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Conformable double Laplace transform method for solving conformable fractional partial differential equations

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Abstract In the present article, we utilize the Conformable Double Laplace Transform Method (CDLTM) to get the exact solutions of a wide class of Conformable fractional differential in mathematical physics. The results obtained show that the proposed method is efficient, reliable and easy to be implemented on related linear problems in applied mathematics and physics. Moreover, the (CDLTM) has a small computational size as compared to other methods.

Keywords. Conformable fractional derivative (CFD), Partial differential equation (PDE), Caputo fractional derivative, Telegraph equation, Laplace transform.

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1. Introduction

Many applications in modern science are modeled by linear fractional partial differential equations (FPDEs) such as in physics, fluids mechanics, chemistry, biology, mathematics, due to this, solving (FPDEs) attracted the interest of many authors[5, 13, 16]. Several definitions of fractional derivatives have been stated to date, such as Rizez, Riemann-Liouville, Caputo, Hadamard and so on. Two of which are the most popular ones the Riemann-Liouville and Caputo, inspite of the popularity of these fractional derivatives, they have a lot of unusual properties such as all the fractional derivatives do not obey chain rule, product and quotient rule of two functions, these properties lead to some flaws in applications of fractional derivatives in mathematical physics and engineering. Recently, Khalil et. al in (2014)[9] extends the familiar limit definition of the derivative of the function and proposed the so-called Conformable fractional derivative (CFD) which satisfies the basic classical properties of the derivatives. During the past few years, a great deal of interest appears in solving (CFPDEs), so several scientists have been implementing the (CFD) in a lot of applications see [4, 7, 11, 17, 18, 19]. The Laplace transform method [2, 10] is one

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of the most popular method for obtaining the approximate and the exact solutions of (FDEs), due to that many authors still working with great efforts in developing and generalizing this transform to be applicable and consistent with the developed fractional derivatives and integrals. For instance, lately, Abedwljawad in (2015) [1] proposed the single Conformable Laplace transform (CLT) to be applicable to solve (FPDEs) in the Conformable fractional derivative sense, this transform has proven to be a powerful technique to obtain the exact and the approximate solutions of a wide class of (CFDEs) that played a very important role in different fields of science and engineering. After the appearance of single (CLT), Ozan özkan and Ali kurt in (2018) [15] introduced a new generalization of double Laplace transform called conformable double Laplace transform (CDLT) and they implemented it to solve Conformable fractional partial heat equation and Conformable fractional partial Telegraph equation. Recently, there are a very extensive works available on the single (CLT) and there is a very little work available on the (CDLT). So the aim of this study is to implement the (CDLTM) to obtain the exact solutions of a class of (CFDEs) that appears in mathematical physics.

2. Preliminaries

Herein, basic definitions of the conformable fractional derivatives (CFDs) are presented.

Definition 2.1. [1] The (CFD) of a function $\Phi : (0, \infty) \to \Re$ of order ν is defined by:

$$D_x^\nu \Phi\left({}^{x^\nu}\!/_{\nu}\right) = \lim_{\lambda\to 0} \frac{\Phi\left({}^{x^\nu}\!/_{\nu} + \lambda x^{1-\nu}\right) - \Phi\left({}^{x^\nu}\!/_{\nu}\right)}{\lambda}, {}^{x^\nu}\!/_{\nu} > 0, 0 < \nu \le 1.$$

Definition 2.2. [17] The (CFPD) of a function $\Phi\left(\frac{x^\nu}{\nu}, \frac{\tau^\beta}{\beta}\right) : \Re \times (0,\infty) \to \Re$ of order ν is defined by:

$$D_x^\nu \Phi\left({}^{x^\nu}\!/_{\nu}, {}^{\tau^\beta}\!/_{\beta}\right) = \lim_{\lambda\to 0} \frac{\Phi\left({}^{x^\nu}\!/_{\nu} + \lambda x^{1-\nu}, {}^{\tau^\beta}\!/_{\beta}\right) - \Phi\left({}^{x^\nu}\!/_{\nu}, {}^{\tau^\beta}\!/_{\beta}\right)}{\lambda},$$

where, $0 < \nu, \beta \le 1, {}^{x^\nu}\!/_{\nu}, {}^{\tau^\beta}\!/_{\beta} > 0$.

Definition 2.3. [17] The (CFPD) of a function $\Phi\left(\frac{x^\nu}{\nu}, \frac{\tau^\beta}{\beta}\right) : \Re \times (0,\infty) \to \Re$ of order β is given by:

$$D_\tau^\beta \Phi\left({}^{x^\nu}\!/_{\nu}, {}^{\tau^\beta}\!/_{\beta}\right) = \lim_{\gamma\to 0} \frac{\Phi\left({}^{x^\nu}\!/_{\nu}, {}^{\tau^\beta}\!/_{\beta} + \gamma\tau^{1-\beta},\right) - \Phi\left({}^{x^\nu}\!/_{\nu}, {}^{\tau^\beta}\!/_{\beta}\right)}{\gamma}$$

where, $0 < \nu, \beta \le 1, {}^{x^\nu}\!/_{\nu}, {}^{\tau^\beta}\!/_{\beta} > 0$.

Definition 2.4. For$x$,$\beta \in$ C, R$(\nu) > 0$, the Mittag-Leffler formula is given by:

$$E_{\nu,\beta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\nu j + \beta)}.$$

Conformable fractional partial derivatives (CFPD) of certain functions:
$\forall m$, n, c, d,λ,ω $\in$,we have

$$\begin{cases} D_x^\nu\left(\left({x^\nu}/{\nu}\right)\left({\tau^\beta}/{\beta}\right)\right)=\left({\tau^\beta}/{\beta}\right), \\ D_\tau^\beta\left(\left({x^\nu}/{\nu}\right)\left({\tau^\beta}/{\beta}\right)\right)=\left({x^\nu}/{\nu}\right), \end{cases}$$

$$\begin{cases} D_x^\nu\left(\left({x^\nu}/{\nu}\right)^n\left({\tau^\beta}/{\beta}\right)^m\right)=n\left({x^\nu}/{\nu}\right)^{n-\nu}\left({\tau^\beta}/{\beta}\right)^m, \\ D_\tau^\beta\left(\left({x^\nu}/{\nu}\right)^n\left({\tau^\beta}/{\beta}\right)^m\right)=m\left({x^\nu}/{\nu}\right)^r\left({\tau^\beta}/{\beta}\right)^{m-\beta}. \end{cases}$$

$$\begin{cases} D_x^\nu\left(e^{c\left({x^\nu}/{\nu}\right)+d\left({\tau^\beta}/{\beta}\right)}\right)=ce^{c\left({x^\nu}/{\nu}\right)+d\left({\tau^\beta}/{\beta}\right)}, \\ D_\tau^\beta\left(e^{c\left({x^\nu}/{\nu}\right)+d\left({\tau^\beta}/{\beta}\right)}\right)=de^{c\left({x^\nu}/{\nu}\right)+d\left({\tau^\beta}/{\beta}\right)}. \end{cases}$$

$$\begin{cases} D_x^\nu\left(\sin\left(c\left({x^\nu}/{\nu}\right)+d\right)\sin\left(\lambda\left({\tau^\beta}/{\beta}\right)+\omega\right)\right)=c.\cos\left(c\left({x^\nu}/{\nu}\right)+d\right)\sin\left(\lambda\left({\tau^\beta}/{\beta}\right)+\omega\right), \\ D_\tau^\beta\left(\sin\left(c\left({x^\nu}/{\nu}\right)+d\right)\sin\left(\lambda\left({\tau^\beta}/{\beta}\right)+\omega\right)\right)=\lambda\sin\left(c\left({x^\nu}/{\nu}\right)+d\right)\cos\left(\lambda\left({\tau^\beta}/{\beta}\right)+\omega\right). \end{cases}$$

$$\begin{cases} D_x^\rho\left(\cos\left(c\left({x^\nu}/{\nu}\right)+d\right)\cos\left(\lambda\left({\tau^\beta}/{\beta}\right)+\omega\right)\right)=-c.\sin\left(c\left({x^\nu}/{\nu}\right)+d\right)\sin\left(\lambda\left({\tau^\beta}/{\beta}\right)+\omega\right), \\ D_\tau^\beta\left(\cos\left(c\left({x^\nu}/{\nu}\right)+d\right)\cos\left(\lambda\left({\tau^\beta}/{\beta}\right)+\omega\right)\right)=-\lambda\cos\left(c\left({x^\nu}/{\nu}\right)+d\right)\sin\left(\lambda\left({\tau^\beta}/{\beta}\right)+\omega\right). \end{cases}$$

3. Conformable Laplace transforms (CLT)

In this section we present a brief Introduction on (CLT).

Definition 3.1. [8] The (CLT) of a real valued function $\Phi : [0, \infty) \to$ R is defined by:

$$\bar{\Phi}(p)=L_x^\nu\left(\Phi\left(\frac{x^\nu}{\nu}\right)\right)=\int_0^\infty e^{-p\frac{x^\nu}{\nu}}\Phi\left(\frac{x^\nu}{\nu}\right)x^{\nu-1}dx, \forall p\in \mathrm{C}.$$

Definition 3.2. [15] The (CDLT) of a piecewise continuous function $\Phi : [0, \infty)\times[0, \infty) \to$ R is defined by

$$\bar{\Phi}(p,q)=L_x^\nu L_\tau^\beta\left(\Phi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)\right)=\int_0^\infty\int_0^\infty e^{-\left(p\frac{x^\nu}{\nu}+q\frac{\tau^\beta}{\beta}\right)}\Phi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)x^{\nu-1}\tau^{\beta-1}dxd\tau,$$

where, $p, q \in C$ $0 < \nu, \beta \leq 1$.

Theorem 3.3. [15] *The (CDLT) of the, conformable partial fractional derivatives* $\frac{\partial^{n\nu}\Phi}{\partial x^{n\nu}}$, $\frac{\partial^{m\beta}\varphi}{\partial\tau^{m\beta}}$, *of the function* Φ *is given by:*

$$L_x^\nu L_\tau^\beta\left(\frac{\partial^{n\nu}\Phi}{\partial x^{n\nu}}\right)=p^n\bar{\Phi}(p,q)-\sum_{i=0}^{n-1}p^{n-1-i}L_\tau^\beta\left(\frac{\partial^{i\nu}}{\partial x^{i\nu}}\Phi\left(0,\frac{\tau^\beta}{\beta}\right)\right),$$

$$L_x^\nu L_\tau^\beta\left(\frac{\partial^{m\beta}\Phi}{\partial\tau^{m\beta}}\right)=q^m\bar{\Phi}(p,q)-\sum_{i=0}^{m-1}q^{m-1-k}L_x^\nu\left(\frac{\partial^{k\beta}}{\partial\tau^{k\beta}}\varphi\left(\frac{x^\nu}{\nu},0\right)\right),$$

Where, $n, m \in$ N, $0 < \nu, \beta \leq 1$.

Theorem 3.4. *Let* $\Phi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right):\mathrm{R}\times(0,\infty)\to\mathrm{R}$ *be a function where* $\bar{\Phi}(p,q)=L_x^\nu L_\tau^\beta\left(\Phi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)\right)$ *exist, then*

$$L_x^\nu L_\tau^\beta\left(\Phi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)\right)=L_xL_\tau\left(\Phi(x,\tau)\right),$$

where, $L_xL_\tau\left(\Phi(x,\tau)\right)=\int_0^\infty\int_0^\infty e^{-px-q\tau}\Phi(x,\tau)dxd\tau.$

Theorem 3.5. *If* $\bar{\Phi}(p,q)=L_x^\nu L_\tau^\beta\left(\Phi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)\right)$*exist for* $p,q>0$*, then for,* $a,b,r,s\in R$*,we have:*

1. $L_x^\nu L_\tau^\beta(1)=\frac{1}{pq}$,
2. $L_x^\nu L_\tau^\beta\left(\left(\frac{x^\nu}{\nu}\right)^r\left(\frac{\tau^\beta}{\beta}\right)^s\right)=\frac{\Gamma(r+1)\Gamma(s+1)}{p^{r+1}q^{s+1}}$,
3. $L_x^\nu L_\tau^\beta\left(e^{c\left(\frac{x^\nu}{\nu}\right)+d\left(\frac{\tau^\beta}{\beta}\right)}\Phi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)\right)=\bar{\Phi}\left(p-c,q-d\right)$,
4. $L_x^\nu L_\tau^\beta\left(\Phi\left(c\frac{x^\nu}{\nu},d\frac{\tau^\beta}{\beta}\right)\right)=\frac{1}{cd}\bar{\Phi}\left(\frac{p}{c},\frac{q}{d}\right)$,
5. $L_x^\nu L_\tau^\beta\left(c\Phi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)+d\Psi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)\right)=c\bar{\Phi}\left(p,q\right)+d\bar{\Psi}\left(p,q\right)$,
6. $L_x^\nu L_\tau^\beta\left(f\left(\frac{x^\nu}{\nu}\right)g\left(\frac{\tau^\beta}{\beta}\right)\right)=\bar{f}(p)\bar{g}(q).$

Theorem 3.2: *the single (CLT) of* $\left(\frac{x^\nu}{\nu}\right)^r E_{\nu,\beta}\left(c\left(\frac{x^\nu}{\nu}\right)^\nu\right)$ *is given by*

$$L_x^\nu\left(\left(\frac{x^\nu}{\nu}\right)^{r-1}E_{\nu,\beta}\left(c\left(\frac{x^\nu}{\nu}\right)^\nu\right)\right)=\frac{p^{\nu-r}}{p^\nu-c},\ \ |c|<|p^\nu|.$$

4. Basic idea of (CDLTM)

This section illustrate the idea of the proposed approach, so we first consider the general form of a linear (CFPDE) of the form:

$$\sum_{j=0}^{n}A_jD_\tau^{j\beta}\Phi\left(\frac{x^\rho}{\rho},\frac{\tau^\beta}{\beta}\right)+\sum_{i=1}^{m}B_iD_x^{i\nu}\Phi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)=h\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right),\tag{4.1}$$

where, $D_\tau^{j\beta}\Phi$ denotes $j\beta-th$ order of CFPDs, $0<\nu\le1,\ 0<\beta\le1,\ (x,\tau)\in\mathrm{R}_+^2,\ \Phi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)$ and $h\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)$ are given functions, A_j,B_i are constants $\forall j,i$.
Herein, we consider the initial and the boundary conditions as follows

$$D_\tau^{j\beta}\Phi\left(\frac{x^\nu}{\nu},0\right)=\phi_j\left(\frac{x^\nu}{\nu}\right),$$

$$D_x^{i\nu}\Phi\left(0,\frac{\tau^\beta}{\beta}\right)=\varphi_i\left(\frac{\tau^\beta}{\beta}\right),j=0,1,2,...,n-1,\ i=0,1,2,...,m-1.\tag{4.2}$$

Applying the (CDLT) on both sides of the equation (4.1), we get

$$\begin{aligned}&\sum_{j=0}^{n}A_j\left[q^j\bar{\Phi}\left(p,q\right)-\sum_{k=0}^{n-1}q^{n-k-1}L_x^\nu\left(D_\tau^{k\beta}\Phi\left(\frac{x^\nu}{\nu},0\right)\right)\right]\\&+\sum_{i=1}^{m}B_i\left[p^i\bar{\Phi}\left(p,q\right)-\sum_{s=0}^{m-1}p^{m-s-1}L_\tau^\beta\left(D_\tau^{k\beta}\Phi\left(0,\frac{\tau^\beta}{\beta}\right)\right)\right]=\bar{h}(p,q).\end{aligned}\tag{4.3}$$

Applying single (CLT) to equation (4.2), we get

$$L_x^{\nu}\left(D_{\tau}^{k\beta}\Phi\left(\frac{x^{\nu}}{\nu},0\right)\right)=\bar{\phi}_j(p),\quad L_{\tau}^{\beta}\left(D_{\tau}^{k\beta}\Phi\left(0,\frac{\tau^{\beta}}{\beta}\right)\right)=\bar{\varphi}_j(q). \tag{4.4}$$

Substituting (4.4) into (4.3), we get

$$\sum_{j=0}^{n}A_j\left[q^j\bar{\Phi}\left(p,q\right)-\sum_{k=0}^{n-1}q^{n-k-1}\bar{\phi}_j(p)\right]+\sum_{i=1}^{m}B_i\left[p^i\bar{\Phi}\left(p,q\right)-\sum_{s=0}^{m-1}p^{m-s-1}\bar{\varphi}_j(q)\right]=\bar{h}(p,q).$$

Simplifying,

$$\begin{aligned}\bar{\Phi}\left(p,q\right)\left[\sum_{j=0}^{n}q^jA_j+\sum_{i=1}^{m}p^iB_i\right]&=\bar{h}(p,q)+\sum_{j=0}^{n}\left(A_j\sum_{k=0}^{n-1}q^{n-k-1}\bar{\phi}_j(p)\right)\\&+\sum_{i=1}^{m}\left(B_i\sum_{s=0}^{m-1}p^{m-s-1}\bar{\varphi}_j(q)\right),\end{aligned}$$

consequently,

$$\bar{\Phi}\left(p,q\right)=\frac{\bar{h}(p,q)+\sum_{j=0}^{n}\left(A_j\sum_{k=0}^{n-1}q^{n-k-1}\bar{\phi}_j(p)\right)+\sum_{i=1}^{m}\left(B_i\sum_{s=0}^{m-1}p^{m-s-1}\bar{\varphi}_j(q)\right)}{\left[\sum_{j=0}^{n}q^jA_j+\sum_{i=1}^{m}p^iB_i\right]}. \tag{4.5}$$

Taking inverse (CDLT) of (4.5), we get

$$\begin{aligned}\Phi\left(\frac{x^{\nu}}{\nu},\frac{\tau^{\beta}}{\beta}\right)=L_x^{-1}L_{\tau}^{-1}\Bigg(&\frac{\bar{h}(p,q)+\sum_{j=0}^{n}\left(A_j\sum_{k=0}^{n-1}q^{n-k-1}\bar{\phi}_j(p)\right)}{\left[\sum_{j=0}^{n}q^jA_j+\sum_{i=1}^{m}p^iB_i\right]}\\&+\frac{\sum_{i=1}^{m}\left(B_i\sum_{s=0}^{m-1}p^{m-s-1}\bar{\varphi}_j(q)\right)}{\left[\sum_{j=0}^{n}q^jA_j+\sum_{i=1}^{m}p^iB_i\right]}\Bigg),\end{aligned} \tag{4.6}$$

which is the exact solution of linear fractional differential equation (4.1) with respect to the initial and boundary conditions (4.2).

5. Applications

Example 5.1. Consider the Advection- diffusion equation of (CFPDs)

$$\frac{\partial^{\beta}\Phi}{\partial\tau^{\beta}}-\frac{\partial^{2\nu}\Phi}{\partial x^{2\nu}}+\frac{\partial^{\nu}\Phi}{\partial x^{\nu}}=0,\quad 0<\nu,\beta\leq 1, \tag{5.1}$$

and the initial and boundary conditions are given by:

$$\Phi\left(\frac{x^{\rho}}{\rho},0\right)=e^{\frac{x^{\nu}}{\nu}}-\frac{x^{\nu}}{\nu},\quad \Phi\left(0,\frac{\tau^{\beta}}{\beta}\right)=1+\frac{\tau^{\beta}}{\beta},\ \frac{\partial^{\nu}}{\partial x^{\nu}}\Phi\left(0,\frac{\tau^{\beta}}{\beta}\right)=0. \tag{5.2}$$

Applying the (CDLT) to equation (5.1) and single (CLT) to initial and boundary conditions given in (5.2) we get

$$q\bar{\Phi}\left(p,q\right)-\bar{\Phi}\left(p,0\right)-p^2\bar{\Phi}\left(p,q\right)+p\bar{\Phi}\left(0,q\right)+\frac{\partial\bar{\Phi}\left(0,q\right)}{\partial x^{\nu}}+p\bar{\Phi}\left(p,q\right)-\bar{\Phi}\left(0,q\right)=0, \tag{5.3}$$

substituting,

$$\bar{\Phi}(p,0)=\frac{1}{p-1}-\frac{1}{p^2},\ \bar{\Phi}(0,q)=p\left(\frac{1}{q}+\frac{1}{q^2}\right),\ \frac{\partial\bar{\Phi}(0,q)}{\partial x^{\nu}}=0.$$

into equation (5.3) we get

$$\left(q-p^2+p\right)\bar{\Phi}(p,q)=\left(\frac{1}{p-1}-\frac{1}{p^2}\right)-p\left(\frac{1}{q}+\frac{1}{q^2}\right)-\left(\frac{1}{q}+\frac{1}{q^2}\right).$$

Consequently,

$$\Phi\left(\frac{x^{\nu}}{\nu},\frac{\tau^{\beta}}{\beta}\right)=L_x^{-1}L_\tau^{-1}\left[\frac{\left(\frac{1}{p-1}-\frac{1}{p^2}\right)-p\left(\frac{1}{q}+\frac{1}{q^2}\right)-\left(\frac{1}{q}+\frac{1}{q^2}\right)}{(q-p^2+p)}\right], \tag{5.4}$$

simplifying,

$$\Phi\left(\frac{x^{\nu}}{\nu},\frac{\tau^{\beta}}{\beta}\right)=L_x^{-1}L_\tau^{-1}\left[\frac{1}{q\,(p-1)}-\frac{1}{qp^2}+\frac{1}{q^2p}\right]=e^{\frac{x^{\nu}}{\nu}}-\frac{x^{\nu}}{\nu}+\frac{\tau^{\beta}}{\beta}. \tag{5.5}$$

When $\nu=1,$ and $\beta=1,$ the solution of equation (5.1) becomes

$\Phi(x,\tau)=e^{x}-x+\tau,$

which agrees with the solution obtained in [12].

Example 5.2. Consider the following equation of (CFPDs)

$$\frac{\partial^{2\beta}\Phi}{\partial\tau^{\beta}}-\frac{\partial^{2\nu}\Phi}{\partial x^{2\nu}}-2\Phi=-2\sin\left(\frac{x^{\nu}}{\nu}\right)\sin\left(\frac{\tau^{\beta}}{\beta}\right),\quad 0<\nu,\beta\leq 1 \tag{5.6}$$

with respect to the following initial and boundary conditions

$$\begin{array}{l}\Phi\left(\frac{x^{\nu}}{\nu},\,0\right)=0,\quad \frac{\partial^{\tau}}{\partial\tau^{\nu}}\Phi\left(\frac{x^{\nu}}{\nu},0\right)=\sin\left(\frac{x^{\nu}}{\nu}\right),\\ \Phi\left(0,\frac{\tau^{\beta}}{\beta}\right)=0,\quad \frac{\partial^{\nu}}{\partial x^{\nu}}\Phi\left(0,\frac{\tau^{\beta}}{\beta}\right)=\sin\left(\frac{\tau^{\beta}}{\beta}\right).\end{array} \tag{5.7}$$

The single (CLT) of the initial and boundary conditions (5.7) is given by

$$\bar{\Phi}(p,0)=0,\ \bar{\Phi}(0,q)=0,\ \frac{\partial^{\nu}\bar{\Phi}(0,q)}{\partial x^{\nu}}=\frac{1}{1+q^2},\ \frac{\partial^{\beta}\bar{\Phi}(p,0)}{\partial\tau^{\beta}}=\frac{1}{1+p^2}.$$

As explained above in the (CDLTM) in equation (4.6), the solution of equation (5.6) can be written as

$$\Phi\left(\frac{x^{\nu}}{\nu},\frac{\tau^{\beta}}{\beta}\right)=L_x^{-1}L_\tau^{-1}\left[\frac{1}{(q^2-p^2-2)}\left(\frac{1}{1+p^2}-\frac{1}{1+q^2}-\frac{2}{(1+q^2)(1+p^2)}\right)\right], \tag{5.8}$$

simplifying,

$$\Phi\left(\frac{x^{\nu}}{\nu},\frac{\tau^{\beta}}{\beta}\right)=L_x^{-1}L_\tau^{-1}\left[\frac{1}{(1+q^2)(1+p^2)}\right]=\sin\left(\frac{x^{\nu}}{\nu}\right)\sin\left(\frac{\tau^{\beta}}{\beta}\right). \tag{5.9}$$

When $\nu=1,$ and $\beta=1,$ the solution of equation (5.6) becomes

$\Phi(x,\tau)=\sin(x)\sin(\tau).$

Which agrees with the solution obtained in [6].

Example 5.3. Consider the following equation of CFPDs

$$\frac{\partial^{\beta}\Phi}{\partial\tau^{\beta}}+\frac{\partial^{3\nu}\Phi}{\partial x^{3\nu}}+\frac{\partial^{\nu}\Phi}{\partial x^{\nu}}=0,\ \ 0<,\beta\leq 1 \tag{5.10}$$

where the initial and boundary conditions are given by

$$\begin{aligned}&\Phi\left(\tfrac{x^{\nu}}{\nu},\,0\right)=e^{-\frac{x^{\nu}}{\nu}},\ \Phi\left(0,\,\tfrac{\tau^{\beta}}{\beta}\ \right)=e^{2\frac{\tau^{\beta}}{\beta}},\ \tfrac{\partial^{\nu}}{\partial x^{\nu}}\Phi\left(0,\,\tfrac{\tau^{\beta}}{\beta}\ \right)=-e^{2\frac{\tau^{\beta}}{\beta}},\\&\tfrac{\partial^{2\nu}}{\partial x^{2\nu}}\Phi\left(0,\,\tfrac{\tau^{\beta}}{\beta}\ \right)=e^{2\frac{\tau^{\beta}}{\beta}}.\end{aligned} \tag{5.11}$$

Substituting the single (CLT) of the conditions (5.11)

$$\bar{\Phi}\left(p,0\right)=\frac{1}{1+p},\ \bar{\Phi}\left(0,q\right)=\frac{1}{-2+q},\ \frac{\partial^{\nu}\bar{\Phi}\left(0,q\right)}{\partial x^{\nu}}=\frac{-1}{-2+q},\ \frac{\partial^{2\nu}\bar{\Phi}\left(0,q\right)}{\partial x^{2\nu}}=\frac{1}{-2+q},$$

in equation (4.6), and for $n=1$, m $=3$, $A_0=0$, $A_1=1$, $B_1=1$, $B_2=0$, $B_3=1$, and h(x,τ) $=0$.
The solution of equation (5.10) can be written as

$$\Phi\left(\frac{x^{\nu}}{\nu},\frac{\tau^{\beta}}{\beta}\right)=L_x^{-1}L_{\tau}^{-1}\left[\frac{1}{(q+p^3+p)}\left(\frac{1}{1+p}+\frac{p^2}{-2+q}-\frac{p}{-2+q}+\frac{1}{-2+q}+\frac{1}{-2+q}\right)\right], \tag{5.12}$$

simplifying, we get

$$\Phi\left(\frac{x^{\nu}}{\nu},\frac{\tau^{\beta}}{\beta}\right)=L_x^{-1}L_{\tau}^{-1}\left[\frac{1}{\left(-2+q\right)\left(1+p\right)}\right]=e^{2\left(\frac{\tau^{\beta}}{\beta}\right)-\left(\frac{x^{\nu}}{\nu}\right)}. \tag{5.13}$$

In the case $\nu=1$ and $\beta=1$, then the exact solutionof equation (5.10) can be written as $\Phi(x,\tau)=e^{2\tau-x}$, which coicides with the exact solution given in [6].

Example 5.4. Consider the Euler -Bernoulli equation of CFPDs

$$\frac{\partial^{4\nu}\Phi}{\partial x^{4\nu}}+\frac{\partial^{2\beta}\Phi}{\partial\tau^{2\beta}}-\left(\frac{x^{\nu}}{\nu}\right)\left(\frac{\tau^{\beta}}{\beta}\right)-\left(\frac{\tau^{\beta}}{\beta}\right)^2=0. \tag{5.14}$$

With the following initial and boundary conditions

$$\begin{aligned}&\Phi\left(\tfrac{x^{\nu}}{\nu},\,0\right)=0,\ ,\tfrac{\partial^{\tau}}{\partial\tau^{\nu}}\Phi\left(\tfrac{x^{\nu}}{\nu},\,0\right)=\tfrac{1}{120}\left(\tfrac{x^{\nu}}{\nu}\right)^5,\\&\Phi\left(0,\tfrac{\tau^{\beta}}{\beta}\ \right)=\tfrac{1}{12}\left(\tfrac{\tau^{\beta}}{\beta}\right)^4,\ \tfrac{\partial^{r\nu}}{\partial x^{r\nu}}\Phi\left(0,\tfrac{\tau^{\beta}}{\beta}\ \right)=0,\ \text{for},\,r=1,2,3.\end{aligned} \tag{5.15}$$

As explained above, substituting

$$\begin{aligned}&\bar{\Phi}\left(p,0\right)=0,\ \frac{\partial^{\beta}\bar{\Phi}\left(p,0\right)}{\partial\tau^{\beta}}=\frac{1}{p^6},\ \ \bar{\Phi}\left(0,q\right)=\frac{2}{q^5},\ \frac{\partial^{r\nu}}{\partial x^{r\nu}}\bar{\Phi}\left(0,q\ \right)=0,\\&for,\,r=1,2,3.\quad m=4,\ n=2,\ A_0=A_1=0,B_1=B_2=B_3=0,\ B_4=1,\\&and\ \bar{\mathrm{h}}(\mathrm{p,q})=\frac{1}{(pq)^2}+\frac{2}{q^3p},\end{aligned}$$

in equation (4.6), we get the solution (5.14)

$$\Phi\left(\frac{x^{\nu}}{\nu},\frac{\tau^{\beta}}{\beta}\right)=L_x^{-1}L_{\tau}^{-1}\left[\frac{1}{(p^4+q^2)}\left(\frac{1}{p^6}+\frac{2p^2}{q^5}+\frac{1}{(pq)^2}+\frac{2}{q^3p}\right)\right], \tag{5.16}$$

simplifying, we get

$$\Phi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)=L_x^{-1}L_\tau^{-1}\left[\left(\frac{1}{p^6q^2}+\frac{2}{q^5p}\right)\right]=\frac{1}{120}\left(\frac{x^\nu}{\nu}\right)^5\left(\frac{\tau^\beta}{\beta}\right)+\frac{1}{12}\left(\frac{\tau^\beta}{\beta}\right)^4, \tag{5.17}$$

when $\nu=\beta=1$, we get

$$\Phi(x,\tau)=\frac{\tau x^5}{120}+\frac{\tau^4}{12},$$

which is fully compatible with the exact solution obtained by [6].

Example 5.5. Consider the Telegraph equation of CFPDs

$$\frac{\partial^{2\nu}\Phi}{\partial x^{2\nu}}-\frac{\partial^{2\beta}\Phi}{\partial \tau^{2\beta}}-\frac{\partial^{\beta}\Phi}{\partial \tau^{\beta}}-\Phi+\left(\frac{x^\nu}{\nu}\right)^2+\left(\frac{\tau^\beta}{\beta}\right)-1=0. \tag{5.18}$$

With respect to the initial and boundary conditions

$$\begin{aligned}&\Phi\left(\tfrac{x^\nu}{\nu},\ 0\right)=\left(\tfrac{x^\nu}{\nu}\right)^2,\ \tfrac{\partial^\tau}{\partial\tau^\nu}\Phi\left(\tfrac{x^\nu}{\nu},\ 0\right)=1,\\ &\Phi\left(0,\tfrac{\tau^\beta}{\beta}\right)=\left(\tfrac{\tau^\beta}{\beta}\right),\ \tfrac{\partial^\nu}{\partial x^\nu}\Phi\left(0,\tfrac{\tau^\beta}{\beta}\right)=0.\end{aligned} \tag{5.19}$$

By applying the (CDLT) to equation (5.18), and the initial and boundary conditions (5.19)

$$\begin{aligned}&p^2\bar{\Phi}(p,q)-p\bar{\Phi}(0,q)-\frac{\partial^\nu\bar{\Phi}(0,q)}{\partial x^\nu}-q^2\bar{\Phi}(p,q)+q\bar{\Phi}(p,0)+\frac{\partial^\beta\bar{\Phi}(p,0)}{\partial\tau^\beta}\\ &-q\bar{\Phi}(p,q)+\bar{\Phi}(p,0)-\bar{\Phi}(p,q)+\frac{2}{qp^3}+\frac{2}{q^2p}-\frac{1}{pq}=0,\end{aligned} \tag{5.20}$$

substituting,

$$\begin{aligned}&\bar{\Phi}(0,q)=\frac{1}{q^2},\ \frac{\partial^\beta\bar{\Phi}(p,0)}{\partial\tau^\beta}=\frac{1}{p},\ \frac{\partial^\nu}{\partial x^\nu}\bar{\Phi}(0,q)=0,\\ &\bar{\Phi}(p,0)=\frac{2}{p^3}-\frac{2p^{2-3}}{p^2-1}+\frac{2p^{2-1}}{p^2-1}-\frac{2}{p}=\frac{2}{p^3},\end{aligned}$$

in equation (5.20), we get

$$\left(p^2-q^2-q-1\right)\bar{\Phi}(p,q)=\left(\frac{p}{q^2}-\frac{2q}{p^3}-\frac{1}{p}-\frac{2}{p^3}-\frac{2}{p^3q}-\frac{1}{pq^2}+\frac{1}{pq}\right), \tag{5.21}$$

simplifying, we obtain

$$\bar{\Phi}(p,q)=\left(\frac{2}{p^3q}+\frac{1}{pq^2}\right), \tag{5.22}$$

taking the inverse (CDLT), we get

$$\Phi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)=L_x^{-1}L_\tau^{-1}\left[\left(\frac{2}{p^3q}+\frac{1}{pq^2}\right)\right]=\left(\frac{x^\nu}{\nu}\right)^2+\left(\frac{\tau^\beta}{\beta}\right), \tag{5.23}$$

for $\nu=\beta=1$, the exact solution of (5.18) becomes

$$\Phi(x,\tau)=x^2+\tau,$$

which is fully compatible with the solution obtained in [3].

Example 5.6. Consider the Telegraph equation of CFPDs

$$\frac{\partial^{2\beta}\Phi}{\partial\tau^{2\beta}} - \frac{\partial^{2\nu}\Phi}{\partial x^{2\nu}} + \frac{\partial^{\beta}\Phi}{\partial\tau^{\beta}} + \Phi = 0, \tag{5.24}$$

with initial and boundary conditions given as follows

$$\begin{aligned} &\Phi\left(\frac{x^{\nu}}{\nu},\, 0\right) = 0\,,\, \frac{\partial^{\beta}\Phi}{\partial\tau^{\beta}}\left(\frac{x^{\nu}}{\nu},\, 0\right) = Exp\left(\frac{x^{\nu}}{\nu}\right), \\ &\Phi\left(0,\, \frac{\tau^{\beta}}{\beta}\right) = \left(\frac{\tau^{\beta}}{\beta}\right) E_{1,2}\left(\left(\frac{-\tau^{\beta}}{\beta}\right)\right),\ \frac{\partial^{\nu}\Phi}{\partial x^{\nu}}\left(0,\, \frac{\tau^{\beta}}{\beta}\right) = \left(\frac{\tau^{\beta}}{\beta}\right) E_{1,2}\left(\left(\frac{-\tau^{\beta}}{\beta}\right)\right). \end{aligned} \tag{5.25}$$

By substituting $n = 2, m = 2, A_0 = A_1 = A_2 = 1, B_1 = 0, B_2 = -1, h = 0$, and the single (CLT) of the initial and boundary conditions

$$\bar{\Phi}(p,0) = 0,,\ \frac{\partial^{\beta}\bar{\Phi}(p,0)}{\partial\tau^{\beta}} = \frac{1}{p-1}, \bar{\Phi}(0,q) = \frac{\partial^{\nu}}{\partial x^{\nu}}\bar{\Phi}(0,q\ ) = \frac{1}{q^2+q},$$

in equation (4.6), we have

$$\Phi\left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta}\right) = L_x^{-1}L_\tau^{-1}\left[\frac{1}{(q^2-p^2+q+1)}\left(\frac{1}{p-1} - \frac{1}{q^2+q} - \frac{p}{q^2+q}\right)\right],$$

simplifying,

$$\Phi\left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta}\right) = L_x^{-1}L_\tau^{-1}\left[\frac{1}{(p-1)\,(q^2+q)}\right] = \left(\frac{\tau^{\beta}}{\beta}\right) e^{\left(\frac{x^{\nu}}{\nu}\right)} E_{1,2}\left(-\left(\frac{\tau^{\beta}}{\beta}\right)\right), \tag{5.26}$$

when $\nu = \beta = 1$, the exact solution of (5.24) becomes

$$\Phi(x,\tau) = \tau e^{x} E_{1,2}(-\tau)\,, \tag{5.27}$$

which agrees with the solution obtained in [14].

Example 5.7. Consider the following boundary value problem of CFPDs

$$\frac{\partial^{2\nu}\Phi}{\partial x^{2\nu}} - \frac{\partial^{2\beta}\Phi}{\partial\tau^{2\beta}} = \sin\left(\frac{\pi x^{\nu}}{\nu}\right),\quad 0 < \frac{x^{\nu}}{\nu} < 1,\ \ \frac{\tau^{\beta}}{\beta} > 0. \tag{5.28}$$

with initial and boundary conditions given by

$$\Phi\left(\frac{x^{\nu}}{\nu},\, 0^+\right) = 0\,,\, \frac{\partial^{\beta}\Phi}{\partial\tau^{\beta}}\left(\frac{x^{\nu}}{\nu},\, 0^+\right) = 0,\ \Phi\left(0, \frac{\tau^{\beta}}{\beta}\right) = 0,\ \Phi\left(1, \frac{\tau^{\beta}}{\beta}\right) = 0. \tag{5.29}$$

Applying the (CDLT) to equation (5.28) and single (CLT) to initial and boundary conditions given in (5.29) we get

$$p^2\bar{\Phi}(p,q) - p\bar{\Phi}(0,q) - \frac{\partial^{\nu}\bar{\Phi}(0,q)}{\partial x^{\nu}} - q^2\bar{\Phi}(p,q) + q\bar{\Phi}(p,0) + \frac{\partial^{\beta}\bar{\Phi}(p,0)}{\partial\tau^{\beta}} = \frac{\pi}{q\,(\pi^2+p^2)}, \tag{5.30}$$

substituting,

$$\bar{\Phi}(p,0) = 0,\ \bar{\Phi}(0,q) = 0,\ \frac{\partial^{\beta}\bar{\Phi}(p,0)}{\partial\tau^{\beta}} = 0, \tag{5.31}$$

in equation (5.30), we get

$$p^2\bar{\Phi}(p,q) - \frac{\partial^{\nu}\bar{\Phi}(0,q)}{\partial x^{\nu}} - q^2\bar{\Phi}(p,q) = \frac{\pi}{q\,(\pi^2+p^2)}, \tag{5.32}$$

simplifying,

$$\bar{\Phi}(p,q)=\frac{1}{(p^2-q^2)}\frac{\partial^\nu\bar{\Phi}(0,q)}{\partial x^\nu}+\frac{1}{(p^2-q^2)}\frac{\pi}{q(\pi^2+p^2)} \tag{5.33}$$

equation (5.33) can be written in the form

$$\bar{\Phi}(p,q)=\frac{1}{(p^2-q^2)}\left[\frac{\partial^\nu\bar{\Phi}(0,q)}{\partial x^\nu}+\frac{\pi}{q(\pi^2+q^2)}\right]-\frac{1}{(\pi^2+q^2)}\frac{\pi}{q(\pi^2+p^2)}, \tag{5.34}$$

or,

$$\bar{\Phi}(p,q)=\frac{1}{2q}\left[\frac{1}{(p-q)}+\frac{1}{(p+q)}\right]\left[\frac{\partial^\nu\bar{\Phi}(0,q)}{\partial x^\nu}+\frac{\pi}{q(\pi^2+q^2)}\right]-\frac{1}{(\pi^2+q^2)}\frac{\pi}{q(\pi^2+p^2)}. \tag{5.35}$$

Applying the inverse Laplace transform with respect to x, we obtain

$$\bar{\Phi}(x,q)=\frac{1}{2q}\left[e^{q\frac{x^\nu}{\nu}}+e^{-q\frac{x^\nu}{\nu}}\right]\left[\frac{\partial^\nu\bar{\Phi}(0,q)}{\partial x^\nu}+\frac{\pi}{q(\pi^2+q^2)}\right]-\frac{1}{q(\pi^2+q^2)}\sin\left(\frac{\pi x^\nu}{\nu}\right), \tag{5.36}$$

as $\frac{x^\nu}{\nu}\to 1, \bar{\Phi}(x,q)\to 0$,, we have $\frac{\partial^\nu\bar{\Phi}(0,q)}{\partial x^\nu}=\frac{-\pi}{q\left(\pi^2+q^2\right)}$.
Thus,

$$\bar{\Phi}(x,q)=-\frac{1}{q(\pi^2+q^2)}\sin\left(\frac{\pi x^\nu}{\nu}\right), \tag{5.37}$$

applying the inverse Laplace transform with respect to τ , we obtain

$$\Phi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)=\frac{1}{\pi^2}\sin\left(\frac{\pi x^\nu}{\nu}\right)\left[\cos\left(\frac{\pi\tau^\beta}{\beta}\right)-1\right]. \tag{5.38}$$

6. Conclusion

In this article, we have successfully implemented the Conformable Double Laplace transform method (CDLTM) to get the exact solutions of fractional partial differential equations (FPDEs) involving conformable fractional derivatives (CFD) that arise in different areas of real life science. All results show that the (CDLTM) is appropriate, efficient, advantageous, reliable and sufficient to acquire the exact solutions of (CFPDEs). Moreover, as a consequence the calculations involved in (CDLTM) have a small computational size as compared to other methods.

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