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PDE-based hyperbolic-parabolic model for image denoising with forward-backward diffusivity

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Abstract In the present study, we propose an effective nonlinear anisotropic diffusion-based hyperbolic parabolic model for image denoising and edge detection. The hyperbolic-parabolic model employs a second-order PDEs and has a second-time derivative to time t. This approach is very effective to preserve sharper edges and better-denoised images of noisy images. Our model is well-posed and it has a unique weak solution under certain conditions, which is obtained by using an iterative finite difference explicit scheme. The results are obtained in terms of peak signal to noise ratio (PSNR) as a metric, using an explicit scheme with forward-backward diffusivities.

Keywords. Image denoising, Nonlinear hyperbolic-parabolic equation, Nonlinear diffusion equation, Explicit scheme, Forward-backward diffusivity.

2010 Mathematics Subject Classification. 35L70, 65M06, 76R50, 68U10.

1. Introduction

The partial differential equations have been increasingly used for solving various image processing and analysis tasks in the last few decades. Perona and Malik [25] have given the first approach to remove the noise of noisy images by the nonlinear anisotropic diffusion equation. Catte et al. [7] introduced an improved Perona and Malik [25] model for image restoration and edge detection by nonlinear anisotropic diffusion model with a Gaussian kernel. Welk et al.[31] introduced an additive noise-based nonlinear anisotropic diffusion model for image restoration and they have explained the forward-backward diffusivities. Many approaches to image restoration by nonlinear diffusion models are suggested by many researchers [1, 2, 11, 12, 15, 16, 17, 20, 21]. One popular nonlinear model for removing noise is the time-dependent model which has introduced by Rudin et al.[27]. Many other techniques such as [7, 9, 10, 13, 14, 18, 19, 23, 26, 30] are proposed to reconstruct

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the image by complex diffusion model, fourth-order equation model, wavelet-based diffusion model, etc.

In recent years many researchers [3, 4, 28, 29] have studied nonlinear diffusion-based hyperbolic-parabolic models for image processing and to achieve a good dealing with the trade-off between image denoising and edge detection. Ratner and Zeevi [28] have given an approach to viewing the image as an elastic sheet by the telegraph-diffusion model. Later, Cao et al.[4] have introduced a diffusion-based hyperbolic-parabolic model to remove the noise of noisy images. Sun et al.[29] proposed a coupled system for image restoration by hyperbolic-parabolic model.

Motivated by [31], in this study, we propose a novel nonlinear anisotropic diffusion-based hyperbolic-parabolic model for additive noise removal. The paper is organized as follows: after abstract and introduction. In section 2, we present an image denoising algorithm. In section 3, the choice of diffusivity for models is discussed. The explicit scheme for the diffusion model and hyperbolic-parabolic model are in section 4. In section 5, the numerical experimental results in terms of PSNR are presented. Finally, a conclusion is given in section 6.

2. Image denoising algorithms

Welk et al.[31] introduced a model for image restoration using a minimizing the energy functional which is as:

$$E(u) = \int_\Omega \phi(|\nabla u|^2)\ dx\ dy + \frac{\lambda}{2}\int_\Omega (u-u_0)^2\ dx\ dy. \tag{2.1}$$

The first integral of (2.1) is the smoothness term or regulariser and the second integral is the data term of the squared error of the denoised image. The equation (2.1) using the Euler-Lagrange equation can be written as:

$$0 = \operatorname{div}(\phi'(|\nabla u|^2)\nabla u) - \lambda(u-u_0). \tag{2.2}$$

The gradient descent leading to a minimizer of E as $t \to \infty$ is given by

$$\frac{\partial u}{\partial t} = \operatorname{div}(g(|\nabla u|^2)\nabla u) - \lambda\ (u-u_0), \tag{2.3}$$

Equation (2.3) known as the diffusion-reaction equation and here the diffusivity $g(s^2)$ is equal to $\phi'(s^2)$ with homogeneous Neumann boundary conditions. Using (2.3), we propose a new approach to remove noise by hyperbolic-parabolic model and it is given by the following equation:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \operatorname{div}(g(|\nabla u|^2)\nabla u) - \lambda\ (u-u_0), \tag{2.4}$$

Equation (2.4) is the diffusion-based hyperbolic-parabolic model, here the diffusivity and boundary conditions are same as (2.3).

The proposed model provides an smoothness additive noise removal images which contains the second derivative to time t, it removes the diffusion effect near of the images boundaries, thus producing much sharper edges and better details.

The model (2.4) is well-posed since it has a unique weak solution under certain conditions. The solution of the model (2.4) represents the denoised image, and it is obtained by using a numerically explicit scheme. The details of the numerical explicit scheme for the hyperbolic-parabolic model are given in section 4.

3. Choice of the diffusivity

It is a very important concept to choose the diffusivity parameter such as total variation (TV) diffusivity, Charbonnier diffusivity and Perona-Malik (PM) diffusivity which are related to the non-convex regularizer function [24, 25, 32]. The importance of diffusivity parameter g is to controlling smoothness and even enhancement of edges. In our numerical experiments, we take the Charbonnier diffusivity $g(s^2)$ is equal to $\frac{1}{\sqrt{1+(|s|^2/K^2)}}$, which is related to the convex regularizer $\phi(s^2) = \sqrt{K^4 + K^2 s^2} - K^2$, as given in [5, 32].

4. Numerical approximation algorithm

In this section, we discretize diffusion model (2.3) and hyperbolic-parabolic model (2.4) by an explicit numerical scheme. Let $u^n_{i,j}$ be the approximation to the value $u(x_i, y_j, t_n)$. The spatial step sizes are Δx and Δy and the time step size is Δt. We define the discrete approximation as which gives. $x_i = i\Delta x$, $y_j = j\Delta x$, i, j=1,2,3.......N, $N\Delta x = 1$, and $t_n = n\Delta t,\ n \geq 1$.

$$\Delta^+_x u^n_{i,j} = \frac{u^n_{i+1,j} - u^n_{i,j}}{\Delta x},\ \Delta^-_x u^n_{i,j} = \frac{u^n_{i-1,j} - u^n_{i,j}}{\Delta x},$$
$$\Delta^+_y u^n_{i,j} = \frac{u^n_{i,j+1} - u^n_{i,j}}{\Delta x},\ \Delta^-_y u^n_{i,j} = \frac{u^n_{i,j-1} - u^n_{i,j}}{\Delta x}$$

and

$$\delta u^n_{i,j} = \frac{u^n_{i,j} - u^{n-1}_{i,j}}{\Delta t},\ \delta^2 u^n_{i,j} = \frac{\delta u^n_{i,j} - \delta u^{n-1}_{i,j}}{\Delta t}$$

The discrete scheme of the diffusion reaction model (2.3) can be written as:

$$u^t_{ij} = \frac{1}{2\Delta x}\ [(g^n_{i+1,j} + g^n_{i,j})\Delta^+_x u^n_{i,j} - (g^n_{i,j} + g^n_{i-1,j})\Delta^-_x u^n_{i,j}]$$
$$+\frac{1}{2\Delta x}\ [(g^n_{i,j+1} + g^n_{i,j})\Delta^+_y u^n_{i,j} - (g^n_{i,j} + g^n_{i,j-1})\Delta^-_y u^n_{i,j}] - \lambda(u^n_{ij} - u^0_{ij}).$$

The discrete scheme of the hyperbolic-parabolic model (2.4) can be written as

$$\begin{aligned} \delta^2 u^{n+1}_{i,j} + \delta u^{n+1}_{ij} &= \frac{1}{2\Delta x}\ [(g^n_{i+1,j} + g^n_{i,j})\Delta^+_x u^n_{i,j} - (g^n_{i,j} + g^n_{i-1,j})\Delta^-_x u^n_{i,j}] \\ &+ \frac{1}{2\Delta x}\ [(g^n_{i,j+1} + g^n_{i,j})\Delta^+_y u^n_{i,j} - (g^n_{i,j} + g^n_{i,j-1})\Delta^-_y u^n_{i,j}] - \lambda(u^n_{ij} - u^0_{ij}). \end{aligned} \tag{4.1}$$

where the diffusivity $g(|\nabla u|^2)$ is discretised by,

$$g^n_{ij} = \phi^{'}\left(\left(\frac{u^n_{i+1,j} - u^n_{i-1,j}}{\Delta x}\right)^2 + \left(\frac{u^n_{i,j+1} - u^n_{i,j-1}}{\Delta x}\right)^2\right).$$

The explicit method is stable and convergent for $\frac{\Delta t}{\Delta x^2} \leq 0.5$, see [22].

The numerical explicit scheme (4.1) is stable and consistent with the diffusion based hyperbolic model. It also converges fast to its weak solution represent the filtered image. It is then used in our numerical experiments which are given in the next section.

5. Numerical implementation

The nonlinear anisotropic diffusion-based hyperbolic technique is applied to many noisy images with different levels of white additive Gaussian noise, which gives better-preserved edges and smooth images. We consider here grayscale images as depicted in Figure 1. The interval [0,255] contains initially pixel values of all images. We use the normal imnoise function in Matlab to add the Gaussian white noise. The intensities of the images are scaled between zero and one. The diffusivity parameter like Charbonnier diffusivity K and Lagrange multiplier λ are taken as 5 and 0.85 respectively as given in [6, 8]. We choose $\Delta t/\Delta x^2 = 0.4$ and $\Delta t = 0.01$ for (2.3) and (2.4) respectively in our all numerical experiments.

To measure the quality of denoised images, we use peak signal to noise ratio (PSNR) which is given by below:

$$\text{PSNR} = 10\log_{10}\left(\frac{R^2}{\frac{1}{mn}\sum_{i,j}^{n}(u(i,j) - u_{new}(i,j))^2}\right), \tag{5.1}$$

where $\{u(i,j) - u_{new}(i,j)\}$ is the difference of the denoised and original images.

6. Figures

(a) (b)

Figure 1. (a-b) Original Lena 256 × 256 and Boat: 256 × 256 respectively.

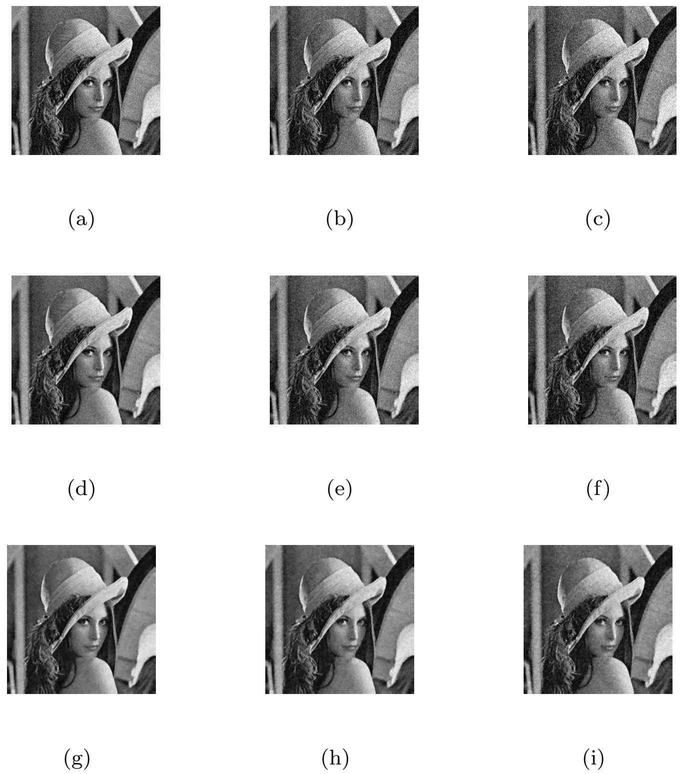

FIGURE 2. (a-c) Depiction of Lena images with various different levels of of Gaussian noise says $\sigma^2 = 0.004,\ 0.006,\ 0.008$, respectively; (d)-(f) corresponding denoised images by model (2.3); (g)-(i) by model (2.4).

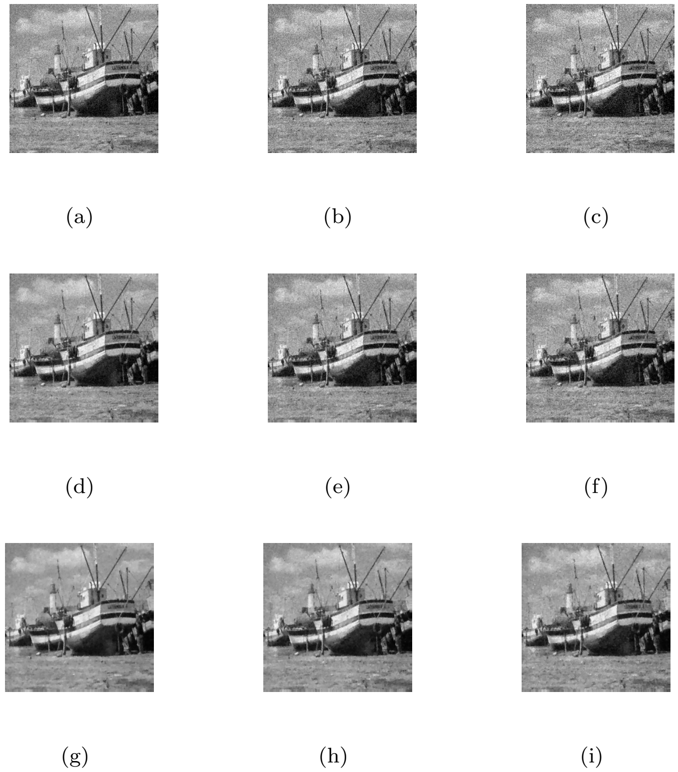

FIGURE 3. (a-c) Depiction of Boat images with various different levels of of Gaussian noise says $\sigma^2 = 0.004,\ 0.006,\ 0.008$, respectively; (d)-(f) corresponding denoised images by model (2.3); (g)-(i) by model (2.4).

7. TABLES

TABLE 1. Figure 2(a-c) represent noisy images with different level of Gaussian noise ($\sigma^2 = 0.004$, 0.006 and 0.008). We applied the models (2.3) and (2.4) to noisy images. We get denoised images figure 2(d-f) and 2(g-i) respectively. The results of denoised images given as PSNR values.

Images in figures	PSNR of noisy images	Images in figures	PSNR for (Model-2.3)	Images in figures	PSNR for (Model-2.4)
2(a)	24.10	2(d)	27.75	2(g)	29.74
2(b)	22.39	2(e)	25.77	2(h)	28.48
2(c)	21.18	2(f)	24.34	2(i)	27.42
-	-	No. of iterations	500	No. of iterations	200

Table 2. Figure 3(a-c) represent noisy images with different level of Gaussian noise ($\sigma^2 = 0.004$, 0.006 and 0.008). We applied the models (2.3) and (2.4) to noisy images. We get denoised images figure 3(d-f) and 3(g-i) respectively. The results of denoised images given as PSNR values.

Images in figures	PSNR of noisy images	Images in figures	PSNR for (Model-2.3)	Images in figures	PSNR for (Model-2.4)
3(a)	24.07	3(d)	27.34	3(g)	28.94
3(b)	22.30	3(e)	25.41	3(h)	27.87
3(c)	21.09	3(f)	24.04	3(i)	26.98
-	-	No. of iterations	500	No. of iterations	200

8. Conclusion

We have proposed a nonlinear diffusion-based hyperbolic-parabolic model for removing additive white Gaussian noise. This model assures a strong feature edge-preserving and removing noise component which have the role of controlling the speed of the diffusion process, and a drift term introduced to enhance the image edges. The unique weak solution of our model has computed using a finite difference method based iterative explicit numerical approximation algorithm that is stable, consistent, and converges quite fast. It has been successfully applied in our restoration experiments which have proved the effectiveness of the proposed denoising approach. Our restoration framework has provided proper smoothing results while avoiding undesirable effects. The nonlinear explicit scheme has been employed to carry out the goal. The hyperbolic-parabolic model has given better results than the diffusion model with a small number of iterations.

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