{n}\right)\left(v_h - I_h^* v_h\right) ds$$
$$- \sum_{K \in \tau_h} \int_K \left(\nabla.\left(A(x) \nabla \varphi - b\varphi\right) + (\lambda + c)\varphi\right)\left(v_h - I_h^* v_h\right) ds, \ \forall v_h \in S_h.$$

The above identity holds true when $A(.,.)$ is replaced by $B(t, s; .,.)$.

Introduce

$$\epsilon_h(f, v_h) = (f, v_h) - \left(f, I_h^* v_h\right), \quad \forall \ v_h \in S_h,$$

and

$$\epsilon_A(u_h, v_h) = A(u_h, v_h) - A_h\left(u_h, I_h^* v_h\right), \quad \forall \ u_h, v_h \in S_h.$$

The bounds for ϵ_h, ϵ_A can be given as follows

Lemma 2.4 ([5],pp 317). *There exists positive constants C independent of h,such that for $v_h \in S_h$,*

$|\epsilon_h(f, v_h)| \leq Ch^{i+j} \|f\|_i \|v_h\|_j, \ \forall \ f \in H^i, \ i, j = 0, 1,$

$|\epsilon_A(V_h u, v_h)| \leq Ch^{i+j} \left(\|u\|_{i+1} + \int_0^t \|u\|_{i+1}\right) \|v_h\|_j, \ \forall \ u \in H^{i+1} \cap H_0^1, \ i, j = 0, 1$

and

$|\epsilon_A(u_h, v_h)| \leq Ch \|u\|_1 \|v_h\|_1, \quad \forall \ v_h \in S_h.$

Lemma 2.5 ([4]). *Assume that $\varphi \in S_h$. Then we have*

$$A(\varphi, \chi) - A\left(\varphi, I_h^* \chi\right) \leq Ch |\varphi|_{1,p} |\chi|_{1,q}.$$

Further for $\varphi \in W_0^{1,p} \cap W^{2,p}$, we have

$$A(\varphi, \chi) - A\left(\varphi, I_h^* \chi\right) \leq Ch \|\varphi\|_{2,p} \|\chi\|_{1,q}.$$

3. Ritz-Volterra projection and related estimates

Following [7, 13], define the Ritz-Volterra projection $V_h(t) : H_0^1 \to S_h$

$$A\left(u - V_h u, I_h^* v_h\right) + \int_0^t B\left(t, s; u(s) - V_h u(s), I_h^* v_h\right) ds = 0,\ t > 0,\ \forall v_h \in S_h. \tag{3.1}$$

This $V_h(t)$ is an elliptic projection with memory of u into S_h^*. It is easy to see that (3.1) is actually a system of integral equations of Volterra type. In fact if $V_h(t)u = \sum_{j=1}^{N_h} \alpha_j(t)\varphi_j(x)$, then (3.1) can be rewritten as

$$A_h \alpha(t) + \int_0^t B_h(t, s)\alpha(s)\, ds = F_h(t) \tag{3.2}$$

where $A_h,\ B_h(t, s)$ are matrices and $\alpha(t),\ F_h(t)$ are vectors, defined via

$$\alpha(t) = \left(\alpha_1(t), \alpha_2(t), ..., \alpha_{N_h}(t)\right)^T$$
$$F_{hk}(t) = A(u, \chi_k) + \int_0^t B(t, s; u(s), \chi_k)\, ds, \qquad k = 1, 2, ..., N_h,$$
$$A_h = A(\varphi_k(x), \chi_l), \qquad B_h(t, s) = B(t, s; \varphi_k(x), \chi_l).$$

From the positivity of A (Lemma 2.2) and the linearity of (3.2) we see that the system (3.2) possesses a unique solution $\alpha(t)$. Consequently $V_h(t)u$ in (3.1) is well defined.

Set $\rho = u - V_h(t)u$. In [13] the following lemma was proved, which shows the H^1 error estimate for ρ and its temporal derivative.

Lemma 3.1 ([13]). *Assume that $D_t^n u \in L^\infty(H_0^1 \cap H^2)$ for all $0 \le n \le k$, for some integer $k \ge 0$. Then for $T > 0$ fixed there is a constant $C = C(T; k) > 0$, independent of h and u, such that for all $0 \le n \le k$ and $0 < t < T$,*

$$\|\rho(t)\|_1 \le Ch\left(\|u\|_2 + \int_0^t \|u\|_2\, ds\right),$$

and

$$\|D_t^n \rho(t)\|_1 \le Ch\left(\sum_{i=0}^n \left\|D_t^i u\right\|_2 + \int_0^t \|u\|_2\, ds\right).$$

Now we establish L^2 error estimate for ρ and its temporal derivative which improves the Theorem 2.2 in [13]. This estimate is optimal with respect to the order of convergence and the regularity of the solution.

Lemma 3.2. *Assume that $f \in H^{-1}(\Omega)$ and for some integer $k \ge 0$, $D_t^{n+1} u \in L^\infty(H_0^1 \cap H^2)$ for all $0 \le n \le k$. Then for fixed $T > 0$, there is a constant $C =$*

$C(f;T;k)>0$, independent of h and u, such that for all $0\leq n\leq k$ and $0<t<T$,

$$\|\rho(t)\|\leq Ch^2\left(\|f\|_1+\|u_t\|_1+\|u\|_2+\int_0^t\|u\|_2\,ds\right)$$

and

$$\begin{aligned}\left\|D_t^n\rho(t)\right\| &\leq Ch^2\left(\sum_{i=0}^{n}\left\|D_t^i f\right\|_1+\sum_{i=0}^{n+1}\left\|D_t^i u\right\|_1+\sum_{i=0}^{n}\left\|D_t^i u\right\|_2+\int_0^t\|u\|_2\,ds\right)\\ &\leq Ch^2\left(\sum_{i=0}^{n}\left\|D_t^i f\right\|_1+\left\|D_t^{n+1}u\right\|_1+\sum_{i=0}^{n}\left\|D_t^i u\right\|_2+\int_0^t\|u\|_2\,ds\right).\end{aligned}$$

Proof. The proof will proceed by duality argument. Let $\psi\in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of

$$\begin{aligned}A^*\psi&=\rho \quad \text{in } \Omega\\ \psi&=0 \quad \text{on } \partial\Omega.\end{aligned}$$

The solution $\psi\in H^2(\Omega)\cap H_0^1(\Omega)$ satisfies the following regularity estimate

$$\|\psi\|_2\leq C\|\rho\|. \tag{3.3}$$

Recalling the Ritz projection, [7, 13], $R_h: H_0^1\cap H^2\to S_h$ associated with a bilinear form $A(.,.)$; that is,

$$A(R_hu-u,v_h)=0,\qquad v_h\in S_h.$$

Multiplying equation 3.3 by ρ and then taking L^2 inner-product over Ω, we obtain

$$\begin{aligned}\|\rho\|^2&=A(\rho,\psi)=A(\rho,\psi-R_h\psi)+A(\rho,R_h\psi-I_h^*(R_h\psi))\\ &-\int_0^t B(t,s;\rho(s),I_h^*R_h\psi-R_h\psi)\,ds-\int_0^t B(t,s;\rho(s),R_h\psi-\psi)\,ds\\ &-\int_0^t B(t,s;\rho(s),\psi)\,ds=I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

We have

$$|I_1|+|I_4|\leq Ch^2\left(\|u\|_2+\int_0^t\|u\|_2\,ds\right)\|\psi\|_2.$$

Applying lemma 2.4, we obtain

$$\begin{aligned}&A(\rho,R_h\psi-I_h^*(R_h\psi))-\int_0^t B(t,s;\rho(s),I_h^*R_h\psi-R_h\psi)\,ds\\ =\;&A(u,R_h\psi-I_h^*(R_h\psi))+\int_0^t B(t,s;u(s),R_h\psi-I_h^*R_h\psi)\,ds\\ &-A(V_hu,R_h\psi-I_h^*(R_h\psi))+\int_0^t B(t,s;V_hu(s),I_h^*R_h\psi-R_h\psi)\,ds\\ =\;&(f-u_t,R_h\psi-I_h^*(R_h\psi))-\epsilon_A(V_hu,R_h\psi)+\int_0^t\epsilon_{B(t,s;.,.)}(V_hu(s),R_h\psi)\,ds\end{aligned}$$

$$|I_2 + I_3| \le Ch^2 \left(\|f\|_1 + \|u_t\|_1 + \|u\|_2 + \int_0^t \|u\|_2 \, ds \right) \|\psi\|_1 .$$

Finally, we have

$$|I_5| \le \int_0^t (\rho(s), B^*(t,s)\psi) \, ds \le C \left(\int_0^t \|\rho\| \, ds \right) \|\psi\|_2 ,$$

then we have

$$\|\rho\| \le Ch^2 \left(\|f\|_1 + \|u_t\|_1 + \|u\|_2 + \int_0^t \|u\|_2 \, ds \right) + C \left(\int_0^t \|\rho\| \, ds \right).$$

Finally, an application of Grönwall's lemma yields the first estimate.
The second inequality follows in a similar fashion. □

Lemma 3.3. *There exists a constant C independent of h such that*

$$\|\rho\|_{1,p} \le Ch^2 \left(\|u\|_{2,p} + \int_0^t \|u\|_{2,p} \, ds \right).$$

Proof. Let ρ_x be an arbitrary component of $\nabla\rho$ with $p$and $q$conjugate indices, we have

$$\|\rho_x\|_p = \sup \left\{ (\rho_x, \varphi) ; \varphi \in \mathcal{C}_0^\infty(\Omega), \|\varphi\|_q = 1 \right\}.$$

For any such φ, let ψ be the solution of

$$\begin{aligned} A^*(\psi, v) &= -(\varphi_x, v) \qquad \forall\, v \in H_0^1(\Omega), \\ \psi &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

It follows from the regularity theory for the elliptic problem that

$$\|\psi\|_{1,q} \le C_p \|\varphi\|_q = C_p.$$

We then have by application of (3.1) that

$$\begin{aligned} (\rho_x, \varphi) &= A(\rho, \psi) = A(\rho, \psi - R_h\psi) + A(\rho, R_h\psi - I_h^*(R_h\psi)) \\ &\quad + \int_0^t B(t,s;\rho(s), I_h^*(R_h\psi)) \, ds \\ &= I_1 + I_2 + I_3, \end{aligned}$$

$$A(\rho, \psi - R_h\psi) = A(R_h u - u, \psi) = -((R_h u - u)_x, \varphi) \le Ch \|u\|_{2,p} .$$

Applying lemma 4, we have

$$I_2 = A(u, R_h\psi - I_h^*(R_h\psi)) - A(V_h u, R_h\psi - I_h^*(R_h\psi)) \le Ch \|u\|_{2,p} .$$

Finally, I_3 is estimated as

$$I_3 = \int_0^t B(t,s;\rho(s), I_h^*(R_h\psi)) \, ds \le C_p \int_0^t \|\rho\|_{1,p} \, ds.$$

Combining these estimates we get

$$\|\rho\|_{1,p} \le Ch \|u\|_{2,p} + C_p \int_0^t \|\rho\|_{1,p} \, ds$$

whence by Grönwall's lemma

$$\|\rho\|_{1,p} \leq Ch\left(\|u\|_{2,p} + \int_0^t \|u\|_{2,p}\, ds\right).$$

□

4. Error estimates for semi-discrete approximations

We split the error $e(t) = u(t) - u_h(t)$ as follows

$$e(t) = (u(t) - V_h u(t)) + (V_h u(t) - u_h(t)) = \rho + \theta.$$

It is easy to see that $\theta = V_h u(t) - u_h(t) \in S_h$ satisfies an error equation of the form

$$(\theta_t, I_h^* v_h) + A(\theta, I_h^* v_h) + \int_0^t B\left(t, s; \theta\left(s\right), I_h^* v_h\right) ds = -\left(\rho_t, I_h^* v_h\right), v_h \in S_h. \tag{4.1}$$

Since the estimates of ρ are already known, it is enough to have estimates for θ. We shall prove a sequence of lemmas which lead to the following result.

Lemma 4.1. *There is a positive constant C independent of h such that*

$$|||\theta\left(t\right)||| \leq C\left(|||\theta\left(0\right)|||^2 + \int_0^t \|\rho_t\|\, ds\right).$$

Proof. Since $\theta \in S_h$, we may take $v_h = \theta$ in (4.1) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|||\theta\left(t\right)|||^2 + c\left\|\theta\right\|_1^2 &\leq \|\rho_t\|\,\|\theta\| + C\int_0^t \|\theta\|_1\, ds\,\|\theta\|_1 \\ &\leq \|\rho_t\|\,\|\theta\| + \frac{1}{2}c\left\|\theta\right\|_1^2 + C\int_0^t \|\theta\|_1^2\, ds\end{aligned}$$

and hence by integration, we have

$$||\theta\left(t\right)||^2 + \int_0^t \|\theta\|_1^2\, ds \leq C\left(|||\theta\left(0\right)|||^2 + \int_0^t \|\rho_t\|\,\|\theta\|\, ds + \int_0^t\int_0^s \|\theta\left(\tau\right)\|_1^2\, d\tau ds\right).$$

Grönwall's lemma now implies

$$\begin{aligned}|||\theta\left(t\right)|||^2 + \int_0^t \|\theta\|_1^2\, ds &\leq C\left(|||\theta\left(0\right)|||^2 + \int_0^t \|\rho_t\|\,\|\theta\|\, ds\right) \\ &\leq C\,|||\theta\left(0\right)|||^2 + \frac{1}{2}\sup_{s\leq t}\|\theta\left(s\right)\|^2 + \left(\int_0^t \|\rho_t\|\, ds\right)^2.\end{aligned}$$

Since this holds for all t, we may conclude that

$$||\theta\left(t\right)|| \leq C\left(|||\theta\left(0\right)||| + \int_0^t \|\rho_t\|\, ds\right).$$

□

Lemma 4.2. *There is a positive constant C independent of h such that*

$$\int_0^t \|\theta_t\|^2 \ ds + \|\theta\|_1^2 \leq C\left(\|\theta(0)\|_1^2 + \int_0^t \|\rho_t\|^2 ds\right).$$

Proof. Set $v_h = \theta_t$ in (4.1) to get

$$\begin{aligned}
|||\theta_t|||^2 + \frac{1}{2}\frac{d}{dt}A(\theta, I_h^*\theta) &= -(\rho_t, I_h^*\theta_t) - \int_0^t B(t,s;\theta(s), I_h^*\theta_t(t))\,ds \\
&\quad + \frac{1}{2}\left[A(\theta_t, I_h^*\theta) - A(\theta, I_h^*\theta_t)\right] \\
&\leq \frac{1}{2}\|\rho_t\|^2 + \frac{1}{2}|||\theta_t|||^2 \\
&\quad + \frac{1}{2}\left[A(\theta_t, I_h^*\theta) - A(\theta, I_h^*\theta_t)\right] \\
&\quad - \frac{d}{dt}\int_0^t B(t,s;\theta(s), I_h^*\theta(t))\,ds \\
&\quad + B(t,t;\theta(t), I_h^*\theta(t)) \\
&\quad + \int_0^t B_t(t,s;\theta(s), I_h^*\theta(t))\,ds. \\
\|\theta_t\|^2 + \frac{d}{dt}A(\theta, I_h^*\theta) &\leq \|\rho_t\|^2 - 2\frac{d}{dt}\int_0^t B(t,s;\theta(s), I_h^*\theta)\,ds \\
&\quad + C\left(\|\theta\|_1^2 + \int_0^t \|\theta(s)\|_1^2 ds\right) \\
&\quad + \left[A(\theta_t, I_h^*\theta) - A(\theta, I_h^*\theta_t)\right].
\end{aligned}$$

In addition, recall that

$$A(u_h, I_h^* v_h) - A(v_h, I_h^* u_h) \leq Ch\|u_h\|_1\|v_h\|_1, \quad \forall\ u_h,\ v_h \in S_h,$$

then applying an inverse inequality and using kickback argument, we obtain

$$\begin{aligned}
\left[A(\theta_t, I_h^*\theta) - A(\theta, I_h^*\theta_t)\right] &\leq Ch\|\theta_t\|_1\|\theta\|_1 \leq C\|\theta_t\|\|\theta\|_1 \\
&\leq \varepsilon\|\theta_t\|^2 + C\|\theta\|_1^2.
\end{aligned}$$

Combining these estimates, we derive

$$\begin{aligned}
\|\theta_t\|^2 + \frac{d}{dt}A(\theta, I_h^*\theta) &\leq \|\rho_t\|^2 - 2\frac{d}{dt}\int_0^t B(t,s;\theta(s), I_h^*\theta)\,ds \\
&\quad + C\left(\|\theta\|_1^2 + \int_0^t \|\theta(s)\|_1^2 ds\right).
\end{aligned}$$

So after integration in time and using the weak coercivity of $A\left(\theta, I_h^*\theta\right)$ we get

$$\begin{aligned}\int_0^t \|\theta_t\|^2 \; ds + c_0 \|\theta\|_1^2 &\leq c_0 \|\theta(0)\|_1^2 + \int_0^t \|\rho_t\|^2 \; ds \\ &\quad - 2\int_0^t B\left(t, s; \theta(s), I_h^*\theta\right) ds + C\int_0^t \|\theta(s)\|_1^2 \, ds \\ &\leq c_0 \|\theta(0)\|_1^2 + \frac{c}{2}\|\theta\|_1^2 + C\left(\int_0^t \|\rho_t\|^2 + \|\theta(s)\|_1^2 \, ds\right)\end{aligned}$$

and by Grönwall's lemma,

$$\int_0^t \|\theta_t\|^2 \; ds + c\|\theta\|_1^2 \leq C\left(\|\theta(0)\|_1^2 + \int_0^t \|\rho_t\|^2 \, ds\right).$$

□

Theorem 4.3. *([1])[Error estimates in L^2 and H^1-norms] Let u, u_h be the solutions of (2.2) and (2.4) respectively. Assume that $u \in L^\infty(H_0^1 \cap H^2)$, $u_t \in L^\infty(H^2)$.*

(a) Let u_{0h} be chosen so that $\|u_{0h} - u_0\| \leq Ch^2 \|u_0\|_2$ and assume that $u_{tt} \in L^\infty(H^1)$ and $f, f_t \in L^\infty(H^1)$. Then for fixed $T > 0$, there is a constant $C = C(T)$ independent of h, such that for all $0 < t < T$,

$$\|u_h(t) - u(t)\| \leq Ch^2\left(\|f(0)\|_1 + \|u_0\|_2 + \|u_t\|_1 + \int_0^t \|f_t\|_1 + \|u_t\|_2 + \|u_{tt}\|_1 \, ds\right)$$

(b) Let u_{0h} be chosen so that $\|u_{0h} - u_0\|_1 \leq Ch\|u_0\|_2$. Then for fixed $T > 0$, there is a constant $C = C(T)$ independent of h, such that for all $0 < t < T$,

$$\|u_h(t) - u(t))\|_1 \leq Ch\left(\|u_0\|_2 + \int_0^t \|u_t\|_2 \, ds\right).$$

Now, we prove the error estimate for FVE approximation in $W^{1,p}$-norm.

Theorem 4.4. *Let u, u_h be the solution of (2.2) and (2.4), respectively. Assume that $u, u_t \in L^\infty(H_0^1 \cap W^{2,p})$. In addition, for dimension $d = 2$, we have for h sufficiently small*

$$\|u - u_h\|_{1,p} \leq Ch\left(\|u_0\|_2 + \|u\|_{2,p} + \int_0^t \|u_t\|_2 \, ds\right).$$

Proof. Given $\varphi \in \mathcal{C}_0^\infty(\Omega)$, find $\psi \in H_0^1(\Omega)$ such that

$$\begin{aligned} A^*\psi &= -\varphi_x \quad \text{in } \Omega \\ \psi &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and

$$\|\psi\|_{1,q} \leq \|\varphi\|_{0,q}.$$

We have

$$\begin{aligned}((u-u_h)_x\,,\varphi) &= A\,(u-u_h,\psi)=A(u-u_h,\psi-R_h\psi)\\ &+A\,(u-u_h,R_h\psi-I_h^*R_h\psi)\\ &-\int_0^t B\,(t,s;(u-u_h)\,(s)\,,I_h^*R_h\psi)\,ds\\ &-((u-u_h)_t\,,I_h^*R_h\psi)\\ &=I_1+I_2+I_3+I_4.\end{aligned}$$

$$|I_1|\le|A(u-R_hu,\psi)|\le C\left\|u-R_hu\right\|_{1,p}\left\|\psi\right\|_{1,q}\le Ch\left\|u\right\|_{2,p}\left\|\psi\right\|_{1,q}.$$

By lemma 2.1, p.468 in [11]

$$|I_2|\le A\,(u-u_h,R_h\psi-I_h^*R_h\psi)\le Ch\left(|u-u_h|_{1,p}+|u|_{2,p}\right)\left\|\psi\right\|_{1,q}.$$

$$|I_3|\le\int_0^t\left\|u-u_h\right\|_{1,p}ds\left\|\psi\right\|_{1,q}$$

$$|I_4|\le(\|u-u_h\|)\left\|\psi\right\|\le Ch^2\left(\|u_0\|_2+\int_0^t\|u_t\|_2\,ds\right)\left\|\psi\right\|_{1,q},$$

where we have used the fact $\|\psi\|\le\|\psi\|_{1,r}\,,r>1$. Combining these estimates we get

$$|((u-u_h)_x\,,\varphi)|\le Ch\left(\|u_0\|_2+\|u\|_{2,p}+\int_0^t\|u_t\|_2\,ds\right)\|\psi\|_{1,q},$$

$$\begin{aligned}\|(u-u_h)_x\|_{0,p} &= \sup\frac{((u-u_h)_x\,,\varphi)}{\|\varphi\|_{0,q}}\\ &\le Ch\,|u-u_h|_{1,p}+Ch\left(\|u_0\|_2+\|u\|_{2,p}+\int_0^t\|u_t\|_2\,ds\right).\end{aligned}$$

Hence using the Poincaré inequality, we have for h sufficiently small

$$\|u-u_h\|_{1,p}\le Ch\left(\|u_0\|_2+\|u\|_{2,p}+\int_0^t\|u_t\|_2\,ds\right).$$

□

We compare the relationship between covolume solution and the Galerkin finite element solution.

Corollary 4.5. *Let $\widetilde{u}_h$ be the finite element solution to* (2.2) *, i.e.,*

$$(\widetilde{u}_{ht},v_h)+A\,(\widetilde{u}_h,v_h)+\int_0^t B\,(t,s;\widetilde{u}_h\,(s)\,,v_h)\,ds=(f,v_h)\,,v_h\in S_h, \qquad \widetilde{u}_h\,(0)=R_hu_0. \tag{4.2}$$

Suppose that $d=2$. Then for sufficiently small $h>0$, one may conclude

$$\begin{aligned}\|(\widetilde{u}_h-u_h)\|_{1,p} &\le C\left(\begin{array}{c}h\left\|u-u_h\right\|_{1,p}+\|(u-u_h)_t\|+\|(\widetilde{u}_h-u)_t\|\\ +\int_0^t\left(\|(u-u_h)\,(s)\|_{1,p}+\|(u-\widetilde{u}_h)\,(s)\|_{1,p}\right)ds\end{array}\right)\\ &\le C\,(u)\,h.\end{aligned}$$

Proof. By (2.2) and (4.2),

$$\left(\left(\widetilde{u}_h - u\right)_t, v_h\right) + A\left(\widetilde{u}_h - u, v_h\right) + \int_0^t B\left(t, s; \left(\widetilde{u}_h - u\right)(s), v_h\right) ds = 0, \quad v_h \in S_h.$$

Consider the following auxiliary problem. For any such φ, let ψ be the solution of

$$\begin{aligned} A^*\psi &= -\varphi_x && \text{in } \Omega \\ \psi &= 0 && \text{on } \partial\Omega, \end{aligned}$$

with

$$\|\psi\|_{1,q} \le \|\varphi\|_{0,q}$$

$$\begin{aligned} \left(\left(\widetilde{u}_h - u_h\right)_x, \varphi\right) &= A\left(\widetilde{u}_h - u_h, \psi\right) \\ &= A\left(\widetilde{u}_h - u_h, \psi - R_h\psi\right) + A\left(u - u_h, R_h\psi\right) \\ &- A\left(u - u_h, I_h^* R_h\psi\right) - \left(\left(u - u_h\right)_t, I_h^* R_h\psi\right) \\ &- \int_0^t B\left(t, s; \left(u - u_h\right)(s), I_h^* R_h\psi\right) ds + A\left(\widetilde{u}_h - u, R_h\psi\right) \\ &= \left[A\left(u - u_h, R_h\psi\right) - A\left(u - u_h, I_h^* R_h\psi\right)\right] \\ &- \left(\left(u - u_h\right)_t, I_h^* R_h\psi\right) - \left(\left(\widetilde{u}_h - u\right)_t, R_h\psi\right) \\ &- \int_0^t B\left(t, s; \left(u - u_h\right)(s), I_h^* R_h\psi\right) ds \\ &- \int_0^t B\left(t, s; \left(\widetilde{u}_h - u\right)(s), R_h\psi\right) ds \\ &= I_1 + I_2 + I_3. \end{aligned}$$

On the other hand,

$$\begin{aligned} |I_1| &\le Ch\left\|u - u_h\right\|_{1,p}\|\psi\|_{1,q} \\ |I_2| &\le C\left(\left\|\left(u - u_h\right)_t\right\| + \left\|\left(\widetilde{u}_h - u\right)_t\right\|\right)\|\psi\| \\ &\le C\left(\left\|\left(u - u_h\right)_t\right\| + \left\|\left(\widetilde{u}_h - u\right)_t\right\|\right)\|\psi\|_{1,q} \end{aligned}$$

where we have used the fact $\|\psi\| \le \|\psi\|_{1,r}$, $r > 1$

$$|I_3| \le \int_0^t \left(\left\|\left(u - u_h\right)(s)\right\|_{1,p} + \left\|\left(u - \widetilde{u}_h\right)(s)\right\|_{1,p}\right) ds\,\|\psi\|_{1,q}$$

$$\begin{aligned} \left\|\left(\widetilde{u}_h - u_h\right)_x\right\|_{0,p} &= \sup_{\varphi \in \mathcal{C}_0^\infty} \frac{\left(\left(\widetilde{u}_h - u_h\right)_x, \varphi\right)}{\|\varphi\|_{0,q}} \\ &\le C\left(\begin{array}{c} h\left\|u - u_h\right\|_{1,p} + \left\|\left(u - u_h\right)_t\right\| + \left\|\left(\widetilde{u}_h - u\right)_t\right\| \\ + \int_0^t \left(\left\|\left(u - u_h\right)(s)\right\|_{1,p} + \left\|\left(u - \widetilde{u}_h\right)(s)\right\|_{1,p}\right) ds \end{array}\right). \end{aligned}$$

We deduce the result from the known finite element estimates. □

Remark 4.6. In order to estimate $\|(u-u_h)_t\|$, by differentiating (4.1) with respect to t we obtain

$$\begin{aligned}(\theta_{tt}, I_h^* v_h) + A(\theta_t, I_h^* v_h) + B(t,t;\theta, I_h^* v_h) &+ \int_0^t B_t(t,s;\theta(s), I_h^* v_h)\,ds \\ &= -(\rho_{tt}, I_h^* v_h).\end{aligned}$$

Setting $v_h = \theta_t$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|||\theta_t|||^2 + c\|\theta_t\|_1^2 \\ &\le \|\rho_{tt}\|\,\|\theta_t\| + \frac{1}{2}c\|\theta_t\|_1^2 + C\|\theta\|_1^2 + \int_0^t \|\theta\|_1^2\,ds \\ &\le \|\rho_{tt}\|\,\|\theta_t\| + \frac{1}{2}c\|\theta_t\|_1^2 + C\int_0^t \|\theta_t\|_1^2\,ds.\end{aligned}$$

Using kickback argument, integrating and applying Grönwall's lemma, we deduce

$$\|\theta_t\| \le C\left(\|\theta_t(0)\| + \int_0^t \|\rho_{tt}\|\,ds\right).$$

5. The lumped mass finite volume element method

In this section, we restrict our study to the case 2-D. A simple way to define the lumped mass scheme is to replace the mass matrix M_h in (2.5) by the diagonal matrix $\overline{M}_h$ obtained by taking for its diagonal elements the numbers $\overline{M}_{hii} = \sum_{j=1}^{N_h} M_{hij}$, or by lumping all masses in one row into the diagonal entry. This makes the inversion of the matrix in front of $\alpha'(t)$ a triviality.

We shall thus study the matrix problem

$$\overline{M}_h \alpha'(t) + A_h \alpha(t) + \int_0^t B_h(t,s)\,\alpha(s)\,ds = F_h(t). \tag{5.1}$$

We know that the lumped mass method defined by (5.1) above is equivalent to

$$(I_h^* u_{ht}, I_h^* v_h) + A(u_h, I_h^* v_h) + \int_0^t B(t,s;u_h(s), I_h^* v_h)\,ds = (f, I_h^* v_h),\, v_h \in S_h. \tag{5.2}$$

Our alternative interpretation of this procedure will be to think of (5.1) as being obtained by evaluating the first term in (5.2) by numerical quadrature. Let K be a triangle of the triangulation T_h, let $x_j, j = 1,2,3$, be its vertices, and consider the quadrature formula

$$Q_{K,h}(f) = \frac{1}{3}\text{area } K \sum_{j=1}^{3} f(x_j) \simeq \int_K f\, dx.$$

We may then define the associated bilinear form in $S_h \times S_h^*$, using the quadrature scheme, by

$$\begin{aligned}(v_h, \eta_h)_h &= \sum_{K \in T_h} Q_{K,h}(v_h \eta_h) \\ &= \sum_{x_i \in N_h^{\grave{a}}} v_h(x_i)\, \eta_h(x_i)\, |V_{x_i}|\,, \forall v_h \in S_h,\ \eta_h \in S_h^*.\end{aligned}$$

We note that $\|v_h\|_h^2 = (v_h, I_h^* v_h)_h$ is a norm in S_h which is equivalent with the L^2-norm uniformly in h, i.e there exist two positive constants C_1 and C_2 such that for all $v_h \in S_h$, we have

$$C_0 \|v_h\| \leq \|v_h\|_h \leq C_1 \|v_h\|\,, \qquad \forall \ \ v_h \in S_h.$$

We note that the above definition $(v_h, \eta_h)_h$ may be used also for $\eta_h \in S_{h,}$, and that $(v_h, w_h)_h = (v_h, I_h^* w_h)_h$ for $v_h, w_h \in S_h$.

The lumped mass method defined by (5.2) above is equivalent to

$$(u_{ht}, I_h^* v_h)_h + A(u_h, I_h^* v_h) + \int_0^t B(t,s; u_h(s), I_h^* v_h)\, ds = (f, I_h^* v_h)\,, \qquad v_h \in S_h. \tag{5.3}$$

We introduce the quadrature error

$$\varepsilon_h(v_h, w_h) = (v_h, w_h)_h - (v_h, w_h)\,,$$

Lemma 5.1 ([22]). *Let $v_h, w_h \in S_h$. Then*

$$|\varepsilon_h(v_h, w_h)| \leq C h^2 \|\nabla v_h\| \|\nabla w_h\|.$$

Theorem 5.2. *Let u_h and u be the solutions of (5.3) and (2.2) respectively, Assume that $u \in L^\infty(H_0^1 \cap H^2)$, $u_t \in L^\infty(H^2), u_{tt} \in L^\infty(H^1)$, $f, f_t \in L^\infty(H^1)$ and $u_h(0) = R_h u_0$. Then we have for the error in the lumped mass semi discrete method, for $t \geq 0$,*

$$\|u_h(t) - u(t)\| \leq C h^2 \left(\int_0^t (\|f\|_1 + \|f_t\|_1 + \|u\|_2 + \|u_t\|_2 + \|u_{tt}\|_1)\, ds \right).$$

Proof. In order to estimate $\|\theta\|$, we write

$$\begin{aligned}&(\theta_t, I_h^* v_h)_h + A(\theta, I_h^* v_h) + \int_0^t B(t,s;\theta(s), I_h^* v_h)\, ds \\ &= (u_{ht}, I_h^* v_h)_h + A(u_h, I_h^* v_h) + \int_0^t B(t,s;u_h(s), I_h^* v_h)\, ds \\ &- ((V_h u)_t, I_h^* v_h)_h - A(V_h u, I_h^* v_h) - \int_0^t B(t,s;V_h u(s), I_h^* v_h)\, ds \\ &= (f, I_h^* v_h) - ((V_h u)_t, I_h^* v_h)_h - A(u, I_h^* v_h) - \int_0^t B(t,s;u(s), I_h^* v_h) \\ &= (u_t, I_h^* v_h) - ((V_h u)_t, I_h^* v_h)_h \\ &= -(\rho_t, I_h^* v_h) - ((V_h u)_t, I_h^* v_h)_h + ((V_h u)_t, I_h^* v_h).\end{aligned}$$

We rewrite

$$\begin{aligned}\left((V_hu)_t\,,I_h^*v_h\right)_h-\left((V_hu)_t\,,I_h^*v_h\right)&=\left((V_hu)_t\,,I_h^*v_h\right)_h-\left((V_hu)_t\,,v_h\right)\\&\quad+\left((V_hu)_t\,,v_h\right)-\left((V_hu)_t\,,I_h^*v_h\right)\\&=\varepsilon_h\left((V_hu)_t\,,v_h\right)+\left((V_hu)_t\,,v_h\right)\\&\quad-\left((V_hu)_t\,,I_h^*v_h\right).\end{aligned}$$

$$\left(\theta_t,I_h^*v_h\right)_h+A\left(\theta,I_h^*v_h\right)+\int_0^t B\left(t,s;\theta\left(s\right),I_h^*v_h\right)ds \tag{5.4}$$

$$=-\left(\rho_t,I_h^*v_h\right)+\varepsilon_h\left((V_hu)_t\,,v_h\right)+\left((V_hu)_t\,,v_h\right)-\left((V_hu)_t\,,I_h^*v_h\right). \tag{5.4}$$

Setting $v_h=\theta$ in (5.4), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\theta\right\|_h^2+c_0\left\|\theta\right\|_1^2\\&\leq\left\|\rho_t\right\|\left\|\theta\right\|+\frac{1}{2}c_0\left\|\theta\right\|_1^2+C\int_0^t\left\|\theta\right\|_1^2ds\\&+\varepsilon_h\left((V_hu)_t\,,\theta\right)+\left((V_hu)_t\,,\theta\right)-\left((V_hu)_t\,,I_h^*\theta\right).\end{aligned}$$

And, using lemma 10 and the inverse estimate, we get

$$\begin{aligned}\left|\varepsilon_h\left((V_hu)_t\,,\theta\right)\right|&\leq Ch^2\left\|\nabla\left(V_hu\right)_t\right\|\left\|\nabla\theta\right\|\\&\leq Ch^2\left(\left\|\rho_t\right\|_1+\left\|u_t\right\|_1\right)\left\|\nabla\theta\right\|\\&\leq Ch\left(\left\|\rho_t\right\|_1+\left\|u_t\right\|_1\right)\left\|\theta\right\|\end{aligned}$$

we have

$$\left|\left((V_hu)_t\,,\theta\right)-\left((V_hu)_t\,,I_h^*\theta\right)\right|\leq Ch\left(\left\|\rho_t\right\|_1+\left\|u_t\right\|_1\right)\left\|\theta\right\|.$$

Using Young's inequality and Grönwall lemma to eliminate $\left\|\theta\right\|_1$ on the right hand side it becomes

$$\frac{1}{2}\frac{d}{dt}\left\|\theta\right\|_h^2+c\left\|\theta\right\|_1\leq\left\|\rho_t\right\|\left\|\theta\right\|+Ch\left(\left\|\rho_t\right\|_1+\left\|u_t\right\|_1\right)\left\|\theta\right\|.$$

Using Young's inequality to eliminate $\left\|\theta\right\|$ on the right hand side it becomes.

Using integration in t, we get the result. □

We will now show that the H^1-norm error bound of theorem remains valid for the Lumped mass method (5.3).

Theorem 5.3. *Let u_h and u be the solutions of (5.3) and (2.2), respectively, under the assumptions of theorem and assume*

$$u_h\left(0\right)=R_hu_0,\qquad\left\|u_{1h}\left(0\right)-u_1\right\|\leq Ch^2\left\|u_1\right\|_2\ .$$

Then we have for the error in the lumped mass semi discrete method, for $t\geq0$,

$$\left\|u_h\left(t\right)-u\left(t\right)\right\|_1\leq Ch\left(\left\|u_0\right\|_2+\left\|u_1\right\|_2+\int_0^t\left\|f\right\|_1+\left\|f_t\right\|_1+\left\|u_{tt}\right\|_2ds\right).$$

Proof. Setting $v_h = \theta_t$ in(5.4) we obtain

$$\begin{aligned}
&\|\theta_t\|_h^2 + \frac{1}{2}\frac{d}{dt}A\left(\theta, I_h^*\theta\right)\\
&= \frac{1}{2}\left[A\left(\theta_t, I_h^*\theta\right) - A\left(\theta, I_h^*\theta_t\right)\right]\\
&B\left(t,t;\theta\left(t\right), I_h^*\theta\left(t\right)\right) + \int_0^t B_t\left(t,s;\theta\left(s\right), I_h^*\theta\left(t\right)\right)ds\\
&-\frac{d}{dt}\int_0^t B\left(t,s;\theta\left(s\right), I_h^*\theta\left(t\right)\right)ds - \left(\rho_t, I_h^*\theta_t\right)\\
&-\varepsilon_h\left(\left(V_hu\right)_t, \theta_t\right) + \left(\left(V_hu\right)_t, \theta\right) - \left(\left(V_hu\right)_t, I_h^*\theta\right).
\end{aligned}$$

It follows thus that and using integration in t and Grönwall lemma, we have

$$\begin{aligned}
\int_0^t \|\theta_t\|_h^2 + \|\theta\|_1^2 \quad &\leq \quad C\left\|\nabla\theta\left(0\right)\right\|^2 + C\int_0^t \left\|\rho_t\right\|^2 + \|\theta_t\| + \|\theta\|\,ds\\
&+Ch\left(\left\|\rho_t\right\|_1 + \left\|u_t\right\|_1\right)\|\theta\|.
\end{aligned}$$

This completes the proof. □

6. Full Discretization

Let $\overline{\partial}U^n = \left(U^n - U^{n-1}\right)/k$ be the backward difference quotient of U^n, assume that $A_h = P_hA$ is a discrete analogue of A (similarly $B_h = P_hB$), where $P_h : L^2\left(\Omega\right) \to S_h^*$ the L^2 projection defined by

$$\left(P_hv, I_h^*v_h\right) = \left(v, I_h^*v_h\right), \qquad v \in L^2\left(\Omega\right),\ v_h \in S_h.$$

In order to define fully-discrete approximation of (2.3), we discretize the time by taking $t_n = nk,\ k > 0,\ n = 1, 2, ...$ and use numerical quadrature

$$\int_0^{t_{n-\frac{1}{2}}} g\left(s\right)ds \sum_{k-1}^{n}\omega_{n,k}g\left(t_{k-1/2}\right), t_{n-\frac{1}{2}} = \left(n - \frac{1}{2}\right)k.$$

Here $\{\omega_{n,k}\}$ are the integration weights and we assume that the following error estimate is valid

$$q^n\left(g\right) = \int_0^{t_{n-\frac{1}{2}}} g\left(s\right)ds - \sum_{k=1}^{n}\omega_{n,k}g\left(t_{k-1/2}\right) \leq Ck^2\int_0^{t_n}\left(|g'| + |g"|\right)ds.$$

Now define our complete discrete finite volume element approximation of (2.3) by: Find $U^n \in S_h$ for $n = 1, 2, ..$, such that for all $v_h \in S_h$

$$\begin{aligned}
\left(\overline{\partial}U^n, I_h^*v_h\right) + A\left(U^{n-\frac{1}{2}}, I_h^*v_h\right) + \sum_{k=1}^{n}\omega_{n,k}B\left(t_{n-\frac{1}{2}}, t_{k-1/2}, U^{k-1/2}, I_h^*v_h\right)\\
= \left(f^{n-\frac{1}{2}}, I_h^*v_h\right), \quad U^0 \in S_h
\end{aligned} \tag{6.1}$$

where $U^{n-\frac{1}{2}} = \left(U^n + U^{n-1}\right)/2$.

Theorem 6.1. *Let $u(t)$ and U^n be the solutions of problem (2.2) and its complete discrete scheme (6.1) respectively. Then for any $T > 0$ there exists a positive constant $C = C(T) > 0$, independent of h, such that for $0 < t_n \leq T$*

$$\|u(t_n) - U^n\| \leq Ch^2 \left(\|u_0\|_2 + \int_0^{t_n} \|u_t\|_2 \, ds\right) + Ck^2 \left(\int_0^{t_n} \left(\|f_t\|_1 + \|u\|_2 + \|u_t\|_2 + \|u_{tt}\|_2 + \|u_{ttt}\|\right) ds\right).$$

Proof. Let us split the error into two parts: $u(t_n) - U^n = \rho^n + \theta^n$, where $\rho^n = u(t_n) - V_h u(t_n)$ and $\theta^n = V_h u(t_n) - U^n$ and let $W = V_h u(t) \in S_h$ be the Ritz-Volterra projection of u. Then from (2.2) and (6.1) we have for all $v_h \in S_h$

$$\left(\overline{\partial}\theta^n, I_h^* v_h\right) + A\left(\theta^{n-1/2}, I_h^* v_h\right) + \sum_{k=1}^{n} \omega_{n,k} B\left(t_{n-\frac{1}{2}}, t_{k-1/2}, \theta^{k-1/2}, I_h^* v_h\right) = -\left(r_n, I_h^* v_h\right), v_h \in S_h \tag{6.2}$$

where

$$r_n = r_n^1 + r_n^2 + r_n^3 + r_n^4,$$

and

$$\begin{aligned}
r_n^1 &= \overline{\partial}\rho^n, \\
r_n^2 &= \overline{\partial}u(t_n) - u_t\left(t_{n-\frac{1}{2}}\right), \\
r_n^3 &= A\left(\left(u(t_n) + u(t_{n-1})\right)/2 - u\left(t_{n-\frac{1}{2}}\right), I_h^* v_h\right), \\
r_n^4 &= q^n\left(B_h W\right) \\
&= \sum_{k=1}^{n} \omega_{n,k} B_h\left(t_{n-\frac{1}{2}}, t_{k-1/2}, W^{k-1/2}, I_h^* v_h\right) \\
&\quad - \int_0^{t_{n-\frac{1}{2}}} B\left(t_n, s, W(s), I_h^* v_h\right) ds.
\end{aligned}$$

In fact, by Taylor expansion,

$$\begin{aligned}
u^{n+1} &= u^n + ku'(t_n) + \int_{t_n}^{t_{n+1}} u"(s)\left(t_{n+1} - s\right) \, ds \\
&= u^n + ku'(t_n) + \frac{k^2}{2} u"(t_n) + \frac{k^3}{6} u^{(3)}(t_n) \\
&+ \frac{1}{6} \int_{t_n}^{t_{n+1}} u^{(4)}(s)\left(t_{n+1} - s\right)^3 \, ds
\end{aligned}$$

we have

$$\left\|r_n^1\right\| = \left\|\overline{\partial}\rho^n\right\| \leq \frac{1}{k} \int_{t_{n-1}}^{t_n} \|\rho_t\| \, ds \leq C\frac{h^2}{k} \int_{t_{n-1}}^{t_n} \left(\|f_t\|_1 + \|u_{tt}\|_2\right) ds$$

$$\begin{aligned}\left\|r_n^2\right\| &= \left\|\overline{\partial} u\left(t_n\right)-u_t\left(t_{n-\frac{1}{2}}\right)\right\| \\ &= \frac{1}{k}\left\|\int_{t_{n-1}}^{t_n}\left(u_t(s)-\ u_t\left(t_{n-\frac{1}{2}}\right)\right)ds\right\| \leq Ck\int_{t_{n-1}}^{t_n}\left\|u^{(3)}\left(s\right)\right\|\ ds\end{aligned}$$

and

$$\begin{aligned}\left\|r_n^3\right\| &= \left\|A\left(\frac{u\left(t_n\right)+u\left(t_{n-1}\right)}{2}-u\left(t_{n-\frac{1}{2}}\right), I_h^* v_h\right)\right\| \\ &\leq Ck\int_{t_{n-1}}^{t_n}\left\|Au_{tt}\left(s\right)\right\| ds \leq Ck\int_{t_{n-1}}^{t_n}\left\|u_{tt}\right\|_2 ds.\end{aligned}$$

In addition, the quadrature error satisfies

$$\begin{aligned}\left\|r_n^4\right\| &= q^{n-\frac{1}{2}}\left(B_h W\right) \\ &= \sum_{k=1}^{n}\omega_{n,k} B\left(t_{n-\frac{1}{2}}, t_{k-1/2}, W^{k-1/2}, I_h^* v_h\right) \\ &- \int_0^{t_{n-\frac{1}{2}}} B\left(t_n, s, W\left(s\right), I_h^* v_h\right) ds \\ &\leq Ck^2\int_0^{t_n}\left\|\left(B_h W\right)_{ss}\right\|\ ds \\ &\leq Ck^2\int_0^{t_n}\left(\left\|u\right\|_2+\left\|u_t\right\|_2+\left\|u_{tt}\right\|_2\right)\ ds.\end{aligned}$$

$$\begin{aligned}k\sum_{n=1}^{N}\left\|r_n\right\| &\leq Ch^2\int_0^{t_n}\left\|u_{tt}\right\|_2 ds \\ &+ Ck^2\int_0^{t_n}\left(\left\|u\right\|_2+\left\|u_t\right\|_2+\left\|u_{tt}\right\|_2+\left\|u^{(3)}\right\|\right)\ ds.\end{aligned}$$

Taking $v_h=\theta^{n-1/2}$ in (6.2) and noting that $\left(\overline{\partial}\theta^n, I_h^*\theta^{n-1/2}\right)=\frac{1}{2}\overline{\partial}\left|\left|\left|\theta^n\right|\right|\right|^2$ there is

$$\begin{aligned}&\left|\left|\left|\theta^n\right|\right|\right|^2-\left|\left|\left|\theta^{n-1}\right|\right|\right|^2+2kc\left|\left|\left|\theta^{n-1/2}\right|\right|\right|_1^2 \\ &\leq Ck^2\sum_{k=1}^{n}\left\|\theta^{k-1/2}\right\|_1\left\|\theta^{n-1/2}\right\|_1+Ck\left\|r_n\right\|\left\|\theta^{n-1/2}\right\| \\ &\leq kc\left|\left|\left|\theta^{n-1/2}\right|\right|\right|_1^2+Ck^2\sum_{k=1}^{n}\left\|\theta^{k-1/2}\right\|_1^2+Ck\left\|r_n\right\|\left\|\theta^{n-1/2}\right\|.\end{aligned}$$

Summing from $n=1$ to N, and then, after cancelling the common factor and using Grönwall lemma, we obtain

$$\left|\left|\left|\theta^N\right|\right|\right|^2 \leq C\left|\left|\left|\theta^0\right|\right|\right|^2+Ck\sum_{k=1}^{N}\left\|r_n\right\|\left(\left\|\theta^k\right\|+\left\|\theta^{k-1/2}\right\|\right)$$

and then

$$|||\theta^N||| \leq C\,|||\theta^0||| + Ck\sum_{n=1}^{N} \|r_n\|\,.$$

The theorem follows from the estimates of ρ^n and r^n. □

7. Conclusion

In this work, our main aim is to construct and analyze the FVE scheme for solving for convection-diffusion-reaction equations with memory by using the transfer operator. We give the detailed construction for the semi-discrete, modified lumped mass, and fully discrete schemes. For the spatially discrete and modified lumped mass schemes, we obtain optimal order error estimates in L^2, H^1, and $W^{1,p}$ norms for $2 \leq p < \infty$. Based on the Crank-Nicolson method, a fully discrete scheme is discussed, and related error estimates are derived.

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