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Toward a new understanding of cohomological method for fractional partial differential equations

Akbar Dehghan Nezhad*
Department of Mathematics,
Iran University of Science and Technology,
P.O.Box, 16846-13114, Narmak, Tehran, Iran.
E-mail: dehghannezhad@iust.ac.ir

Mina Moghaddam
Department of Mathematics,
Iran University of Science and Technology,
P.O.Box, 16846-13114, Narmak, Tehran, Iran.
E-mail: m.moghaddam.zeabadi@gmail.com

Abstract One of the aims of this article is to investigate the solvability and unsolvability conditions for fractional cohomological equation $\psi^{\alpha} f = g$, on $\mathbb{T}^n$. We prove that if f is not analytic, then fractional integro-differential equation $I_t^{1-\alpha} D_x^{\alpha} u(x,t) + iI_x^{1-\alpha} D_t^{\alpha} u(x,t) = f(t)$ has no solution in $C^1(B)$ with $0 < \alpha \leq 1$. We also obtain solutions for the space-time fractional heat equations on $\mathbb{S}^1$ and $\mathbb{T}^n$. At the end of this article, there are examples of fractional partial differential equations and a fractional integral equation together with their solutions.

Keywords. Fractional calculus, Fractional cohomological equations, Space-time-fractional heat equation, Solvable and unsolvable fractional differential equations.

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1. Introduction

In recent years fractional differential equations (or FDE) has been a fruitful field of research in science and engineering. In fact, many scientific areas are currently paying attention to the FDE concepts. We can refer to its adoption in modeling and identification, electromagnetism, chaos and fractals, heat transfer, physics, chemistry, biology, electronics, signal processing, robotics, system identification, traffic systems, genetic algorithms, telecommunications, irreversibility, control systems as well as economy, finance, etc [8, 13, 17, 18, 19, 21, 22].

There are equations in dynamical systems that are known as cohomological equations. These equations first introduced by Livšic [10, 11]. First, we introduce the symbols which we will need.

The subset $\mathbb{S}^m = \{x \in \mathbb{R}^{m+1} : |x| = 1\} \subseteq \mathbb{R}^{m+1}$ of $(m+1)$-Euclidean space $\mathbb{R}^{m+1}$ is a m-dimensional smooth manifold that is called m-sphere. Let $\mathbb{T}^l = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (l times). Product manifold $\mathbb{T}^l$ is called an l-dimensional torus.

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* corresponding.

A lie group (topological group) G is a smooth manifold (topological space) that also is a group such that the multiplication map $m : G \times G \longrightarrow G$ and inversion map $i : G \longrightarrow G$

$$m(g_1, g_2) = g_1 g_2, \qquad i(g) = g^{-1}, \quad g, g_1, g_2 \in G,$$

are both smooth (continuous) maps. Let M be an n-dimensional closed Riemannian manifold, let G be a group and let Γ be a complete metric space and topological group with identity element e such that G acts smoothly on M. Map $f : M \times G \longrightarrow \Gamma$ is cocycle if

$$f(x, g_1) f(xg_1, g_2) = f(x, g_1 g_2), \qquad x \in M, \quad g_1, g_2 \in G\cdot$$

Two cocycles f, h are cohomologous, if for some continuous function $\varphi : M \longrightarrow \Gamma$,

$$f(x, g) = \varphi^{-1}(x) h(x, g) \varphi(xg), \qquad x \in M, \quad g \in G\cdot$$

The cocycle f is called coboundary if and only if f be cohomologous to the trivial cocycle $g = e$. If f be coboundary, then there exists continuous function $\varphi : M \longrightarrow \Gamma$ such that

$$f(x, y) = \varphi^{-1}(x) \varphi(xg), \qquad x \in M, \quad g \in G\cdot \tag{1.1}$$

Eq. (1.1) is said to be a cohomological equation where $\varphi : M \longrightarrow \Gamma$ is an unknown function.
Consider a smooth flow $\phi^t : M \longrightarrow M$ as a dynamical system generated by the vector field ψ. The following equation

$$\psi(h) = k, \tag{1.2}$$

is a cohomological equation where k is a known function and h is an unknown function. The reader interested in further details concerning the cohomology for dynamical systems is referred to [5, 7].

In this article, the cohomological equation associated to the fractional-order dynamical system is investigated.
Let $(U, \Phi = (x_1, \cdots, x_n))$ be a coordinate chart of M. Consider the following fractional-order dynamical system [23]

$$\begin{aligned} \frac{d^\alpha x_1}{dt^\alpha} &= v_1(x_1, \cdots, x_n), \\ &\vdots \\ \frac{d^\alpha x_n}{dt^\alpha} &= v_n(x_1, \cdots, x_n), \qquad 0 < \alpha \in \mathbb{R}, \end{aligned}$$

where for every $1 \leq i \leq n$,

$$\frac{d^\alpha x_i}{dt^\alpha} = \frac{1}{\Gamma(L - \alpha)} (\frac{d}{dt})^L \int_{-\infty}^{t} \frac{x_i(s)\, ds}{(t - s)^{\alpha - L + 1}}, \quad L - 1 \leq \alpha < L,$$

is Liouville fractional derivative of x_i of order $\alpha \in \mathbb{R}$ with respect to t. Associated with this fractional-order dynamical system, a fractional-order vector field is defined as follows

$$\psi^\alpha = \sum_{i=1}^{n} v_i(x_1, \cdots, x_n) \frac{\partial^\alpha}{\partial x_i^\alpha}. \tag{1.3}$$

In Eq. (1.3), for every $1 \leq i \leq n$, the operator $\dfrac{\partial^\alpha}{\partial x_i^\alpha}$ is defined by

$$\begin{cases} \dfrac{\partial^\alpha}{\partial x_i^\alpha} : C^\infty\infty(U) \longrightarrow \mathbb{R}, \\ \qquad f \longmapsto \dfrac{\partial^\alpha f}{\partial x_i^\alpha}. \end{cases} \tag{1.4}$$

In Eq. (1.4),

$$\left(\frac{\partial^\alpha f}{\partial x_i^\alpha}\right)(x) := \frac{1}{\Gamma(L-\alpha)} \left(\frac{\partial}{\partial x_i}\right)^L \int_{-\infty}^{x_i} \frac{f \circ \varphi^{-1}(x_1, \cdots, x_{i-1}, t, x_{i+1}, \cdots, x_n)dt}{(x_i - t)^{\alpha - L + 1}},$$

is the partial Liouville fractional derivative of real-valued smooth function $f : U \longrightarrow \mathbb{R}$ of order $\alpha \in \mathbb{R}$ with respect to x_i where $L - 1 \leq \alpha < L$. For a deeper discussion of fractional calculus, we refer the reader to [6, 9, 12, 14, 15, 16, 24]. Therefore, fractional cohomological equation (or FCE) is introduced as follows

$$\psi^\alpha(f) = g, \tag{1.5}$$

where g, f are a known function and an unknown function, respectively. The determination of solvability and unsolvability of differential equations is of specific importance. The solvability and unsolvability conditions already obtained for the cohomological equation [1, 3]. Recent advances in FDE have aided our understanding of FCE. In our study, we decide to investigate FCE $\psi^\alpha f = g$ on $\mathbb{T}^n$ where ψ^α is a fractional-order vector field and $0 < \alpha \in \mathbb{R}$.
Therefore, we have the following problem.

Problem 1.1. (The principal problem) Let $g \in C^\infty(U)$ and let $\psi^\alpha : C^\infty(U) \longrightarrow C^\infty(U)$ be a fractional-order vector field. Is there any $f \in C^\infty(U)$ such that $\psi^\alpha(f) = g$?

This paper is organized in this way. In section 2, results on the FCE $\psi^\alpha(f) = g$ is described, where ψ^α is a fractional-order area-preserving vector field on $\mathbb{T}^n$.
In the third section, we first restrict the discussion to solve the space-time fractional heat equation on $\mathbb{S}^1$

$$(D_t^\alpha + D_\theta^\beta) f(t, \theta) = 0, \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 2.$$

For solving this equation, we use Fourier decomposition

$$f(t, \theta) = \sum_{n \in \mathbb{Z}} a_n(t) e^{in\theta},$$

for $f(t, \theta)$ where $a_n(0) = a_n$ is the n^{th} Fourier coefficient for $f(t, \theta)$. In the same manner, we can see that the space-time fractional heat equation on $\mathbb{T}^n$

$$(D_t^\alpha + D_\Theta^\beta) f(t, \Theta) = 0, \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 2, \quad \Theta = (\theta_1, \cdots, \theta_n),$$

is solvable.

Section 4 contains an example of unsolvable fractional nonlinear integro-differential equation. Finally, in this section, we will give some solvable FDE and a solvable fractional integral equation.

2. The FCE $\psi^\alpha(f) = g$

In this section, all the objects are assumed to be smooth (of class C^∞). A function on $\mathbb{T}^n$ is just a real-valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfies in the relation $f(x + L) = f(x)$ for every $x \in \mathbb{R}^n$ and every $L \in \mathbb{Z}^n$. The results of the FCE $\psi^\alpha(f) = g$ will be described where ψ^α is a fractional-order area-preserving vector field on $\mathbb{T}^n$. Indeed by using the Diophantine and Liouville conditions, the solvability and unsolvability conditions will be revealed. Our purpose in this section is to solve the following problem.

Problem 2.1. Let g be a real-valued smooth function on $\mathbb{T}^n$. Can we find a real-valued smooth function f on $\mathbb{T}^n$ such that $\psi^\alpha(f) = g$?

For solving this problem, we need to recall the definitions of Diophantine condition and Liouville condition.

Definition 2.2. [20] Let $V \in \mathbb{R}^n$ be a vector. We say

(1) $V \in \mathbb{R}^n$ is Diophantine if there exist $\delta > 0$ and $\eta > 0$ such that for every $L \in \mathbb{Z}^n - \{0\}$, we have; $|\langle V, L\rangle| \geq \dfrac{\delta}{|L|^\eta}$ (Diophantine condition),

(2) $V \in \mathbb{R}^n$ is Liouville if there exists $\delta > 0$ such that for every $\eta > 0$ there exists an infinite series (L_η) in $\mathbb{Z}^n - \{0\}$ which satisfies in $|\langle V, L_\eta\rangle| \leq \dfrac{\delta}{|L_\eta|^\eta}$ (Liouville condition).

Let $x = (x_1, \cdots, x_n) \in \mathbb{T}^n$. For every $L = (l_1, \cdots, l_n) \in \mathbb{Z}^n$, we denote $E_L(x) = e^{2i\pi\langle x,L\rangle}$ where $\langle x, L\rangle = x_1 l_1 + \cdots + x_n l_n$. The functions $f, g \in C^\infty(\mathbb{T}^n)$ can be expanded as Fourier series. Set, $f(x) = \sum\limits_{L\in\mathbb{Z}^n} f_L E_L(x)$ and $g(x) = \sum\limits_{L\in\mathbb{Z}^n} g_L E_L(x)$ where f_L and g_L are the Fourier coefficients of the functions f and g, respectively. Therefore f_L and g_L are given by the following integral formulas

$$f_L = \int_{\mathbb{T}^n} f(x)e^{-2i\pi\langle x,L\rangle}dx, \quad g_L = \int_{\mathbb{T}^n} g(x)e^{-2i\pi\langle x,L\rangle}dx.$$

If f and g are square-integrable, then the coefficients f_L and g_L satisfy the following conditions

$$\sum_{L\in\mathbb{Z}^n} |f_L|^2 < \infty, \quad \sum_{L\in\mathbb{Z}^n} |g_L|^2 < \infty.$$

Every distribution T on $\mathbb{T}^n$ can be written as $T(x) = \sum\limits_{L\in\mathbb{Z}^n} T_L E_L(x)$. The family of real numbers T_L is of polynomial growth, that is, there exist $r \in \mathbb{N}$ and a constant $C > 0$ such that $|T_L| \leq C|L|^r$ for every $L \in \mathbb{Z}^n$.

Proposition 2.3. [1, 3] *Let* $T = \sum_{L\in\mathbb{Z}^n} T_L E_L$ *be a series (where* T_L *are real numbers). Then the following assertions are equivalent.*

1. T *is a regular distribution (*T *is a smooth function).*
2. *For every* $r \in \mathbb{N}$*, the series* $\sum_{L\in\mathbb{Z}^n} |f_L|^{2r}|T_L|^2$ *is convergence.*
3. *For every* $r \in \mathbb{N}$*, the series* $\sum_{L\in\mathbb{Z}^n} |f_L|^{r}|T_L|$ *is convergence.*

We consider a fractional-order vector field on n-torus $\mathbb{T}^n$ of the form $\psi^\alpha = \sum_{i=1}^n v_i \frac{\partial^\alpha}{\partial x_i^\alpha}$. The following definition is important.

Definition 2.4. [20] A fractional-order vector field $\psi^\alpha = \sum_{i=1}^n v_i \frac{\partial^\alpha}{\partial x_i^\alpha}$ is Diophantine (or Liouville) if $V = (v_1, \cdots, v_n)$ satisfies Diophantine (or Liouville) condition.

Theorem 2.5. *Let* $\psi^\alpha = \sum_{i=1}^n v_i \frac{\partial^\alpha}{\partial x_i^\alpha}$ *be a fractional-order vector field on* $\mathbb{T}^n$ *where* $v_1, \cdots, v_n$ *are Linearly independent over* $\mathbb{Q}$*. Let the subgroup generated by* $V = (v_1, \cdots, v_n)$ *be dense in* $\mathbb{T}^n$*.*

1. *Suppose* $V = (v_1, \cdots, v_n)$ *is Diophantine. Then the equation* $\psi^\alpha(f) = g$ *has a solution* $f \in C^\infty(\mathbb{T}^n)$ *if and only if* $\int_{\mathbb{T}^n} g(x)dx = 0$*.*
2. *Suppose* $V = (v_1, \cdots, v_n)$ *is Liouville. Therefore there exists an infinite family of Linearly independent functions* g *satisfying* $\int_{\mathbb{T}^n} g(x)dx = 0$ *such that the equation* $\psi^\alpha(f) = g$ *has no solution.*

Proof. Let $g \in C^\infty(\mathbb{T}^n)$ such that $\int_{\mathbb{T}^n} g(x)dx = 0$. Let us consider the Fourier development of f, g as follows

$$f(x) = \sum_{L\in\mathbb{Z}^n} f_L E_L(x), \quad g(x) = \sum_{L\in\mathbb{Z}^n} g_L E_L(x)\cdot$$

According to the properties of the Liouville fractional derivative, we have

$$D_+^\alpha e^{\lambda x} = \lambda^\alpha e^{\lambda x}\cdot$$

With the notation $E_L(x) = e^{2i\pi\langle x,L\rangle}$, we have

$$\frac{\partial^\alpha}{\partial x_i^\alpha}(E_L(x)) = (2\pi i l_k)^\alpha (E_L(x))\cdot$$

Under the hypotheses of Theorem, we obtain

$$
\begin{aligned}
\sum_{L\in\mathbb{Z}^n} g_L E_L(x) &= \psi^{\alpha}(\sum_{L\in\mathbb{Z}^n} f_L E_L(x)) \\
&= \sum_{k=1}^{n} v_k \frac{\partial^{\alpha}}{\partial x_k^{\alpha}}(\sum_{L\in\mathbb{Z}^n} f_L E_L(x)) \\
&= \sum_{L\in\mathbb{Z}^n}(\sum_{k=1}^{n} v_k \frac{\partial^{\alpha}}{\partial x_k^{\alpha}}(f_L E_L(x))) \\
&= \sum_{L\in\mathbb{Z}^n} f_L(\sum_{k=1}^{n} v_k \frac{\partial^{\alpha}}{\partial x_k^{\alpha}}(E_L(x))) \\
&= \sum_{L\in\mathbb{Z}^n} f_L(\sum_{k=1}^{n} v_k (2\pi i l_k)^{\alpha}(E_L(x))) \\
&= \sum_{L\in\mathbb{Z}^n} f_L(\sum_{k=1}^{n} v_k (2\pi i l_k)^{\alpha}) E_L(x)\cdot
\end{aligned}
$$

It follows that

$$
f_L(\sum_{k=1}^{n} v_k (2\pi i l_k)^{\alpha}) = g_L \quad with \quad L\in\mathbb{Z}^n\cdot
$$

We conclude from the necessary condition $\int_{\mathbb{T}^n} g(x)dx = 0$ that $g_0 = 0$.
As a conclusion of the recent process, we have

$$
f_L = \begin{cases} 0 & if \quad L = 0, \\ \frac{g_L}{(\sum\limits_{k=1}^{n} v_k (2\pi i l_k)^{\alpha})} & if \quad L \neq 0\cdot \end{cases} \tag{2.1}
$$

Therefore the function f is formally given by its Fourier coefficients $(f_L)_{L\in\mathbb{Z}^n}$.

Let us check its regularity. We will divide the proof of regularity into two parts $\alpha = 1$ and $\alpha \in (0,1)$.

The part I) If we take $\alpha = 1$, then analysis similar to [1, 3]. To read more about the details, we refer the reader to [1, 3].

The part II) Let $\alpha \in (0,1)$. For every $L \in \mathbb{Z}^n$, we denoted $[L^{\alpha}] = ([|l_1^{\alpha}|], \cdots, [|l_n^{\alpha}|])$.

Let $r \in \mathbb{N}$ and $|L|^{2r}|g_L|^2 < +\infty$. For regularity of f, we have

$$\begin{aligned}
| L |^{2r}| f_L |^2 &=| L |^{2r}| \frac{g_L}{(\sum_{k=1}^{n} v_k(2\pi i l_k)^{\alpha})} |^2 \\
&=| L |^{2r}| \frac{g_L}{(2\pi i)^{\alpha}(\sum_{k=1}^{n} v_k(l_k)^{\alpha})} |^2 \\
&=| L |^{2r} \frac{| g_L |^2}{| (2\pi i)^{\alpha} |^2| \sum_{k=1}^{n} v_k(l_k)^{\alpha} |^2} \\
&\leq| L |^{2r} \frac{| g_L |^2}{| (2\pi i)^{\alpha} |^2| \sum_{k=1}^{n} v_k[| (l_k)^{\alpha} |] |^2} \\
&\leq| L |^{2r} \frac{| g_L |^2}{| (2\pi i)^{\alpha} |^2 < V, [L^{\alpha}] >}\cdot
\end{aligned}$$

In the rest of the proof, we face the following two cases.
(The case I) Let $V = (v_1, \cdots, v_n)$ be Diophantine. Consequently, there exist $\delta > 0$ and $\eta > 0$ such that

$$|L|^{\eta} \geq \frac{\delta}{|\langle V, L\rangle|},$$

for every $L \in \mathbb{Z}^n - \{0\}$ (Diophantine condition).
Now, it is easy to verify that

$$\begin{aligned}
| L |^{2r}| f_L |^2 &\leq| L |^{2r} \frac{| g_L |^2}{| (2\pi i)^{\alpha} |^2 < V, [L^{\alpha}] >} \\
&\leq| L |^{2r}| g_L |^2 \frac{1}{| (2\pi i)^{\alpha} |^2 < V, [L^{\alpha}] >} \\
&\leq| [L^{\alpha}] |^{\eta}| L |^{2r}| g_L |^2 \frac{1}{| (2\pi i)^{\alpha} |^2} \\
&\leq| L |^{2r+\eta}| g_L |^2 \frac{1}{| (2\pi i)^{\alpha} |^2} < +\infty\cdot
\end{aligned}$$

(The case II) Let $V = (v_1, \cdots, v_n)$ be Liouville. Therefore, there exists $\delta > 0$ such that for every $\eta > 0$ there exists an infinite series (L_η) in $\mathbb{Z}^n - \{0\}$ that (L_η) satisfies in the following condition (Liouville condition)

$$|L_\eta|^{\eta} \leq \frac{\delta}{|\langle V, L_\eta\rangle|}\cdot$$

Let $(\eta_k)_k$ be increasing series in $\mathbb{N}^*$. The corresponding L_{η_k} will be denoted by L_k. Since V is Liouville, then we can find a function g by its Fourier coefficients as

$$g_L = \begin{cases} |L_k|^{-\eta_k/2} & if \quad L \neq L_k, \\ 0 & if\ not \quad L = 0\cdot \end{cases} \tag{2.2}$$

The function g is smooth and satisfies in $\int_{\mathbb{T}^n} g(x)dx = 0$. According to the above remark, we have

$$\begin{aligned} | f_{L_k} |^2 &= \left| \frac{g_{L_k}}{(\sum\limits_{k=1}^{n} v_k(2\pi i l_k)^\alpha)} \right|^2 \\ &= \left| \frac{| L_k |^{\eta_k/2}}{(\sum\limits_{k=1}^{n} v_k(2\pi i l_k)^\alpha)} \right|^2 \\ &\geq \frac{1}{\delta^2} | L_k |^{\eta_k} \cdot \end{aligned}$$

In this way, we can construct an infinite family of smooth Linearly independent functions g that satisfies in $\int_{\mathbb{T}^n} g(x)dx = 0$. So the proof is complete. □

3. The space-time fractional heat equations

In the first subsection, we investigate the space-time fractional heat equation on $\mathbb{S}^1$. In the second subsection, by using a solution of the space-time fractional heat equation, we can solve the space-time fractional heat equation on $\mathbb{T}^n$.

3.1. **On $\mathbb{S}^1$.** Let $L^2([0,\infty) \times \mathbb{S}^1) = L^2([0,\infty) \times \mathbb{S}^1, \mathbb{C})$ denote the space of complex-valued L^2 functions. Consider an initial distribution $f(\theta) = f(0,\theta)$ of the space-time fractional heat on $\mathbb{S}^1$ that is considered to be perfectly insulated. The distribution $f(t,\theta) \in L^2([0,\infty) \times \mathbb{S}^1)$ of fractional heat at time t is governed by the space-time fractional heat equation

$$(D_t^\alpha + D_\theta^\beta) f(t,\theta) = 0, \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 2\cdot$$

If $f(t,\theta) = \sum\limits_{n\in\mathbb{Z}} a_n(t) e^{in\theta}$ is the Fourier decomposition for $f(t,\theta)$ with $a_n(0) = a_n$ as the n^{th} Fourier coefficient for f, then

$$\begin{aligned} 0 &= \sum_{n\in\mathbb{Z}} ((D_t^\alpha a_n(t) e^{in\theta}) + (D_\theta^\beta a_n(t) e^{in\theta})) \\ &= \sum_{n\in\mathbb{Z}} (e^{in\theta}(D_t^\alpha a_n(t)) + (a_n(t) D_\theta^\beta e^{in\theta}))\cdot \end{aligned}$$

By using the Liouville fractional derivatives of exponential functions, we obtain

$$D_\theta^\beta e^{in\theta} = i^\beta (n)^\beta e^{in\theta} = (n)^\beta (cos\frac{\pi\beta}{2} + i\ sin\frac{\pi\beta}{2}) e^{in\theta}\cdot$$

Consequently,

$$\begin{aligned} 0 &= \sum_{n\in\mathbb{Z}} (e^{in\theta} D_t^\alpha a_n(t) + (a_n(t) i^\beta (n)^\beta e^{in\theta})) \\ &= \sum_{n\in\mathbb{Z}} (D_t^\alpha a_n(t) + a_n(t) i^\beta (n)^\beta) e^{in\theta}\cdot \end{aligned}$$

It follows that for every $n \in \mathbb{Z}$,

$$D_t^\alpha a_n(t) + a_n(t) i^\beta (n)^\beta = 0, \tag{3.1}$$

where α, β were assumed constant. We solve the above equation with the help of the Laplace transform. If we apply Laplace transformation to both sides of Eq. (3.1), then we have

$$\mathcal{L}(D_t^{\alpha} a_n(t) + a_n(t) i^{\beta} (n)^{\beta}) = 0\cdot$$

With the notation $\mathcal{L}(a_n(t)) = A_n(s)$, we have

$$s^{\alpha} A_n(s) - \sum_{k=1}^{m} b_k s^{k-1} + i^{\beta} (n)^{\beta} A_n(s) = 0, \tag{3.2}$$

where $b_k = [D_t^{\alpha-k} a_n(t)]_{t=0}$ for $k = 1, \cdots, m$ and m is an integer such that $m-1 < \alpha \leq m$. From Eq. (3.2) we conclude that

$$A_n(s) = \frac{\sum_{k=1}^{m} b_k s^{k-1}}{s^{\alpha} + i^{\beta}(-n)^{\beta}}\cdot$$

By the inverse Laplace transform, we yield the solutions

$$a_n(t) = \sum_{k=1}^{m} b_k t^{\alpha-k} E_{\alpha, \alpha-k+1}(-i^{\beta} n^{\beta} t^{\alpha})\cdot$$

It follows that

$$f(t, \theta) = \sum_{n \in \mathbb{Z}} \sum_{k=1}^{m} b_k t^{\alpha-k} E_{\alpha, \alpha-k+1}(-i^{\beta} n^{\beta} t^{\alpha}) e^{in\theta},$$

where $E_{\alpha,\alpha-k+1}$ is the two-parameter Mittag-Leffler function. Now, we solve the space-time fractional heat equation for $\alpha = \frac{1}{2}$ and $0 < \beta \leq 2$. From the above discussion, if we assume $[D_t^{-\frac{1}{2}} a_n(t)]_{t=0} = C$, then we have the solution

$$a_n(t) = C t^{-\frac{1}{2}} E_{\frac{1}{2}, \frac{1}{2}}(-i^{\beta} n^{\beta} \sqrt{t})\cdot$$

3.2. **On $\mathbb{T}^n$.** We recall that the n-torus $\mathbb{T}^n$ is a Lie group. The n-torus $\mathbb{T}^n$ is the n times product of the Lie group $\mathbb{S}^1$ with itself, i.e. $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. Consider the space of the complex-valued L^2 functions $L^2([0,\infty) \times \mathbb{T}^n) = L^2([0,\infty) \times \mathbb{T}^n, \mathbb{C})$. Let $f(\Theta) = f(0, \Theta)$ be an initial distribution of the space-time fractional heat on $\mathbb{T}^n$ that is considered perfectly insulated. The distribution $f(t, \Theta) \in L^2([0,\infty) \times \mathbb{T}^n)$ of the fractional heat at time t is governed by the space-time fractional heat equation

$$(D_t^{\alpha} + D_{\Theta}^{\beta}) f(t, \Theta) = 0, \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 2, \; \Theta = (\theta_1, \cdots, \theta_n)\cdot \tag{3.3}$$

If $f(t, \Theta) = \sum_{L \in \mathbb{Z}^n} a_L(t) e^{i\langle L, \Theta \rangle}$ is the Fourier decomposition for $f(t, \Theta)$, then from now on $a_L(0) = a_L$ is the L^{th} Fourier coefficient for f. If $L = (l_1, \cdots, l_n)$ and $\langle L, \Theta \rangle = l_1\theta_1 + \cdots + l_n\theta_n$. Then from Eq. (3.3) and the Fourier decomposition, we obtain

$$0 = \sum_{L \in \mathbb{Z}^n} e^{i\langle L, \Theta \rangle} (D_t^{\alpha} a_L(t)) + \sum_{L \in \mathbb{Z}^n} a_L(t) D_{\theta}^{\beta} e^{i\langle L, \Theta \rangle})\cdot$$

Derived formulas for the fractional derivatives of trigonometric functions are based on the following form

$$\begin{aligned}
D_\Theta^\beta e^{i\langle L,\Theta\rangle} &= D_\Theta^\beta e^{i(l_1\theta_1+\cdots+l_n\theta_n)}\\
&= (\frac{\partial^\beta}{\partial\theta_1}+\cdots+\frac{\partial^\beta}{\partial\theta_n})e^{i(l_1\theta_1+\cdots+l_n\theta_n)}\\
&= \frac{\partial^\beta}{\partial\theta_1}e^{i(l_1\theta_1+\cdots+l_n\theta_n)}+\cdots+\frac{\partial^\beta}{\partial\theta_n}e^{i(l_1\theta_1+\cdots+l_n\theta_n)}\\
&= i^\beta l_1^\beta e^{i(l_1\theta_1+\cdots l_n\theta_n)}+\cdots+i^\beta l_n^\beta e^{i(l_1\theta_1+\cdots l_n\theta_n)}\\
&= i^\beta(l_1^\beta+\cdots+l_n^\beta)e^{i\langle L,\Theta\rangle}.
\end{aligned}$$

Therefore, we can write

$$0=\sum_{L\in\mathbb{Z}^n} e^{i\langle L,\theta\rangle}(D_t^\alpha a_L(t)+a_L(t)(i^\beta(l_1^\beta+\cdots+_n^\beta)).$$

So, for every $L\in\mathbb{Z}^n$

$$D_t^\alpha a_L(t)+(i^\beta(l_1^\beta+\cdots+l_n^\beta))a_L(t)=0\cdot \tag{3.4}$$

Where α and β are constants. We solve the above equation by using the Laplace transform. For this purpose, we apply Laplace transform on both sides of the Eq. (3.4)

$$\mathcal{L}(D_t^\alpha a_L(t)+(i^\beta(l_1^\beta+\cdots+l_n^\beta)a_L(t))=0\cdot$$

Set $\mathcal{L}(a_L(t))=A_L(s)$ and

$$i^\beta(l_1^\beta+\cdots+l_n^\beta)=(cos\frac{\pi\beta}{2}+i\ sin\frac{\pi\beta}{2})(l_1^\beta+\cdots+l_n^\beta)=B\cdot$$

Consequently,

$$s^\alpha A_L(s)-\sum_{k=1}^m b_k s^{k-1}+BA_L(s)=0, \tag{3.5}$$

where $b_k=[D_t^{\alpha-k}a_L(t)]_{t=0}$, $k=1,\cdots,m$, and m is an integer such that $m-1<\alpha\le m$.
From the Eq. (3.5),

$$A_L(s)=\frac{\sum\limits_{k=1}^m b_k s^{k-1}}{s^\alpha+B}\cdot$$

By applying the inverse Laplace transform, we can conclude that

$$a_L(t)=\sum_{k=1}^m b_k t^{\alpha-k}E_{\alpha,\alpha-k+1}(-Bt^\alpha)\cdot$$

As a conclusion of former technique, we have

$$f(t,\Theta)=\sum_{L\in\mathbb{Z}^n}\sum_{k=1}^m b_k t^{\alpha-k}E_{\alpha,\alpha-k+1}(-Bt^\alpha)e^{i\langle L,\Theta\rangle}.$$

4. Examples of solvable and unsolvable FDE's

In this section, we were encouraged to construct a fractional integro-differential equation without a solution in Theorem 4.1. In the following examples, we analyze the system of solvable FDEs and an example of the solvable FIE.

Let B be the set $\{(x,t) : |x| < a, \ |t| < b\}$ where a and b are two fixed positive numbers.

Proposition 4.1. *There exists a smooth function $f(t)$ such that the fractional integro-differential equation*

$$I^{1-\alpha}_{+,t} D^{\alpha}_{+,x} u(x,t) + i I^{1-\alpha}_{+,x} D^{\alpha}_{+,t} u(x,t) = f(t), \tag{4.1}$$

has no solution in $C^1(B)$.

Where $0 < \alpha < 1$ and $I^{1-\alpha}_{+,x}$, $I^{1-\alpha}_{+,t}$ are the following partial Liouville fractional integrals

$$I^{1-\alpha}_{+,x} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{u(y,t)}{(x-y)^{\alpha}} dy,$$

and

$$I^{1-\alpha}_{+,t} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(x,s)}{(t-s)^{\alpha}} ds\cdot$$

Also, $D^{\alpha}_{+,x} u(x,t)$, $D^{\alpha}_{+,t} u(x,t)$ are the following partial Liouville fractional derivatives

$$D^{\alpha}_{+,x} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_0^x \frac{u(y,t)}{(x-y)^{\alpha}} dy,$$

and

$$D^{\alpha}_{+,t} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,s)}{(t-s)^{\alpha}} ds\cdot$$

Proof. Suppose $u \in C^1(B)$ is a solution of Eq. (4.1). We set

$$v(x,t) = \frac{1}{\Gamma(1-\alpha)^2} \int_0^x \int_0^t \frac{u(y,s)}{(x-y)^{\alpha}(t-s)^{\alpha}} ds dy\cdot \tag{4.2}$$

By calculating the partial derivatives of $v(x,t)$, we have

$$\begin{aligned}
v_x &= \frac{\partial}{\partial x} \frac{1}{\Gamma(1-\alpha)^2} \int_0^x \int_0^t \frac{u(y,s)}{(x-y)^{\alpha}(t-s)^{\alpha}} ds dy \\
&= \frac{\partial}{\partial x} \frac{1}{\Gamma(1-\alpha)^2} \int_0^t \int_0^x \frac{u(y,s)}{(x-y)^{\alpha}(t-c)^{\alpha}} dy ds \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^{\alpha}} \left(\frac{\partial}{\partial x} \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{u(y,s)}{(x-y)^{\alpha}} dy \right) ds \\
&= I^{1-\alpha}_{+,t} D^{\alpha}_{+,x} u(x,t),
\end{aligned}$$

and

$$\begin{aligned} v_t &= \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)^2}\int_0^x\int_0^t \frac{u(y,s)}{(x-y)^\alpha(t-s)^\alpha}dsdy \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{1}{(x-y)^\alpha}(\frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u(y,s)}{(t-s)^\alpha}ds)dy \\ &= I^{1-\alpha}_{+,x}D^{\alpha}_{+,t}u(x,t)\cdot \end{aligned}$$

From Eq. (4.1) we conclude that

$$v_x + iv_t = I^{1-\alpha}_{+,t}D^{\alpha}_{+,x}u(x,t) + iI^{1-\alpha}_{+,x}D^{\alpha}_{+,t}u(x,t) = f(t),$$

for $|x| < a$, $|t| < b$. The function g is defined by $g'(t) = f(t)$. Therefore g is a smooth real-valued function of t. Let

$$\omega(x,t) = v(x,t) + ig(t)\cdot$$

Since

$$\omega_x + i\omega_t = 0, \qquad |x| < a, \qquad |t| < b,$$

the map ω is an analytic function. Applying Eq. (4.2) we conclude that $v(0,t) = 0$. Therefore,

$$Re(\omega(0,t)) = 0, \qquad |t| < b\cdot$$

Consequently, $\omega(0,t) = ig(t)$ is an analytic function of t in B. Thus we have shown that f is necessarily analytic a function of t. If f is smooth, then Eq. (4.1) has no solution in $C^1(B)$. □

Example 4.2. [2] Let $b > 0$ be a fixed time. Let us denote by $*$ the classical convolution. Suppose that $0 < \alpha \leq 1$, $a \neq 0$, and $0 < t \leq b$. The ordinary fractional differential equation

$$(1 + aD^{\alpha}_t)f(t) = g(t), \; g(t) < \infty,$$

has a unique solution given by $f(t) = A_\alpha(a,t) * g(t)$ where

$$A_\alpha(a,t) := a^{-1}t^{\alpha-1}E_{\alpha,\alpha}(-a^{-1}t^\alpha)\cdot$$

The plot of the exact solution is shown in FIGURE 1, For $a = 10$, $b = 50$, $\alpha = \dfrac{1}{2}$, $g(t) = \sin(t)$.

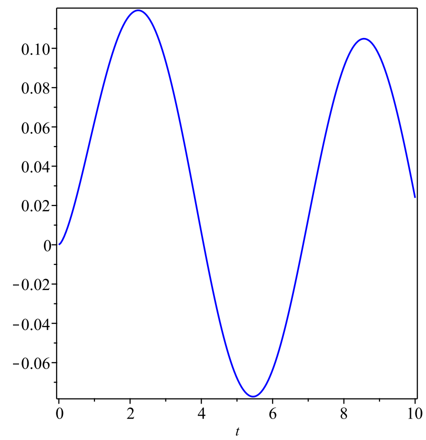

FIGURE 1. Plot of an instance of Example 4.2

Example 4.3. Consider the following FDE of the order $n+\alpha$

$$(D_x^{n+\alpha}y)(x) = c\beta(\beta-1)\cdots(\beta-n+1)x^{\beta-n}y^2 + cx^\beta\left(\sum_{k=0}^{n}\binom{n}{k}y^{(k)}(x)y^{(n-k)}(x)\right),$$

$$n\in\mathbb{N},\quad \alpha\in\mathbb{C},\quad Re(\alpha)>0,\quad x>0,\quad c,\beta\in\mathbb{R},\quad c\neq 0,\quad \beta\neq 0,1,\cdots,n-1\cdot$$

If $\alpha+\beta<1$, then the above FDE has the following solution

$$y(x) = \frac{\Gamma(1-\alpha-\beta)}{c\Gamma(1-2\alpha-\beta)}x^{-(\alpha+\beta)}.$$

The fractional derivative of $y(x)$ of order α is

$$(D_x^\alpha y)(x) = \left(\frac{\Gamma(1-\alpha-\beta)}{c\Gamma(1-2\alpha-\beta)}\right)^2 x^{2\alpha-\beta},$$

hence $(D_x^\alpha y)(x) = cx^\beta y^2$. We know

$$(y^2)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}y^{(k)}(x)y^{(n-k)}(x),$$

consequently,

$$(D_x^{\alpha+n}y)(x) = c\beta(\beta-1)\cdots(\beta-n+1)x^{\beta-n}y^2 + cx^\beta\left(\sum_{k=0}^{n}\binom{n}{k}y^{(k)}(x)y^{(n-k)}(x)\right)\cdot$$

We conclude that $y(x)$ is the solution of the above FDE.
The plot of the exact solution is shown in FIGURE 2, For $c=4$, $\alpha=\frac{1}{7}$, $\beta=\frac{3}{7}$.

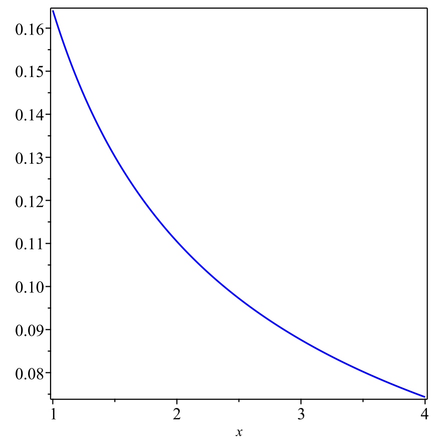

FIGURE 2. Plot of an instance of Example 4.3

Let g, h be two real-valued functions and $n \in \mathbb{N}$. The Faà Bi Bruno's formula is

$$\frac{d^n}{dx^n} g(h(x)) = \sum \frac{n!}{k_1!k_2! \cdots k_n!} g^{(k_1+k_2+\cdots+k_n)}(h(x)) \prod_{i=1}^{n} \left(\frac{g^{(i)}(x)}{i!} \right)^{k_i}.$$

The sum is over n-tuples $(k_1, k_2, \cdots, k_n)$ of nonnegative integers that satisfy $1.k_1 + 2.k_2 + \cdots + n.k_n = n$ [4].

Example 4.4. Consider the following FDE of order $\alpha + n$

$$\begin{aligned}(D^{n+\alpha} y)(x) &= cx^{\beta-n} \sqrt{y} \prod_{i=0}^{n-1} (\beta - i) \\ &+ cx^{\beta} \sum \frac{n!}{k_1!k_2! \cdots k_n!} \frac{1}{2}(-\frac{1}{2}) \cdots (\frac{3}{2} - k) y^{\frac{1}{2}-k} \prod_{i=1}^{n} \left(\frac{y^{(i)}}{i!} \right)^{k_i}.\end{aligned}$$

Where $k := k_1 + \cdots + k_n$ and $k_1 + 2k_2 + \cdots + nk_n = n$. If $2(\alpha + \beta) > -1$, then the above FDE has the following solution

$$y(x) = \left[\frac{c\Gamma(\alpha + 2\beta + 1)}{\Gamma(2\alpha + 2\beta + 1)} \right]^2 (x - \alpha)^{2(\alpha+\beta)},$$

where $x > 0$, $c, \alpha, \beta \in \mathbb{R}$,$\alpha > 0$, $c \neq 0$, and $\beta \neq 0, 1, \cdots, n - 1$. The fractional derivative of $y(x)$ of order $\alpha \in \mathbb{R}$ is

$$(D_x^{\alpha} y)(x) = c^2 \left[\frac{\Gamma(\alpha + 2\beta + 1)}{\Gamma(2\alpha + 2\beta + 1)} \right] x^{\alpha+2\beta}.$$

Therefore, $(D_x^\alpha y)(x) = cx^\beta [y(x)]^{\frac{1}{2}}$. By using the Faà di Bruno's formula if $n \in \mathbb{N}$, then

$$\begin{aligned}(D^{n+\alpha} y)(x) &= cx^{\beta-n}\sqrt{y}\prod_{i=0}^{n-1}(\beta - i)\\ &+ cx^\beta \sum \frac{n!}{k_1!k_2!\cdots k_n!}\frac{1}{2}(-\frac{1}{2})\cdots(\frac{3}{2}-k)y^{\frac{1}{2}-k}\prod_{i=1}^{n}\left(\frac{y^{(i)}}{i!}\right)^{k_i}.\end{aligned}$$

Consequently, $y(x)$ is the solution of the above FDE.
The plot of the exact solution is shown in FIGURE 3, For $c = -\frac{1}{3}$, $\alpha = \frac{10}{17}$, $\beta = \frac{3}{2}$.

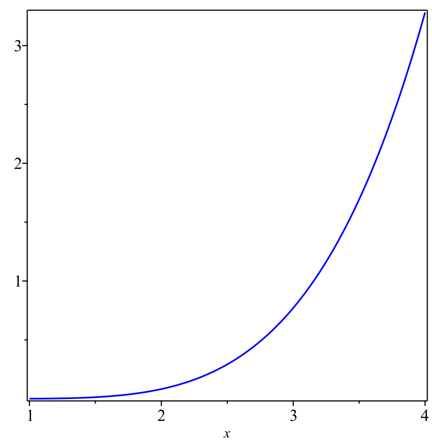

FIGURE 3. Plot of an instance of Example 4.4

Example 4.5. We want to solve the FIE

$$(I_x^\alpha y)(x) = x^{\beta-1} \tag{4.3}$$

where $\alpha, \beta \in \mathbb{R}$ and $\beta \geq \alpha \geq 0$. We apply Laplace transformation to Eq. (4.3). If we apply inverse Laplace transformation to the obtained equation, we can show that

$$y(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}x^{\beta-\alpha-1},$$

is the solution of this FIE.
The plot of the exact solution is shown in FIGURE 4, For $\alpha = \frac{5}{3}$, $\beta = \frac{17}{5}$.

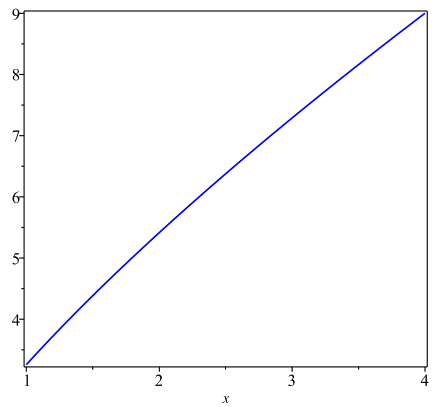

FIGURE 4. Plot of an instance of Example 4.5

Example 4.6. We want to solve the following FDE

$$D^{\alpha} f(x) = af(x) + \frac{x^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)},$$

where $\alpha, \beta \in \mathbb{R}$, $0 < \alpha < 1$, and $\beta \geq \alpha \geq 0$. In a similar argument of Example (4.5), we have

$$f(x) = bx^{\alpha-1} E_{\alpha,\alpha}(ax^2) + x^{\beta-1} E_{\alpha,\beta}(ax^2), \qquad b \in \mathbb{R},$$

as the solution of this FDE.
The plot of the exact solution is shown in FIGURE 5, For $a = 1.5$, $b = \dfrac{17}{5}$, $\alpha = \dfrac{5}{11}$, $\beta = \dfrac{5}{4}$.

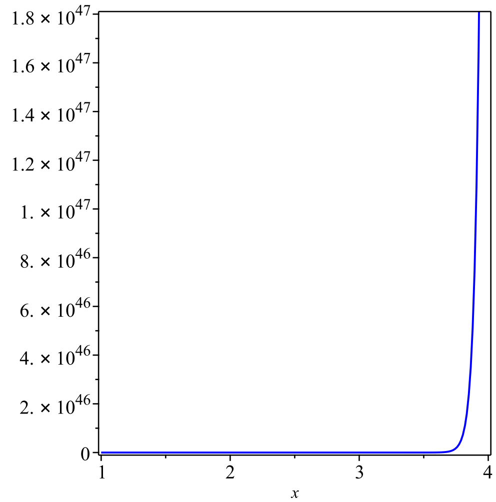

FIGURE 5. Plot of an instance of Example 4.6

REFERENCES

[1] E. K. Alaoui, H. Hmili, and et al., *Cohomological equations and invariant distributions on a compact lie group*, Hokkaido Mathematical Journal, *2* (2014), 151–173.

[2] E. Bazhlekovaand and I. Bazhlekov, *Viscoelastic flows with fractional derivative models: computational approach by convolutional calculus of dimovski*, Fractional Calculus and Applied Analysis, *4* (2014), 954–976.

[3] A. Dehghan-Nezhad and A. E. K. ALAOUI, *Equations cohomologiques de flots riemanniens et de diffeomorphismes danosov*, Journal of the Mathematical Society of Japan, *4* (2007), 1105–1134.

[4] F. F. Di Bruno, *Note sur une nouvelle formule de calcul differentiel*, Quarterly J. Pure Appl. Math, *1* (1857), 359–360.

[5] B. Hasselblatt and A. Katok, *Handbook of dynamical systems* (2002), 287-289.

[6] R. Hilfer and et al., *Applications of fractional calculus in physics*, World scientific Singapore, *12* (2000), 1-87.

[7] A. B. Katok, *Combinatorial constructions in ergodic theory and dynamics*, *30* (2003), 53-117.

[8] A. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science Limited, *204* (2006), 449-469.

[9] A. Livsic, *Homology properties of y-systems*, Mathematical notes of the Academy of Sciences of the USSR, *5* (1971), 69-135.

[10] A. Livsic, *Cohomology of dynamical systems*, Mathematics of the USSR-Izvestiya, *6* (1972).

[11] R. L. Magin, *Fractional calculus in bioengineering*, Begell House Redding, *6* (2006), 1278-1301.

[12] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, (1993), 1-104.

[13] K. Oldham and J. Spanier, *The fractional calculus theory and applications of differentiation and integration to arbitrary order*, Elsevier, (1974), 87-93.

[14] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, (1998), 21-105.

[15] Y. A. Rossikhin and M. V. Shitikova, *Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results*, Applied Mechanics Reviews, *1* (2010), 25-45.
[16] H. Rudolf, *Applications of fractional calculus in physics*, World Scientific, (2000), 41-121.
[17] S. G. Samko, A. A. Kilbas, O. I. Marichev, and et al., *Fractional integrals and derivatives*, Gordon and Breach Science Publishers, (1993), 1-52 .
[18] W. M. Schmidt, *Diophantine approximation*, Springer Science and Business Media, (1996), 331-429.
[19] V. E. Tarasov, *On history of mathematical economics: Application of fractional calculus*, Mathematics, *7* (2019), 26-35.
[20] V. E. Tarasov, *Applications in physics*, Walter de Gruyter GmbH and Co KG, 2019.
[21] M. S. Tavazoei, *Fractional order chaotic systems: history, achievements, applications, and future challenges*, The European Physical Journal Special Topics, *229* (2020), 1-28.
[22] M. Unser, S. Horbelt, and T. Blu, *Fractional derivatives, splines and tomography*, 2000 10th European Signal Processing Conference, (2000), 1-4.