Accelerated fitted operator finite difference method for singularly perturbed parabolic reaction-diffusion problems

Tesfaye Aga Bullo*
Department of Mathematics, Jimma University, Ethiopia.
E-mail: `tesfayeaga2@gmail.com`

Gemechis File Duressa
Department of Mathematics, Jimma University, Ethiopia.
E-mail: `gammeef@gmail.com`

Guy Aymard Degla
Institut De Mathematiques et de sciences physiques,
Universit D'Abomey Calavi, Benin.
E-mail: `gdegla@gmail.com`

Abstract   This paper deals with the numerical treatment of singularly perturbed parabolic reaction-diffusion initial boundary value problems. Introducing a fitting parameter into the asymptotic solution and applying average finite difference approximation, a fitted operator finite difference method is developed for solving the problem. To accelerate the rate of convergence of the method, Richardson extrapolation technique is applied. The consistency and stability of the proposed method have been established very well to ensure the convergence of the method. Numerical experimentation is carried out on some model problems and both the results are presented in tables and graphs. The numerical results are compared with findings of some methods existing in the literature and found to be more accurate. Generally, the formulated method is consistent, stable, and more accurate than some methods existing in the literature for solving singularly perturbed parabolic reaction-diffusion initial boundary value problems.

1. Introduction

The mathematical model of most problems in science regarding the rates of change concerning two or more independent variables, usually time and length leads to partial differential equations. The quantities of attention in many areas of applied mathematics are often to be found as the solution of certain partial differential equations together with prescribed boundary and/or initial conditions. For instance, several problems emerging from real-life phenomena such as systems of differential equations and partial differential equations have been thoroughly discussed and novel numerical

* corresponding author.

methods has been well develoed in this regard. The details of such developments are presented in [5, 8, 12, 13, 14, 15, 16, 17, 27] and references there in. Basic problems related to fractional differential equations are further treated in [1, 2, 3, 4]. Parabolic partial differential equations arise in various branches of science and engineering, such as fluid dynamics, heat flow, diffusion, elastic vibrations etc. These equations are subject to the initial and boundary conditions often occur owing to the nature of certain physical phenomena such as small viscosity in the Navier stokes equations, modeling and analysis of heat and mass transfer process in the thermal conductivity when diffusion coefficients are small and the rate of reaction is large, one can refer [11, 23, 26].

Boundary layers happen in the solution of singularly perturbed problems when the singular perturbation parameter multiplies the terms involving the highest derivatives in the differential equation tends to zero. These boundary layers are the neighborhood of the boundary of the domain, where the solution has a very steep gradient [22]. If one attempts to solve singularly perturbed parabolic initial boundary value problems (IBVPs) using standard numerical methods, then inaccurate solutions are obtained unless the mesh discretization used is extremely fine. Even in this context, careful numerical experiments show that the classical computational methods fail to decrease the maximum point-wise error as the mesh is refined; until the mesh size and the perturbation parameter have the same order of magnitude. Subsequently, the size of the system of algebraic equations will be growing more as the dimension of the problem increases. Hence, this results in the huge computational cost. This drawback motivates the researcher to develop and analyze different numerical methods.

More recently, Gupta *et al.* [10] have established a parameter-uniform numerical method to solve singularly perturbed parabolic problems with two parameters. These authors developed and analyzed the method using asymptotic behavior of the solution and a decomposition of solution into its regular and singular parts. To approximate the solution, they considered the implicit Euler method for time stepping on a uniform mesh and a special hybrid monotone difference operator for spatial discretization on a specially designed piecewise uniform Shishkin mesh. They improved the order of convergence using the Richardson extrapolation technique used in a temporal variable only and the resulting scheme was proved to be uniformly convergent of order two in both the spatial and temporal variables.

As a result, in the past few decades, various uniformly convergent numerical schemes are proposed in the literature for singular perturbation problems (SPPs). The numerical methods for SPPs are broadly classified into fitted operators and fitted mesh methods. In fitted operator methods, exponential fitting factors or artificial viscosity will be used to control the rapid growth or decay of the numerical solution in the boundary layer regions, [18, 19, 21]. While, fitted mesh methods use nonuniform meshes, which will be dense in the boundary layer regions and coarse outside the layer regions. For the reason that small values of the perturbation parameter, the boundary layer may appear to give rise to difficulties when classical methods are applied on a uniform mesh. Moreover, the error in the approximate solution depends on the variable perturbation parameter. An adapted placement of the nodes or artificial viscosity is needed to ensure that the error is independent of the parameter

value and depends only on the number of nodes in the mesh. The discretization with this property is stated as a uniformly convergent numerical method. Here, both fitted operators and fitted mesh methods help to get uniformly convergent numerical methods.

From too many methods had been constructed to find the numerical solution of singularly perturbed parabolic IBVPs, few of the recent works are; parameter uniform numerical method [18], higher-order uniformly convergent method with Richardson extrapolation in time [9, 10]; A fitted numerical method [24], numerical approximation and an iterative technique [20, 25] respectively. From these developed numerical methods, we observe that a large amount of work has been done on singularly perturbed parabolic reaction-diffusion IBVPs as far as designing and analyzing numerical methods for their integration is concerned. In these works, fitted mesh finite difference methods have been adopted, but the obtained numerical solution yet not satisfactory with regards to the order of convergence. Hence, it is necessary to develop stable and convergent methods that produce a more accurate numerical solution with a higher order of convergence to solve singularly perturbed parabolic reaction-diffusion IBVPs. Thus, in this work, we formulate, analyze and implement accelerated fitted operator finite difference methods to solve singularly perturbed parabolic IBVPs and provide its new substantial contribution as the proposed scheme produce a more accurate solution.

2. Formulation of the method

We consider the singularly perturbed parabolic reaction-diffusion IBVP

$$\left(\varepsilon \frac{\partial^2 u}{\partial x^2} - bu - \frac{\partial u}{\partial t}\right)(x,t) = f(x,t), \forall (x,t) \in D := (0,1) \times (0,1]. \tag{2.1}$$

subject to the initial and boundary conditions

$$\begin{cases} u(x,0) = s(x), \forall x \in \bar{\Omega} := [0,1]. \\ u(0,t) = q_0(t), u(1,t) = q_1(t), \forall t \in [0,1], \end{cases} \tag{2.2}$$

where ε is perturbation parameter that satisfies $0 < \varepsilon << 1$ and assume that the coefficient function $b(x,t) \geq \beta > 0$ is sufficiently smooth. Under sufficient smoothness and compatibility conditions imposed on the functions $s(x), q_0(t), q_1(t)$ and $f(x,t)$, the initial-boundary value problem admits a unique solution $u(x,t)$ which exhibits twin boundary layer of width $O(\sqrt{\varepsilon})$ neighboring the boundaries $x = 0$ and $x = 1$.

To formulate the method, let us take the singularly perturbed homogeneous differential equation:

$$\varepsilon \frac{d^2 u}{dx^2} - \beta u = 0, \tag{2.3}$$

subject to the boundary conditions $u(0) = q_0(t), u(1) = q_1(t)$ and its solution for constant C is

$$u(x) = C exp(\sqrt{\frac{\beta}{\varepsilon}} x). \tag{2.4}$$

For ordinary differential equation case, representing the approximate solution $u(x)$ at the gird point x_m by u_m with the mesh size $h = \frac{1}{M}$, we have

$$x_m = mh, m = 0, 1, ..., M;$$

for M been positive integer. The central finite difference approximation for (2.3) is

$$\frac{\varepsilon}{h^2}(u_{m+1} - 2u_m + u_{m-1}) - \beta u_m = 0. \tag{2.5}$$

Introducing the fitting parameter σ on (2.5), denoting $\rho = \frac{h}{\sqrt{\varepsilon}}$ and evaluating limits on both sides of (2.5) yields $\frac{\sigma}{\rho^2} = \frac{\beta \lim\limits_{h \to 0} u_m}{\lim\limits_{h \to 0}(u_{m+1} - 2u_m + u_{m-1})}$ and considering (2.4) on discreet domain of $\bar{\Omega}$, then we get the value of fitting parameter $\sigma = \frac{\beta\rho^2}{exp(\sqrt{\beta}\rho) + exp(-\sqrt{\beta}\rho) - 2}$ which is equal to

$$\sigma = \frac{\beta\rho^2}{4}\left(csch(\frac{\sqrt{\beta}\rho}{4})\right)^2. \tag{2.6}$$

Let N be a positive integer when working on $\bar{D}$, we custom a rectangular grid D_h^k whose nodes are (x_m, t_n) for $0 = x_0 < x_1 < ... < x_M = 1, 0 = t_0 < t_1 < ... < x_N = 1, x_m = mh, m = \frac{1}{M}, m = 0, 1, ..., M, t_n = nk, k = \frac{1}{N}, n = 0, 1, ..., N$. Consequently, let denote the approximate solution $u_m^n \simeq u(x_m, t_n)$ at an arbitrary point (x_m, t_n). To obtain a finite difference scheme, we need to approximate the derivatives in (2.1) after introducing the fitting parameter in the finite difference approximations. Assume that (2.1) is satisfied at $(m, n+\frac{1}{2})^{th}$ - level and with the introduced fitted parameter, which is written as

$$\varepsilon\sigma\frac{\partial^2 u_m^{n+\frac{1}{2}}}{\partial x^2} - b_m^{n+\frac{1}{2}}u_m^{n+\frac{1}{2}} - \frac{\partial u_m^{n+\frac{1}{2}}}{\partial t} = f_m^{n+\frac{1}{2}}. \tag{2.7}$$

For the derivatives concerning t, Taylor series expansion yields:

$$u_m^{n+1} = u_m^{n+\frac{1}{2}} + \frac{k}{2}\frac{\partial u_m^{n+\frac{1}{2}}}{\partial t} + \frac{k^2}{8}\frac{\partial^2 u_m^{n+\frac{1}{2}}}{\partial t^2} + \frac{k^3}{48}\frac{\partial^3 u_m^{n+\frac{1}{2}}}{\partial t^3} + O(k^4). \tag{2.8}$$

$$u_m^n = u_m^{n+\frac{1}{2}} - \frac{k}{2}\frac{\partial u_m^{n+\frac{1}{2}}}{\partial t} + \frac{k^2}{8}\frac{\partial^2 u_m^{n+\frac{1}{2}}}{\partial t^2} - \frac{k^3}{48}\frac{\partial^3 u_m^{n+\frac{1}{2}}}{\partial t^3} + O(k^4). \tag{2.9}$$

Subtracting (2.9) from (2.8), gives the central difference approximation in such a point as

$$\frac{\partial u_m^{n+\frac{1}{2}}}{\partial t} = \frac{u_m^{n+1} - u_m^n}{k} + \tau_1, \tag{2.10}$$

where the truncation term $\tau_1 = -\frac{k^2}{24}\frac{\partial^3 u_m^{n+\frac{1}{2}}}{\partial t^3}$.

Considering all terms of (2.7) except $\frac{\partial u_m^{n+\frac{1}{2}}}{\partial t}$ at the average of nth and $n + 1^{st}$ time level, we have

$$\varepsilon\sigma\frac{\partial^2 u_m^{n+\frac{1}{2}}}{\partial x^2} - b_m^{n+\frac{1}{2}}u_m^{n+\frac{1}{2}} - f_m^{n+\frac{1}{2}} = \frac{L_x^N\left(u_m^{n+1} + u_m^n\right)}{2}. \tag{2.11}$$

where $L_x^N u_m^n = \varepsilon\sigma \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} - b_m^n u_m^n - f_m^n + \tau_2$; for $\tau_2 = -\frac{\varepsilon\sigma h^2}{12}\frac{\partial^4 u_m^n}{\partial x^4}$,

Substituting both (2.10) and (2.11) into (2.7) gives:

$$
\begin{aligned}
\varepsilon\sigma\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2} - b_m^{n+1}u_m^{n+1} + \varepsilon\sigma\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} \\
- b_m^n u_m^n - \frac{2}{k}\left(u_m^{n+1} - u_m^n\right) = f_m^{n+1} + f_m^n + \tau_3,
\end{aligned}
\tag{2.12}
$$

where $\tau_3 = -2\tau_1 - \tau_2$.

This can be re-written as three term recurrence relation

$$
E_m^{n+1}u_{m+1}^{n+1} - F_m^{n+1}u_m^{n+1} + G_m^{n+1}u_{m-1}^{n+1} = H_m^{n+1},
\tag{2.13}
$$

where $E_m^{n+1} = \frac{\varepsilon\sigma}{h^2} = G_m^{n+1}$, $F_m^{n+1} = 2\frac{\varepsilon\sigma}{h^2} + \frac{2}{k} + b_m^{n+1}$,

$$
H_m^{n+1} = f_m^{n+1} + f_m^n - \frac{\varepsilon\sigma}{h^2}\left(u_{m+1}^n u_{m-1}^n\right) + \left(2\frac{\varepsilon\sigma}{h^2} + b_m^n - \frac{2}{k}\right)u_m^n.
$$

Here, equation (2.13) is the tridiagonal system of equations concerning the $x-$direction and the coefficients $E_m^{n+1}, F_m^{n+1}, G_m^{n+1}$ and the right-hand side H_m^{n+1} are given that they satisfy the conditions $|E_m^{n+1}| > 0, |F_m^{n+1}| > 0, |G_m^{n+1}| > 0$ with $|F_m^{n+1}| > |E_m^{n+1}| + |G_m^{n+1}|$ at each $(n+1)th$ level. These situations guarantee that the system is diagonally dominant. Thus, (2.13) can be solved by Thomas algorithm and stable.

3. RICHARDSON EXTRAPOLATION

This technique is a convergence acceleration technique that involves a combination of two computed approximations of a solution. The combination goes out to be a better approximation. Truncation error of the schemes given in (2.10) - (2.12) is $\tau_3 = -2\tau_1 - \tau_2 \equiv O(h^2 + k^2)$.

As h and k closer and closer to zero, the truncation term is also become zero. This implies that the developed numerical method in (2.13) is consistent. Hence, we have

$$
|u(x_m, t_{n+1}) - U_m^{n+1}| \le C(h^2 + k^2),
\tag{3.1}
$$

where $u(x_m, t_{n+1})$ and U_m^{n+1} are exact and approximate solutions respectively, C is constant free from mesh sizes h and k.

Let D_{2M}^{2N} be the mesh found by dividing each mesh interval D_M^N into two and symbolize the calculation of the solution on D_{2M}^{2N} by $\bar{U}_m^{n+1}$. Equation (3.1) works for any $h, k \ne 0$, which implies:

$$
u(x_m, t_{n+1}) - U_m^{n+1} \le C(h^2 + k^2) + R_M^N, \forall(x_m, t_{n+1}) \in D_M^N.
\tag{3.2}
$$

So that, it works for any $\frac{h}{2}, \frac{k}{2} \ne 0$ yields:

$$
u(x_m, t_{n+1}) - \bar{U}_m^{n+1} \le C(\frac{h^2}{4} + \frac{k^2}{4}) + R_{2M}^{2N}, \forall(x_m, t_{n+1}) \in D_{2M}^{2N},
\tag{3.3}
$$

where the remainders, R_M^N and R_{2M}^{2N} are convergent of fourt-order or $O(h^4 + k^4)$. Combination of inequalities in (3.2) and (3.3) leads to $3u(x_m, t_{n+1}) - \left(4\bar{U}_m^{n+1} - U_m^{n+1}\right) \equiv$

$O(h^4 + k^4)$. Hence, we have

$$\left(U_m^{n+1}\right)^{ext} = \frac{1}{3}\left(4\bar{U}_m^{n+1} - U_m^{n+1}\right). \tag{3.4}$$

By means of this approximation to estimate the truncation error, we obtain

$$|u(x_m, t_{n+1}) - \left(U_m^{n+1}\right)^{ext}| \le C(h^4 + k^4), \tag{3.5}$$

where C is free of mesh sizes h and k. Thus, the obtained accelerated fitted operator method is to convergent of fourth-order.

4. Stability and Consistency of the method

The analysis of the proposed method is easily accomplished by the use of Fourier analysis. As authors of the books in [28, 29] provided detail reasons, the Von Neumann stability method is applied to investigate the stability of the developed scheme in (2.13), by assuming that its solution, at the grid point (x_m, t_{n+1}) is given by

$$U_m^n = \xi^n exp(im\theta), \tag{4.1}$$

where $i = \sqrt{-1}, \theta$ is the real number and ξ is the amplitude factor.

Now, putting (4.1) into the homogeneous part of (2.13) yields the amplitude factor,

$$\xi = \frac{-\varepsilon\sigma\left(exp(i\theta) + exp(-i\theta) - 2\right) + h^2 b_m^n}{\varepsilon\sigma\left(exp(i\theta) + exp(-i\theta) - 2\right) - h^2 b_m^{n+1}}.$$

For sufficiently small h, the condition of stability is $|\xi| \le 1$ that can be satisfied,

$$|-\varepsilon\sigma\left(exp(i\theta) + exp(-i\theta) - 2\right)| \le |\varepsilon\sigma\left(exp(i\theta) + exp(-i\theta) - 2\right)|.$$

Therefore, $|\xi| \le 1$. Hence, the scheme given in (2.13) is stable and, we can say the formulated scheme is unconditionally stable.

To investigate the consistency of the method, we have considered both (3.1) and (3.5), then truncation terms vanishes as $h \to 0$ and $k \to 0$. Hence, the scheme is consistent with the order of $O(h^2 + k^2)$ before Richardson extrapolation and order of $O(h^4 + k^4)$ after Richardson extrapolation respectively. Therefore, the constructed scheme is convergent by Laxs equivalence theorem, [6, 7, 28, 29].

5. Numerical Results and Discussion

In this section, we provide numerical examples and results for problems of type (2.1) and (2.2) to validate the applicability of the schemes in Eq. (13) before extrapolation and after extrapolation by (3.4)as follow:

Example 5.1. Consider the singularly perturbed parabolic IBVP

$$\varepsilon\frac{\partial^2 u}{\partial x^2} - (1 + xexp(-t))u(x,t) - \frac{\partial u}{\partial t} = f(x,t), \forall(x,t) \in (0,1) \times (0,1],$$

subject to the conditions $u(x,0) = 0, \forall x \in [0,1], u(0,t) = u(1,t) = 0, \forall t \in [0,1]$. Where the source function $f(x,t)$ is occupied such that the exact solution is

$$u(x,t) = (1 - exp(-t))\left(\frac{exp(-\frac{x}{\sqrt{\varepsilon}}) + exp(-\frac{1-x}{\sqrt{\varepsilon}})}{1 + exp(-\frac{1}{\sqrt{\varepsilon}})} - (cos\pi x)^2\right).$$

For this example, the maximum absolute error evaluated before and after Richardson extrapolation respectively by

$$E_\varepsilon^{M,N} = \max_{(x_m, t_{n+1}) \in \bar{D}_M^N} |u(x_m, t_{n+1}) - u_m^{n+1}|$$

and

$$E_\varepsilon^{M,N} = \max_{(x_m, t_{n+1}) \in \bar{D}_M^N} |u(x_m, t_{n+1}) - (U_m^{n+1})^{ext}|,$$

where $u(x_m, t_{n+1})$ is an exact solution, u_m^{n+1} is approximated solution before extrapolation and $(U_m^{n+1})^{ext}$ is also an approximated solution after Richardson extrapolation. The corresponding order of convergence is determined by

$$P_\varepsilon^{M,N} = \frac{log E_\varepsilon^{M,N} - log E_\varepsilon^{2M,2N}}{log 2}.$$

The numerical results are given below in Tables 1 - 4 and Figures 1 and 2.

TABLE 1. Maximum absolute errors for Example 5.1 at the number of interval $M = N$.

$\varepsilon \downarrow N \rightarrow$	16	32	64	128	256
After					
2^{-5}	2.9109e-06	2.1164e-07	1.3395e-08	8.4080e-10	4.5679e-11
2^{-6}	6.6145e-06	4.1973e-07	2.6385e-08	1.6535e-09	1.0346e-10
2^{-7}	1.2425e-05	8.5139e-07	5.3716e-08	3.3727e-09	2.1095e-10
2^{-8}	2.4780e-05	1.6801e-06	1.0710e-07	6.7530e-09	4.2260e-10
2^{-9}	4.4030e-05	3.1975e-06	2.1164e-07	1.3395e-08	8.4080e-10
Before					
2^{-5}	2.5652e-03	6.4322e-04	1.6092e-04	4.0238e-05	1.0060e-05
2^{-6}	2.1268e-03	5.3666e-04	1.3448e-04	3.3639e-05	8.4110e-06
2^{-7}	1.8440e-03	4.7001e-04	1.1807e-04	2.9555e-05	7.3917e-06
2^{-8}	1.6579e-03	4.3054e-04	1.0867e-04	2.7234e-05	6.8138e-06
2^{-9}	1.5084e-03	4.0599e-04	1.0343e-04	2.5982e-05	6.5044e-06

Example 5.2. Consider the singularly perturbed parabolic IBVP

$$\varepsilon \frac{\partial^2 u}{\partial x^2} - b(x,t)u(x,t) - \frac{\partial u}{\partial t} = f(x,t), \forall (x,t) \in (0,1) \times (0,1]$$

for $b(x,t) = 1 + x^2 + t^2 exp(t)$ and $f(x,t) = exp(t) - 1 + sin(\pi x)$; subject to the conditions

$$u(x,0) = 0, \forall x \in [0,1], u(0,t) = u(1,t) = 0, \forall t \in [0,1].$$

For this example the exact solution is not accessible, so that its maximum absolute error calculated by

$$E_\varepsilon^{M,N} = \max_{(x_m, t_{n+1}) \in \bar{D}_M^N} |(U_{2m}^{2n+1})^{ext} - (U_m^{n+1})^{ext}|.$$
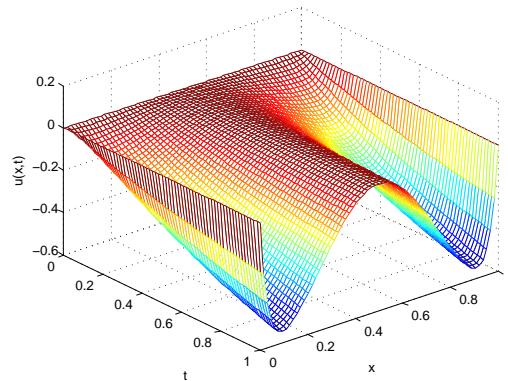
FIGURE 1. Behavior of the numerical solution for Example 5.1 at $M = N = 64$ and $\varepsilon = 2^{-10}$.

TABLE 2. Rate of convergence of the numerical methods for Example 5.1.

$\varepsilon \downarrow N \rightarrow$	8	16	32	64	128
After Extrapolation					
2^{-5}	3.9200	3.7818	3.9818	3.9938	4.2022
2^{-6}	3.9747	3.9781	3.9917	3.9961	3.9984
2^{-7}	3.8639	3.8673	3.9864	3.9934	3.9989
2^{-8}	3.5968	3.8826	3.9715	3.9873	3.9982
2^{-9}	3.4276	3.7835	3.9173	3.9818	3.9938
Before Extrapolation					
2^{-5}	1.9816	1.9957	1.9990	1.9997	1.9999
2^{-6}	1.9461	1.9866	1.9966	1.9992	1.9998
2^{-7}	1.8897	1.9721	1.9930	1.9982	1.9994
2^{-8}	1.7897	1.9451	1.9862	1.9965	1.9989
2^{-9}	1.6125	1.8935	1.9728	1.9931	1.9980

Numerical results are given in Table 5 and Figures 3 and 4.

The results in Tables 1 - 2, depicts the effects of Richardson extrapolation technique on the solution profile and observed that it produce more accurate numerical solutions and a corresponding higher rate of convergence for singularly perturbed parabolic reaction-diffusion IBVPs. Tables 4 and 5 reveals that the proposed method gives a more accurate numerical solution than some existing methods in the literature. Furthermore, to realize another contribution of the method, one can observe results presented in Table 3. Figures 1 and 3 indicates the physical behavior of numerical solutions for Examples 5.1 and 5.2 which have twin boundary layers at each end of the
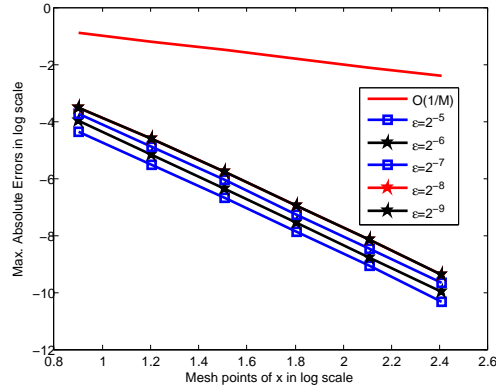
FIGURE 2. Log-log plot of maximum point-wise error of the solution for Example 5.1 (using results in Table 1).

space domain. The numerical solutions obtained by the present method have been log-log plotted in Figures 2 and 4 to indicate that the maximum absolute errors decrease as the number of mesh points increases and maximum absolute errors increase as perturbation parameters decreases.

TABLE 3. Maximum absolute errors obtained with introduced with fitting parameter (W.f.f) and without fitting parameter(W.O.f.f) for Example 5.1 at the number of intervals $M = N$.

$\varepsilon \downarrow N \rightarrow$	16	32	64	128	256
W.f.f					
10^{-2}	9.8018e-06	6.6348e-07	4.1741e-08	2.6219e-09	1.6398e-10
10^{-4}	1.9485e-04	3.3309e-05	3.2604e-06	2.4053e-07	1.6035e-08
10^{-6}	4.8623e-06	5.0916e-06	5.0085e-06	3.0287e-06	6.6473e-07
W.O.f.f					
10^{-2}	2.8316e-05	1.9013e-06	1.2102e-07	7.5984e-09	4.7551e-10
10^{-4}	8.1658e-04	1.2081e-04	3.9369e-04	6.8959e-05	4.8197e-06
10^{-6}	2.2210e-05	8.7943e-05	3.2929e-04	8.1702e-04	2.9043e-04

6. CONCLUSION

The main purpose of this work is to design and investigate accelerated fitted operator finite difference method to solve singularly perturbed parabolic reaction-diffusion initial boundary value problems whose solution exhibits twin boundary layers. By taking the homogeneous ordinary differential equation part of the governing problem and introducing a fitting parameter on the central finite difference approximation, we obtained fitted operator finite difference which in turn gives two-level time direction

TABLE 4. Comparison of maximum absolute errors for Example 5.1.

$\varepsilon\downarrow$	M=128 N=4	M=256 N=8	M=512 N=16	M=1024 N=32
Present Method				
2^{-2}	1.3043e-05	9.4533e-07	6.0175e-08	3.9740e-09
2^{-6}	5.8129e-06	3.3566e-07	1.9870e-08	1.2328e-09
2^{-10}	6.3666e-06	3.9622e-07	2.4571e-08	1.5340e-09
2^{-14}	8.5693e-06	5.9224e-07	3.7924e-08	2.3874e-09
2^{-18}	6.6480e-06	1.8974e-06	4.0403e-07	2.8810e-08
Results in [10]				
2^{-2}	0.956e03	0.382e03	0.131e03	0.401e04
2^{-6}	0.116e02	0.392e03	0.118e03	0.332e04
2^{-10}	0.236e02	0.709e03	0.206e03	0.559e04
2^{-14}	0.268e02	0.794e03	0.231e03	0.626e04
2^{-18}	0.273e02	0.809e03	0.235e03	0.639e04

TABLE 5. :Maximum absolute errors for Example 5.2 and its comparison .

$\varepsilon\downarrow$	M=128 N=4	M=256 N=8	M=512 N=16	M=1024 N=32
Present Method				
2^{-4}	3.9138e-04	9.5343e-05	2.3353e-05	5.8317e-06
2^{-8}	6.5366e-04	1.6276e-04	4.0499e-05	1.0111e-05
2^{-12}	6.6576e-04	1.6709e-04	4.1915e-05	1.0552e-05
2^{-16}	6.6691e-04	3.1446e-04	2.7007e-04	9.6167e-05
Results in [10]				
2^{-4}	0.347e02	0.122e02	0.373e03	0.105e03
2^{-8}	0.433e02	0.143e02	0.420e03	0.114e03
2^{-12}	0.440e02	0.145e02	0.423e03	0.115e03
2^{-16}	0.440e02	0.145e02	0.423e03	0.115e03

and three-term recurrence relations in spatial derivatives that can easily be solved by Thomas algorithm. Then, applying the Richardson extrapolation on the method, we obtain accelerated version of the scheme. Consistency and stability of the proposed method have been established very well and guaranteed that our method is of fourth order convergent. It is evident from the tabular results that the proposed method gives more accurate numerical solution than some others. The results of numerical simulation further confims that the numerical solution obtained is in agreement with the theoretical results that the solution of the problem has twin boundary layers (Figures 1 and 3). In a concise manner, the developed method is consistent, stable and more accurate than some existing methods for solving singularly perturbed parabolic initial boundary value problems.
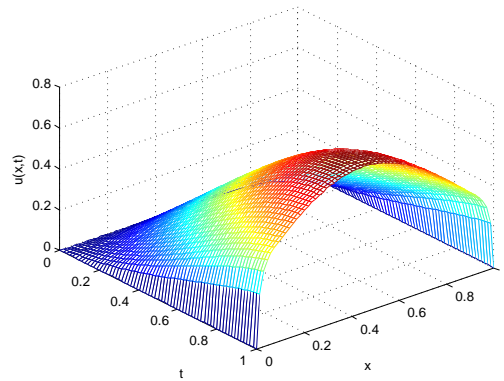
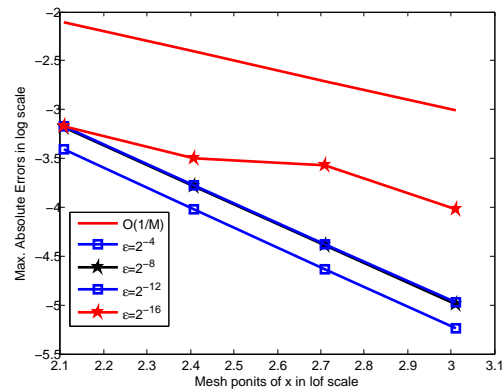FIGURE 3. Behavior of the numerical solution for Example 5.2 at $M = N = 64$ and $\varepsilon = 10^{-4}$.

FIGURE 4. Log-log plot of maximum point-wise error of the solution for Example 5.2.

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