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Dynamical behaviours of Bazykin-Berezovskaya model with fractional-order and its discretization

Mohammad Hossein Akrami
Department of Mathematics, Yazd University, 89195-741 Yazd, Iran.
E-mail: akrami@yazd.ac.ir

Abstract This paper is devoted to study dynamical behaviours of the fractional-order Bazykin-Berezovskaya model and its discretization. The fractional derivative has been described in the Caputo sense. We show that the discretized system, exhibits more complicated dynamical behaviours than its corresponding fractional-order model. Specially, in the discretized model Neimark-Sacker and flip bifurcations and also chaos phenomena will happen. In the final part, some numerical simulations verify the analytical results.

Keywords. Fractional-order dynamical system, Neimark-Sacker bifurcation, Flip bifurcation, Chaos.
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1. Introduction

The predator-prey model is one of the most widely used mathematical models in the biological sciences that deals with the relationships between the two species. This model provides a useful tool to help ecologists understand the population dynamics. The predator-prey model is the result of the Lotka and Volterra works [19, 30]. This model was later extended by many researchers, for example see [11, 21, 24, 26, 28]. However, today a lot of extended predator-prey models are available in the literature, but some of these models less considered by authors. One of them is Bazykin-Berezovskaya or BB-model proposed in 1998 by Bazykin and Berezovskaya [6].

An important phenomenon that indicates the threshold of population decline is the Allee effect. An Allee effect is a phenomenon in population dynamics attributed to Warden Clyde Allee [2]. He found that the birth rate in low-density populations was reduced. Moreover, strong Allee effect means that the per capita growth rate is negative when density is zero. As a result, when the population density falls below the critical density, the population is declining to extinction and above which it may increase [18].

Voorn et al. [29] considered the following BB-model by strong Allee effects

$$\begin{cases} \dot{x} = x(x-l)(k-x) - xy, \\ \dot{y} = c(x-m)y, \end{cases} \tag{1.1}$$

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where x, y are the sizes of the prey and predator population respectively, k is the carrying capacity, l is the strong Allee effect threshold, m is the predator mortality rate and c is the feeding efficiency of Lotka-Volterra model [6].

On the other hand, the use of fractional differential equations in mathematical modelling and control of biological phenomena has attracted much attention in recent decades, for example see [1, 4, 14, 27]. Fractional differential equations help us to reduce the errors coming from the neglected parameters in modeling real life phenomena [4]. Moreover, the fractional differential equations are intrinsically return to systems with memory and the hereditary properties, which existed in most biological systems [4, 25].

In this paper, we investigate the dynamical behaviors of fractional-order BB-model

$$\begin{cases} {}^{C}D_t^{\alpha} x(t) = x(t)(x(t) - l)(1 - x(t)) - x(t)y(t), \\ {}^{C}D_t^{\alpha} y(t) = (x(t) - m)y(t), \end{cases} \tag{1.2}$$

where ${}^{C}D_t^{\alpha}$ denotes the Caputo fractional derivative of order $0 < \alpha \leq 1$. For the biologically meaningful conditions, we have $x \geq 0$ and $y \geq 0$. We also, suppose $k = c = 1$.

Many authors study the dynamics of continuous models using discrete techniques. They first convert the continuous model into a discrete model and then analyze its dynamics, see [5, 10, 15]. On this basis, we first decide to discretize BB-model with fractional order and then investigate its dynamical behaviours.

Several definitions of fractional derivative and integral are existed in the literature, such as Caputo, Grünwald- Letnikov and Riemann-Liouville [25]. Here, we need to recall some definition.

Definition 1.1. A real function $f(x), x > 0$, is said to be in the space C_μ with $\mu \in \mathbb{R}$, if there exists a real number $p(> \mu)$ such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty)$.

Definition 1.2. [25] Let $f \in C_\mu$ and $\mu \geq 1$. Then Caputo's definition of the fractional-order derivative is defined as

$${}^{C}D_t^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}} \mathrm{d}t. \tag{1.3}$$

Hence, $\alpha > 0$ is the order of derivative and $n = \lceil \alpha \rceil$, when ceiling function $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

2. Dynamical behaviours of fractional BB-model

In this section, we investigate the dynamics of the fractional-order BB-model (1.2)

First, we present some concepts in fractional-order differential equation theory. Consider autonomous fractional-order system

$${}^{C}D_t^{\alpha} x(t) = f(x), \qquad 0 \leq \alpha < 1, \tag{2.1}$$

where, $x \in \mathbb{R}^n$. Suppose x^* is an equilibrium point of system (2.1) and $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ are the eigenvalues of Jacobian matrix Df, through the equilibrium point x^*. The local stability of equilibrium point x^* can be obtained by Matignon's results [20]:

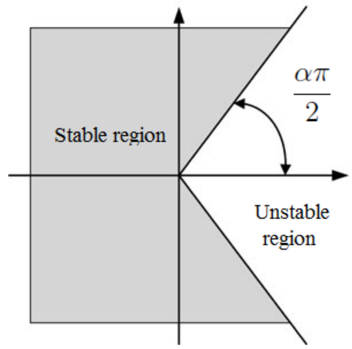


FIGURE 1. Stability regions of the fractional-order system [23].

Lemma 2.1. *The equilibrium point x^* for system (2.1) is locally asymptotically stable iff*

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}, \quad i = 1, 2, \cdots, n.$$

According to the above Lemma, the stability region of system (2.1) has been depicted in figure 1.

Next, we investigate the local stability of system (1.2). Fixed points of model (1.2) are as follow:

$$E_0 = (0, 0), \quad E_1 = (1, 0), \quad E_2 = (l, 0), \quad E_3 = (m, (m-l)(1-m)).$$

The fixed point E_3 is positive iff $l < m < 1$. For $m = l$ or $m = 1$ the model has 3 boundary fixed points. The Jacobian matrix of system (1.2) is given by

$$J = \begin{pmatrix} -3x^2 + 2(1+l)x - l - y & -x \\ y & x - m \end{pmatrix}. \tag{2.2}$$

The eigenvalues corresponding to the equilibrium point E_0 are $\lambda_1 = 0$ and $\lambda_2 = -m$. Therefore, the equilibrium point E_0 is stable point. The eigenvalues corresponding to the equilibrium point E_1 are $\lambda_1 = l - 1 < 0$ and $\lambda_2 = 1 - m > 0$. Thus, the equilibrium point E_1 is saddle point. The equilibrium point E_2 is also saddle point, because the eigenvalues corresponding to this equilibrium point are $\lambda_1 = l - l^2 < 0$ and $\lambda_2 = l - m > 0$. Moreover, the characteristic equation corresponding to the equilibrium point E_3 is given by

$$\lambda^2 - m(l + 1 - 2m)\lambda + m(m-l)(1-m) = 0. \tag{2.3}$$

Hence, the eigenvalues of characteristic equation (2.3) are

$$\lambda_{1,2} = \frac{m(1 + l - 2m) \pm \sqrt{\Delta}}{2},$$

where,

$$\Delta := m^2(1 + l - 2m)^2 - 4m(m-l)(1-m). \tag{2.4}$$

Thus, if $\Delta \geq 0$, then two real eigenvalues $\lambda_{1,2}$ are negative for $m > \frac{l+1}{2}$ and positive for $m < \frac{l+1}{2}$, i.e. the equilibrium point E_3 is locally asymptotically for $m > \frac{l+1}{2}$ and unstable for $m < \frac{l+1}{2}$. For $\Delta < 0$, let $\theta := |\arg(\lambda_{1,2})| = \arctan(\frac{\sqrt{-\Delta}}{m(1+l-2m)})$. In this case, according to Lemma 2.1, if $\theta > \frac{\alpha\pi}{2}$ the equilibrium point E_3 is locally asymptotically stable and if $\theta < \frac{\alpha\pi}{2}$ it is unstable for all $0 < \alpha < 1$.

3. Discretized fractional-order BB-model

In this section, we construct a discrete system corresponding to the fractional-order model. Suppose $x(0) = x_0$ and $y(0) = y_0$ are the initial conditions of System (1.2). By discretization process with piecewise constant arguments, presented in [10] the discrete model can be obtained as

$$\begin{cases} x_{n+1} = x_n + \frac{s^\alpha}{\Gamma(\alpha+1)}\big(x_n(x_n - l)(1 - x_n) - x_n y_n\big), \\ y_{n+1} = y_n + \frac{s^\alpha}{\Gamma(\alpha+1)}(x_n - m)y_n. \end{cases} \tag{3.1}$$

It is noticed that if $\alpha \to 1$ in (3.1), we obtain the well-known Euler discretization of the model.

We next investigate the local stability of fixed points of system (3.1). Note that the fixed points of System (3.1) are equivalent to the equilibrium points of (1.2). The Jacobian matrix associated with (3.1) is given by

$$J(x,y) = \begin{pmatrix} 1 + \frac{s^\alpha}{\Gamma(\alpha+1)}\big(-3x^2 + 2(1+l)x - l - y\big) & \frac{-s^\alpha}{\Gamma(\alpha+1)}x \\ \frac{s^\alpha}{\Gamma(\alpha+1)}y & 1 + \frac{s^\alpha}{\Gamma(\alpha+1)}(x - m) \end{pmatrix}. \tag{3.2}$$

Proposition 3.1. *The fixed point E_0 is a locally asymptotically stable fixed point if $0 < s < \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{m}}$, it is an unstable point if $s > \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{l}}$, it is a saddle if $\sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{m}} < s < \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{l}}$ and it is non-hyperbolic if $s = \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{l}}$ or $s = \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{m}}$.*

Proof. According to (3.2) at $E_0 = (0,0)$, we get

$$J(0,0) = \begin{pmatrix} 1 - \frac{ls^\alpha}{\Gamma(\alpha+1)} & 0 \\ 0 & 1 - \frac{ms^\alpha}{\Gamma(\alpha+1)} \end{pmatrix}. \tag{3.3}$$

Therefore, the eigenvalues are $\lambda_1 = 1 - \frac{ls^\alpha}{\Gamma(\alpha+1)}$ and $\lambda_2 = 1 - \frac{ms^\alpha}{\Gamma(\alpha+1)}$. Since $0 < l < m$, then $\lambda_2 < \lambda_1 < 1$. Hence, $|\lambda_{1,2}| < 1$ if $\lambda_2 > -1$ or $0 < s < \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{m}}$. Also, $|\lambda_{1,2}| > 1$ if $\lambda_1 < -1$ or $s > \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{l}}$. Consequently, the origin is saddle if $\lambda_2 < -1 < \lambda_1$ or equivalently $\sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{m}} < s < \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{l}}$. Finally, if $\lambda_1 = \lambda_2 = -1$, then E_0 is non-hyperbolic and this completes the proof. $\square$

Proposition 3.2. *The fixed point E_1 is an unstable fixed point if $s > \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{1-l}}$, it is a saddle if $s < \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{1-l}}$ and it is non-hyperbolic if $s = \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{1-l}}$.*

Proof. The Jacobian matrix (3.2) at E_1 is

$$J(1,0) = \begin{pmatrix} 1 + \frac{s^\alpha}{\Gamma(\alpha+1)}(l-1) & -\frac{s^\alpha}{\Gamma(\alpha+1)} \\ 0 & 1 + \frac{s^\alpha}{\Gamma(\alpha+1)}(1-m) \end{pmatrix}. \tag{3.4}$$

Therefore, the eigenvalues are $\lambda_1 = 1 - \frac{ls^\alpha}{\Gamma(\alpha+1)}(1-l)$ and $\lambda_2 = 1 + \frac{s^\alpha}{\Gamma(\alpha+1)}(1-m) > 1$. Now, if $s > \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{1-l}}$ then $\lambda_1 < -1$ and if $s < \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{1-l}}$, then $-1 < \lambda_1 < 1$. Also, if $s = \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{1-l}}$ then $\lambda_1 = -1$ and the proof is completed. □

Proposition 3.3. *The fixed point E_2 is an unstable fixed point if $s > \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{m-l}}$, it is a saddle if $s < \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{m-l}}$ and it is non-hyperbolic if $s = \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{m-l}}$.*

Proof. The Jacobian matrix (3.2) at E_2 is

$$J(l,0) = \begin{pmatrix} 1 + \frac{ls^\alpha}{\Gamma(\alpha+1)}(1-l) & -\frac{ls^\alpha}{\Gamma(\alpha+1)} \\ 0 & 1 + \frac{s^\alpha}{\Gamma(\alpha+1)}(l-m) \end{pmatrix}. \tag{3.5}$$

Therefore, the eigenvalues are $\lambda_1 = 1 + \frac{ls^\alpha}{\Gamma(\alpha+1)}(1-l) > 1$ and $\lambda_2 = 1 + \frac{s^\alpha}{\Gamma(\alpha+1)}(l-m)$.

Since $0 < l < m$, then $\lambda_2 < 1$. Therefore, if $s < \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{m-l}}$, then $-1 < \lambda_2 < 1$, if $s = \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{m-l}}$, then $\lambda_2 = -1$ and for $s > \sqrt[\alpha]{\frac{2\Gamma(\alpha+1)}{m-l}}$, we have $\lambda_2 < -1$.

□

As mentioned above for $0 < l < m < 1$, the model has a unique positive fixed point E_3.

Positive fixed point. In order to study the stability analysis and bifurcation of fixed point E_3 we recall the following lemma from [17].

Lemma 3.4. *Let $F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are two real constants. Suppose $F(1) > 0$; λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then the following statements hold.*

- *(i)* $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $C < 1$;
- *(ii)* $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*) if and only if* $F(-1) < 0$;
- *(iii)* $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $C > 1$;
- *(iv)* $\lambda_1 = -1$ *and* $\lambda_2 \neq -1$ *if and only if* $F(-1) = 0$ *and* $B \neq 0, 2$;
- *(v)* λ_1 *and* λ_2 *are a pair of conjugate complex roots and,* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $-2 < B < 2$ *and* $C = 1$;
- *(vi)* $\lambda_1 = \lambda_2 = -1$ *if and only if* $F(-1) = 0$ *and* $B = 2$.

Note that if λ_1 and λ_2 are the eigenvalues of Jacobian matrix at the fixed point (x^*, y^*), then, the fixed point (x^*, y^*) is a sink or locally asymptotically stable if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. It is a source or locally unstable if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. It is a saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$ and it is non-hyperbolic fixed point if $|\lambda_1| = 1$ or $|\lambda_2| = 1$

The Jacobian matrix (3.2) at $E_3 = (m, (m-l)(1-m))$ is

$$J_3 := J(m, (m-l)(1-m)) = \begin{pmatrix} 1 + \frac{ms^\alpha}{\Gamma(\alpha+1)}(1+l-2m) & -\frac{ms^\alpha}{\Gamma(\alpha+1)} \\ \frac{s^\alpha}{\Gamma(\alpha+1)}(m-l)(1-m) & 1 \end{pmatrix}. \tag{3.6}$$

For simplicity, let $\gamma = \frac{s^\alpha}{\Gamma(\alpha+1)}$. Therefore,

$$tr(J_3) = 2 + m\gamma(1+l-2m),$$
$$det(J_3) = 1 + m\gamma(1+l-2m) + \gamma^2 m(m-l)(1-m).$$

The characteristic polynomial of the Jacobian matrix (3.6) is

$$F(\lambda) = \lambda^2 - tr(J_3)\lambda + det(J_3). \tag{3.7}$$

Since, $0 < l < m < 1$, so one can see that

$$F(1) = \gamma^2 m(m-l)(1-m) > 0. \tag{3.8}$$

According to Lemma 3.4, we can obtain the following proposition.

Proposition 3.5. *Suppose that the fixed point $E_3 = (m, (m-l)(1-m))$ exists and $2m > l+1$. Then E_3;*

- (i) *is a sink if and only if $s < \sqrt[\alpha]{\frac{\Gamma(\alpha+1)(2m-l-1)}{(m-l)(1-m)}}$; and either $(1+l-2m)^2 < \frac{4(m-l)(1-m)}{m}$ or $(1+l-2m)^2 > \frac{4(m-l)(1-m)}{m}$ and $\frac{m(2m-l-1)-\sqrt{\Delta}}{m(m-l)(1-m)} < \gamma < \frac{m(2m-l-1)+\sqrt{\Delta}}{m(m-l)(1-m)}$.*
- (ii) *is a source if and only if $s > \sqrt[\alpha]{\frac{\Gamma(\alpha+1)(2m-l-1)}{(m-l)(1-m)}}$; and either $(1+l-2m)^2 < \frac{4(m-l)(1-m)}{m}$ or $(1+l-2m)^2 > \frac{4(m-l)(1-m)}{m}$ and $\frac{m(2m-l-1)-\sqrt{\Delta}}{m(m-l)(1-m)} < \gamma < \frac{m(2m-l-1)+\sqrt{\Delta}}{m(m-l)(1-m)}$.*
- (iii) *is a saddle if and only if $\gamma > \frac{m(2m-l-1)+\sqrt{\Delta}}{m(m-l)(1-m)}$ or $\gamma < \frac{m(2m-l-1)-\sqrt{\Delta}}{m(m-l)(1-m)}$.*
- (iv) *is non-hyperbolic if one of the following conditions holds*
  - $(iv.1)$ $(1+l-2m)^2 > \frac{4(m-l)(1-m)}{m}$ *and* $\gamma = \frac{m(2m-l-1)\pm\sqrt{\Delta}}{m(m-l)(1-m)}$.
  - $(iv.2)$ $(1+l-2m)^2 < \frac{4(m-l)(1-m)}{m}$ *and* $\gamma = \frac{(2m-l-1)}{(m-l)(1-m)}$.

Proof. By simple calculation we have

$$F(-1) = m(m-l)(1-m)\gamma^2 + 2m(1+l-2m)\gamma + 4. \tag{3.9}$$

Quadratic polynomial (3.9) with respect to γ is positive if $\Delta < 0$, or $\Delta > 0$ and

$$\frac{m(2m-l-1)-\sqrt{\Delta}}{m(m-l)(1-m)} < \gamma < \frac{m(2m-l-1)+\sqrt{\Delta}}{m(m-l)(1-m)},$$

conditionally $2m > 1+l$. We also have $det(J_3) < 1$, if $\gamma < \frac{2m-l-1}{(m-l)(1-m)}$ or equivalently

$$s < \sqrt[\alpha]{\frac{\Gamma(\alpha+1)(2m-l-1)}{(m-l)(1-m)}}.$$

Hence, by a straightforward calculation, all of statements in this proposition will be proven. □

From Lemma 3.4 and Proposition 3.5, we deduce if condition $(iv.1)$ holds, one of the eigenvalues of fixed point E_3 is -1 and the other is neither 1 nor -1. We can rewrite the condition $(iv.1)$ of Proposition 3.5 as the following sets

$$FB_1 = \Big\{(\alpha, m, l, s) : (1+l-2m)^2 > \frac{4(m-l)(1-m)}{m}, \\ \gamma = \frac{m(2m-l-1)+\sqrt{\Delta}}{m(m-l)(1-m)}\Big\},$$

or

$$FB_2 = \Big\{(\alpha, m, l, s) : (1+l-2m)^2 > \frac{4(m-l)(1-m)}{m}, \\ \gamma = \frac{m(2m-l-1)-\sqrt{\Delta}}{m(m-l)(1-m)}\Big\}.$$

Also if condition $(iv.2)$ holds, the eigenvalues of E_3 are a pair of conjugate complex numbers with module one. We can rewrite the condition $(iv.2)$ of Proposition 3.5 as the following set

$$NS = \Big\{(\alpha, m, l, s) : (1+l-2m)^2 < \frac{4(m-l)(1-m)}{m}, \\ \gamma = \frac{(2m-l-1)}{(m-l)(1-m)}\Big\}.$$

In the next section, we will study the period-doubling or flip bifurcation of the positive fixed point E_3 when parameters of the system vary in the small neighborhood of FB_1or FB_2 and the Neimark-Sacker bifurcation if parameters vary in the small neighborhood of NS.

4. Bifurcations

According to the previous section, we claim that two different bifurcations occur in the discretized fractional-order BB-model.

4.1. **Neimark-Sacker bifurcation.**

Let $\gamma^* = \frac{(2m-l-1)}{(m-l)(1-m)}$. Using $x = \hat{x} + m$, $y = \hat{y} + (m-l)(1-m)$ and $\gamma = \delta + \gamma^*$, we transform the fixed point $(m, (m-l)(1-m))$ to the origin and consider the parameter δ as the new bifurcation parameter. Then, system (3.1) is equivalent to the following system

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \mapsto \begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} a_{11}\hat{x}\hat{y} + a_{20}\hat{x}^2 + a_{30}\hat{x}^3 \\ b_{11}\hat{x}\hat{y} \end{pmatrix}, \tag{4.1}$$

where,

$$a_{10} = (\delta+\gamma^*)m(l-2m+1)+1, \quad a_{01} = -(\delta+\gamma^*)m,$$
$$b_{10} = (\delta+\gamma^*)(m-l)(1-m), \quad b_{01} = 1, \quad a_{11} = -(\delta+\gamma^*),$$
$$a_{20} = (\delta+\gamma^*)(l-3m+1), \quad b_{11} = -a_{30} = (\delta+\gamma^*).$$

Therefore, the eigenvalues of the linear part of system (4.1) for $\delta = 0$ at fixed point $(0, 0)$ are

$$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2} := \eta \pm i\sigma,$$

where, $T = a_{10} + b_{01}$ and $D = a_{10}b_{01} - a_{01}b_{10}$. Thus, since parameters belong to NS, we have

$$|\lambda_{1,2}| = \sqrt{1 + m(\gamma^* + \delta)(1 + l - 2m) + (\gamma^* + \delta)^2 m(m - l)(1 - m)},$$

and we have

$$\frac{d|\lambda|}{d\delta}|_{\delta=0} = \frac{m(1 + l - 2m) + 2m(m - l)(1 - m)}{2\sqrt{1 + m\gamma^*)(1 + l - 2m) + (\gamma^*)^2 m(m - l)(1 - m)}} \neq 0.$$

Moreover, if we define $w = \cos^{-1}(1 - \frac{m(2m-l-1)^2}{2(m-l)(1-m)})$, then we have $\lambda_{1,2} = e^{\pm iw}$. By definition of set NS, it is easy to see that $-1 < 1 - \frac{m(2m-l-1)^2}{2(m-l)(1-m)} < 1$. Hence, $-1 < \cos w < 1$, or $w \neq 0, \pi$. This means $\lambda^k \neq 1$ for $k = 1, 2$.
On the other hand if $\frac{m(2m-l-1)^2}{2(m-l)(1-m)} \neq 1, \frac{3}{2}$, then $\lambda^k \neq 1$ for $k = 3, 4$. We can summarize these results in the following Lemma.

Lemma 4.1. *If $(\alpha, m, l, s) \in NS$ and $w = \cos^{-1}(1 - \frac{m(2m-l-1)^2}{2(m-l)(1-m)})$, then the eigenvalues of the Jacobian matrix at fixed point E_3 are $\lambda_{1,2} = e^{\pm iw}$. Moreover,*

(i) $\frac{d|\lambda|}{d\delta}|_{\delta=0} \neq 0$,
(ii) $\lambda^k \neq 1$ *for* $k = 1, 2$,
(iii) $\lambda^k \neq 1$ *for* $k = 3, 4$, *if* $\frac{m(2m-l-1)^2}{2(m-l)(1-m)} \neq 1, \frac{3}{2}$.

Here, using linear transformation

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} a_{01} & 0 \\ \eta - a_{10} & -\sigma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

brings the linear part of system (4.1) into normal form

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \eta & -\sigma \\ \sigma & \eta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \bar{f}(u, v) \\ \bar{g}(u, v) \end{pmatrix} \tag{4.2}$$

where,

$$\begin{aligned} \bar{f}(u, v) =& (a_{11}(\eta - a_{10}) + a_{20}a_{01})u^2 - a_{11}\sigma uv + a_{30}a_{01}^2 u^3, \\ \bar{g}(u, v) =& \frac{(\eta - a_{10})\Big(a_{11}(\eta - a_{10}) + a_{01}(a_{20} - b_{11})\Big)}{\sigma} u^2 \\ & + (b_{11}a_{01} - a_{11}(\eta - a_{10})uv + \frac{a_{01}^2 a_{03}(\eta - a_{10})}{\sigma} u^3. \end{aligned}$$

In order to guarantee the Neimark-Sacker bifurcation for (4.2), we require that the following discriminatory quantity is not zero (see [12, 16]):

$$a = Re\left[\frac{(1 - 2\lambda)\bar{\lambda}^2}{1 - \lambda}\xi_{11}\xi_{20}\right] - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + Re(\bar{\lambda}\xi_{21}), \tag{4.3}$$

where

$$\xi_{20} = \frac{1}{8}\left[(\bar{f}_{uu} - \bar{f}_{vv} + 2\bar{g}_{uv}) + \mathrm{i}(\bar{g}_{uu} - \bar{g}_{vv} - 2\bar{f}_{uv})\right]|_{(0,0)},$$
$$\xi_{11} = \frac{1}{4}\left[(\bar{f}_{uu} + \bar{f}_{vv}) + \mathrm{i}(\bar{g}_{uu} + \bar{g}_{vv})\right]|_{(0,0)},$$
$$\xi_{02} = \frac{1}{8}\left[(\bar{f}_{uu} - \bar{f}_{vv} - 2\bar{g}_{uv}) + \mathrm{i}(\bar{g}_{uu} - \bar{g}_{vv} + 2\bar{f}_{uv})\right]|_{(0,0)},$$
$$\xi_{21} = \frac{1}{16}\left[(\bar{f}_{uuu} + \bar{f}_{uvv} - \bar{f}_{uuv} + \bar{g}_{vvv}) + \mathrm{i}(\bar{g}_{uuu} + \bar{g}_{uvv} - \bar{f}_{vvv})\right]|_{(0,0)}.$$

After calculating, we get

$$\begin{aligned}\xi_{20} =& \frac{1}{4}(a_{20}a_{01} + b_{11}a_{01}) + \frac{i}{4\sigma}\Big((\eta - a_{10})\big(a_{11}(\eta - a_{10}) + a_{01}(a_{20} - b_{11})\big)\\ & - \sigma b_{11}a_{01} + \sigma a_{11}(\eta - a_{10}) + \sigma^2 a_{11}\Big),\\ \xi_{11} =& \frac{1}{2}\Big(a_{11}(\eta - a_{10}) + a_{01}a_{20}\Big)\\ & + \frac{i}{2\sigma}\Big((\eta - a_{10})\Big(a_{11}(\eta - a_{10}) + a_{01}(a_{20} - b_{11})\Big),\\ \xi_{02} =& \frac{1}{4}\Big(2a_{11}(\eta - a_{10}) + (a_{20}a_{01} - a_{01}b_{11})\Big)\\ & + \frac{i}{4\sigma}\Big((\eta - a_{10})\Big(a_{11}(\eta - a_{10}) + a_{01}(a_{20} - b_{11}) - a_{11}\sigma^2\Big),\\ \xi_{21} =& \frac{3}{8}(a_{30}a_{01}^2) + \frac{3i}{8\sigma}(a_{30}a_{01}^2(\eta - a_{10}).\end{aligned}$$

Analyzing above expressions and the Neimark-Sacker bifurcation conditions discussed in [12, 16], we can write the main result as follows:

Proposition 4.2. *If the conditions of Lemma 4.1 hold and* $a \neq 0$, *then the map (3.1) undergoes Neimark-Sacker bifurcation at the positive fixed point* E_3 *when the parameter* γ *varies in a small neighborhood of* γ^*. *Moreover, if* $a < 0$ *(respectively* $a > 0$*), then an attracting (respectively repelling) invariant closed curve bifurcates from the fixed point* (x^*, y^*) *for* $\gamma > \gamma^*$ *(respectively* $\gamma < \gamma^*$*).*

4.2. **Flip Bifurcation.** Here, we choose γ as the (flip) bifurcation parameter and study the flip bifurcation of fixed point E_3. Take parameters (α, m, l, s) arbitrarily from FB_1. The case of FB_2 can be handled similarly. Let $\gamma_0 = \frac{m(2m-l-1)+\sqrt{\Delta}}{m(m-l)(1-m)}$. Using $x = \bar{x} + m$, $y = \bar{y} + (m - l)(1 - m)$ and $\gamma = \gamma_0 + \gamma_*$, we transform the fixed point $(m, (m - l)(1 - m))$ to the origin and consider the parameter γ_* as the new bifurcation parameter. Therefore, System (3.1) is equivalent to the following system

$$\begin{aligned}\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} \mapsto & \begin{pmatrix} a_{10} & a_{01}\\ b_{10} & 1\end{pmatrix}\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}\\ & + \begin{pmatrix} a_{11}\bar{x}\bar{y} + a_{31}\gamma_*\bar{x} + a_{20}\bar{x}^2 + a_{32}\gamma_*\bar{x}^2 + a_{30}\bar{x}^3 + \gamma_*\bar{x}\bar{y} - \gamma_*\bar{x}^3\\ b_{11}\bar{x}\bar{y} + b_{31}\gamma_*\bar{x} + \gamma_*\bar{x}\bar{y}\end{pmatrix},\end{aligned} \tag{4.4}$$

where,

$$\begin{aligned}
&a_{10} = \gamma_0 m(l-2m+1)+1, \quad a_{01} = -\gamma_0 m, \quad a_{31} = m(l+1-2m),\\
&b_{10} = \gamma_0(m-l)(1-m),, \quad a_{11} = a_{30} = -b_{11} = -\gamma_0,\\
&a_{20} = \gamma_0(l-3m+1), \quad b_{31} = (m-l)(1-m).
\end{aligned}$$

Now, we construct an invertible matrix

$$T = \begin{pmatrix} a_{10} & a_{10} \\ -1-a_{01} & \lambda_2 - a_{01} \end{pmatrix}.$$

Under transformation $\binom{u}{v} = T\binom{\bar{x}}{\bar{y}}$, the mapping (4.4) becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v,\gamma_*) \\ g(u,v,\gamma_*) \end{pmatrix}, \tag{4.5}$$

where,

$$\begin{aligned}
f(u,v,\gamma_*) =& \frac{1}{a_{10}(\lambda_2+1)}\Big((a_{11}(\lambda_2-a_{01})-a_{10}b_{11})\bar{x}\bar{y}\\
&+(a_{31}(\lambda_2-a_{01})-a_{10}b_{31})\gamma_*\bar{x}\\
&+a_{20}(\lambda_2-a_{01})\bar{x}^2 + a_{32}(\lambda_2-a_{01})\gamma_*\bar{x}^2 + a_{30}(\lambda_2-a_{01})\bar{x}^3\\
&+((\lambda_2-a_{01})-a_{10})\gamma_*\bar{x}\bar{y} - (\lambda_2-a_{01})\gamma_*\bar{x}^3\Big),\\
g(u,v,\gamma_*) =& \frac{1}{\lambda_2+1}\Big((a_{11}(1+a_{01})+a_{10}b_{11})\bar{x}\bar{y}\\
&+(a_{31}(1+a_{01})+a_{10}b_{31})\gamma_*\bar{x}\\
&+a_{20}(1+a_{01})\bar{x}^2 + a_{32}(1+a_{01})\gamma_*\bar{x}^2 + a_{30}(1+a_{01})\bar{x}^3\\
&+((1+a_{01})+a_{10})\gamma_*\bar{x}\bar{y} - (1+a_{01})\gamma_*\bar{x}^3\Big),
\end{aligned}$$

and

$$\bar{x} = a_{10}(u+v), \quad \bar{y} = (\lambda_2-a_{01})v - (1+a_{01})u.$$

Now, we obtain the center manifold of (4.5) at the origin in a small neighborhood of γ_*. The center manifold can be written as:

$$W^c(0) = \{(u,v,\gamma_*) \in \mathbb{R}^3 : v = h(u,\gamma_*), h(0,0)=0, Dh(0,0)=0\}. \tag{4.6}$$

In other words, the center manifold is given by the graph of

$$v = h(u,v) = h_{20}u^2 + h_{11}\gamma_* u + h_{02}\gamma_*^2 + O((|u|+|\gamma_*|)^3). \tag{4.7}$$

Thus, center manifold (4.7) has to satisfy

$$h(u+f(u,h(u,\gamma_*)),\gamma_*) - \lambda_2 h(u,\gamma_*) - g(u,h(u,\gamma_*)),\gamma_*) = 0. \tag{4.8}$$

Substituting (4.7) into (4.8), implies

$$h_{20} = \frac{(1+a_{10})\big(a_{11}(1+a_{10}) + a_{01}(b_{11}-a_{20})\big)}{\lambda_2^2 - 1},$$
$$h_{11} = -\frac{a_{12}a_{31}(1+a_{10}) + a_{01}^2 b_{31}}{a_{01}(1+\lambda_2)^2},$$
$$h_{02} = 0.$$

Hence, the dynamics restricted to the center manifold is given by the map

$$F : u \mapsto -u + f_1u^2 + f_2\gamma_* u + f_3\gamma_* u^2 + f_4\gamma_*^2 u + f_5 u^3 + O((|u| + |\gamma_*|)^4), \quad (4.9)$$

where

$$\begin{aligned}
f_1 =& \frac{1}{\lambda_2+1}\big(a_{01}a_{20}(\lambda_2 - a_{11}) - a_{11}(\lambda_2 - a_{10})(1+a_{10}) \\
&+ a_{01}b_{11}(1+a_{10})\big),\\
f_2 =& \frac{1}{a_{01}(\lambda_2+1)}\big(a_{01}a_{31}(\lambda_2 - a_{10}) - a_{01}^2 b_{31}\big),\\
f_3 =& \frac{1}{a_{01}(\lambda_2+1)}\Big(h_{11}\big(a_{01}a_{11}(\lambda_2 \\
&- a_{10}(\lambda_2 - 2a_{10} - 1) - a_{01}^2 b_{11}(\lambda_2 - 2a_{10} - 1) \\
&+ 2a_{01}^2 a_{20}(\lambda_2 - a_{10})\big) + h_{20}\big(a_{01}a_{31}(\lambda_2 - a_{10}) - a_{01}^2 b_{31}\big)\Big),\\
f_4 =& \frac{h_{11}}{a_{01}(\lambda_2+1)}\big(a_{01}a_{31}(\lambda_2 - a_{10}) - a_{01}^2 b_{31}\big),\\
f_5 =& \frac{h_{20}}{(\lambda_2+1)}\big(a_{11}(\lambda_2 - a_{10})(\lambda_2 - 2a_{10} - 1) + 2a_{01}a_{20}(\lambda_2 - a_{10})\big).
\end{aligned}$$

The map (4.9) undergoes a flip bifurcation, if it satisfies the following two conditions

$$\alpha_1 = \left[\frac{\partial F}{\partial \gamma_*}.\frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u \partial \gamma}\right]|_{(0,0)} \neq 0,$$
$$\alpha_2 = \left[\frac{1}{2}\left(\frac{\partial^2 F}{\partial u^2}\right)^2 + \frac{1}{3}\frac{\partial^3 F}{\partial u^3}\right]|_{(0,0)} \neq 0.$$

By simple calculation, we obtain

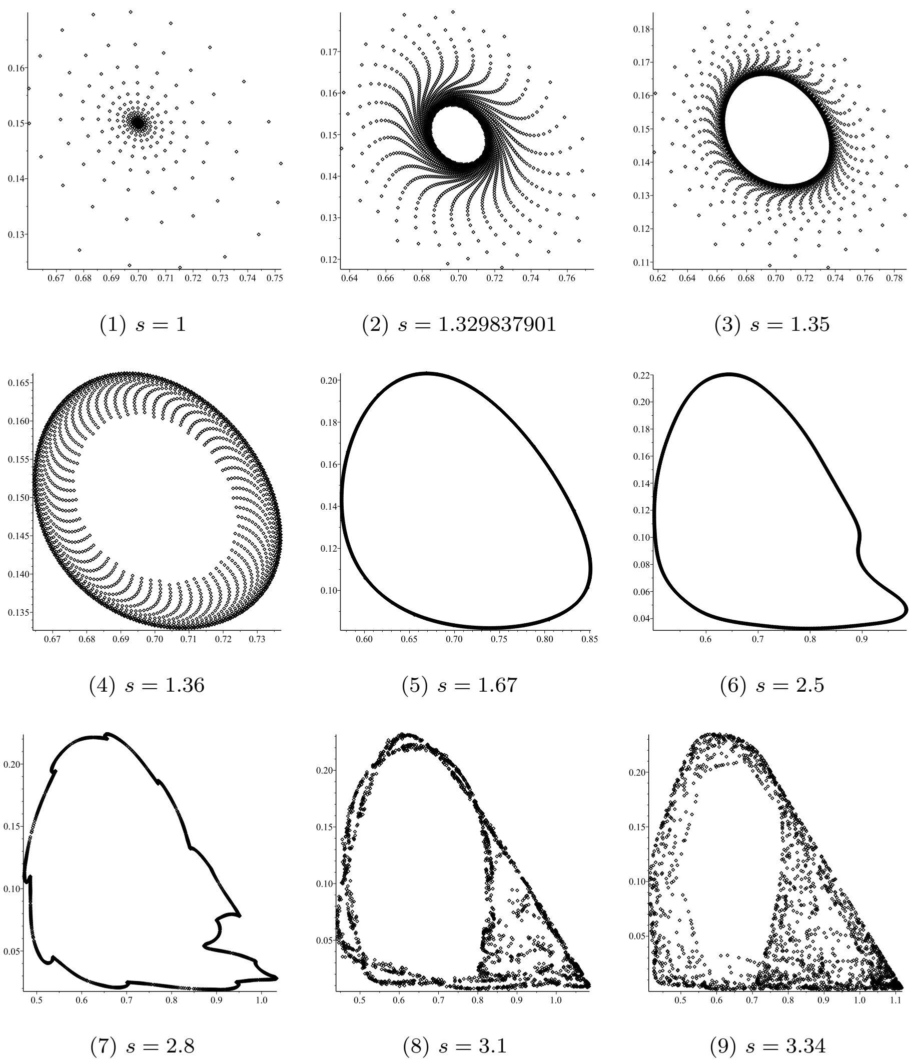
$$\alpha_1 = f_2 \neq 0, \quad \alpha_2 = f_1^2 + f_5 \neq 0.$$

We summarize the above discussion into the following theorem.

Theorem 4.3. *The map (3.1) undergoes a flip bifurcation at the interior fixed point E_3 if the following conditions are satisfied*

$$\alpha_1 \neq 0, \quad \text{and} \quad \alpha_2 \neq 0.$$

Moreover, if $\alpha_2 > 0$ (resp. $\alpha_2 < 0$), the flip bifurcation is supercritical (resp. subcritical). This means, the 2-period points that bifurcate from this point are stable (resp. unstable).



(1) $s = 1$ (2) $s = 1.329837901$ (3) $s = 1.35$

(4) $s = 1.36$ (5) $s = 1.67$ (6) $s = 2.5$

(7) $s = 2.8$ (8) $s = 3.1$ (9) $s = 3.34$

FIGURE 2. Phase portraits of system (3.1) for $\alpha = 0.98, l = 0.2, m = 0.7$.

5. Numerical simulation

In this section we give some numerical simulations of model (3.1) to illustrate the above theoretical analysis. Let $\alpha = 0.98, l = 0.2, m = 0.7$. In this case, $E_3 = (0.7, 0.15)$ is a unique positive fixed point. Then, from non-hyperbolic condition of Proposition 3.5, the Neimark-Sacker bifurcation appears from the fixed point E_3 at $s = 1.329837901$ (or equivalently $\gamma = 1.333333333$). In theoretical point of view, the unique positive fixed point E_3 is stable if $s < 1.329837901$, it loses its stability at

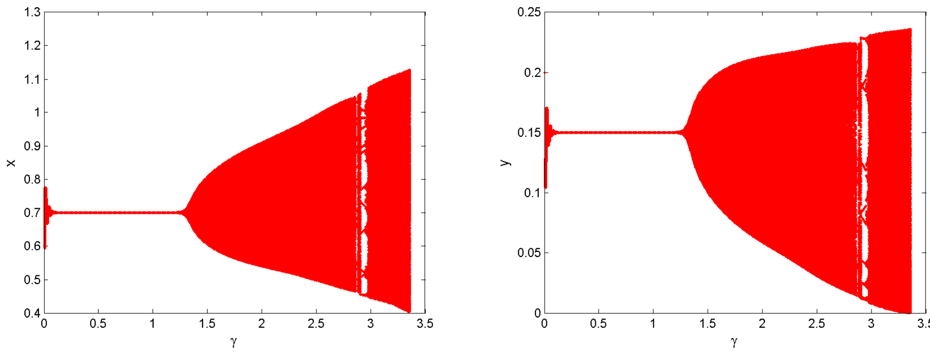


FIGURE 3. The bifurcation diagrams of the model for $\alpha = 0.98, m = 0.9, l = 0.2$, and $(x_0, y_0) = (0.8, 0.02)$.

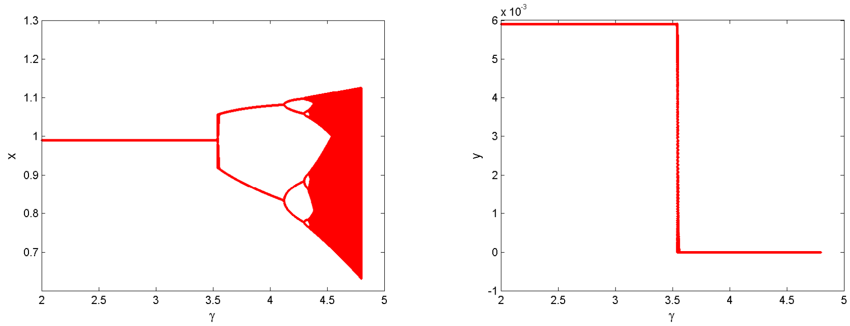
$s = 1.329837901$ and an attracting invariant close curve (or a stable cycle) appears from the positive fixed point when $s < 1.329837901$. By changing s (or equivalently γ) we plot the phase portrait of system (3.1) in Figure 2. Also, the bifurcation diagram for the Neimark-Sacker bifurcation is shown in Figure 3.

In other cases, consider the parameter values as $\alpha = 0.98, l = 0.4, m = 0.99$. Therefore, the unique positive fixed point is $E_3 = (0.99, 0.0059)$. Thus, from non-hyperbolic condition of Proposition 3.5, the values of fold bifurcation is $s = 3.609156303$ and $s = 213.1357619$ (or equivalently $\gamma = 3.547101215$ and $\gamma = 193.0630683$ respectively). The bifurcation diagram for the fold bifurcation is shown in Figure 4. It is seen that the positive fixed point is stable for $\gamma < 3.547101215$, then it loses its stability at the flip bifurcation parameter $\gamma = 3.547101215$. We observe that there is a cascade of period doubling for $3.547101215 < \gamma < 4.8$.

Specially, Figure 4 implies that when the prey is in chaotic, the predator ultimately tends to extinction. Generally, a cascade of period-doubling or fold bifurcation leads to chaos [8, 9]. Hence, since there is the fold bifurcation in this system, we are looking for chaos in the system. One of the criteria that can help us to find the chaos is the Maximum Lyapunov exponent (MLE). In general the positive MLE usually implies that the system is chaotic [3, 13]. The Maximum Lyapunov exponent corresponding to Figure 4 is given in Figure 5. Thus, Figure 5 confirms the existence of the chaotic sets for $\gamma > 3.547101215$.

6. Conclusion

This paper is related to the dynamical behaviors of fractional-order Bazykin-Berezovskaya model and its discretization. The fractional-order model included only stable and unstable equilibria. But discretized counterpart model can produce more complicated dynamical behaviour such as Neimark-Sacker bifurcation, flip bifurcation, and chaos. It is worth noting, when the value of the fractional order approaches to one, the discretization becomes the well-known Eulers discretization. The analytical results have also been verified with some numerical simulations.



(a) (γ, x) (b) (γ, y)

FIGURE 4. Bifurcation diagram for parameters $\alpha = 0.98, l = 0.4, m = 0.99$ and initial condition $(x_0, y_0) = (0.89, 0.011)$.

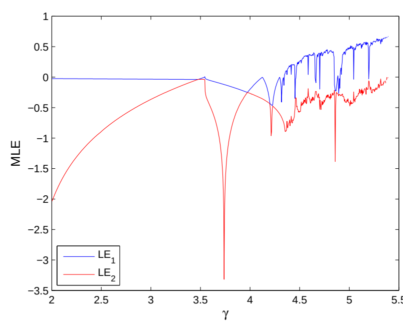


FIGURE 5. Maximum Lyapunov exponents corresponding to bifurcation diagrams Figure 4.

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