



Exact solutions of the combined Hirota-LPD equation with variable coefficients

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Abstract

In this paper, we construct exact families of traveling wave (periodic wave, singular wave, singular-periodic wave, singular-solitary wave and shock wave) solutions of a well-known equation of nonlinear PDE, the variable coefficients combined Hirota-Lakshmanan-Porsezian-Daniel (Hirota-LPD) equation with the fourth nonlinearity, which describes an important development, and application of soliton dispersion management experiment in nonlinear optics is considered, and as an achievement, a series of exact traveling wave solutions for the aforementioned equation is formally extracted. This nonlinear equation is solved by using the extended trial equation method (ETEM) and the improved $\tan(\phi/2)$ -expansion method (ITEM). Meanwhile, the mechanical features of some families are explained through offering the physical descriptions. Analytical treatment to find the nonautonomous rogue waves are investigated for the combined Hirota-LPD equation.

Keywords. Combined Hirota-Lakshmanan-Porsezian-Daniel equation; Nonautonomous rogue wave; Extended trial equation method; Improved $\tan(\phi/2)$ -expansion method.

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1. INTRODUCTION

In recent years, the construction of a special kind of analytical solutions (named traveling wave solutions) for the governing differential equations of many nonlinear physical phenomena has gained remarkable popularity. These phenomena are integrated fields of physical sciences such as optical fibers, plasma physics, ion acoustic plasma waves, mathematical biology, magneto hydro dynamics, fluid mechanics, protein biology, solid state physics, chemical physics, elastic media, etc. In literature, several approaches have been described to acquire the traveling wave solutions of nonlinear governing differential equations. These include the Exp-function method

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[20, 36, 39], the generalized Kudryashov method [61], the extended Jacobi elliptic function expansion method [21], the improved $\tan(\phi/2)$ -expansion method [38], the G'/G -expansion method [49], the Bernoulli sub-equation function method [7], the sine-Gordon expansion method [11], the Riccati equation expansion [62], the rational function transformations [32], the multiple exp-function method [33], the invariant subspace method [35], the formal linearization method [41], the Lie symmetry [54] and many more [14]-[47].

The study of rogue waves has many important applications in some fields such as thunderstorms, earthquakes, and hurricanes. Rogue waves are called extreme waves, which are catastrophic natural physical phenomena [1]. Searching for the rogue waves or rational solutions of the higher-order nonlinear Schrödinger equation is an interesting work, which can describe new physical phenomena [3]. Among the rogue waves, which can generate higher-order rational solutions and can collide with solitons are known soliton waves such as Ma solitons [1] and Akhmediev breathers [19], which are considered as rational solutions and can appear more abundant in new physical phenomena. The studies of rogue waves and solving the higher dimensional equations can help in creating useful rogue waves in optical fibers [50]. Yu [60] has solved a nonautonomous the combined Hirota-LPD equation with variable coefficients through the generalized Darboux transformation and considered the controllable behaviors of the nonautonomous rogue wave solution with the nonlinearity management function and gain/loss coefficient.

The ETEM is one of the robust techniques to look for the exact solutions of nonlinear partial differential equations that has received special interest owing to its fairly great performance. For example, Mohyud-Din and Irshad [44] explored new exact solitary wave solutions of some nonlinear PDEs arising in electronics using the extended trial equation method. Mirzazadeh et al. [42] adopted the extended trial equation method to obtain analytical solutions to the generalized resonant dispersive nonlinear Schrödinger's equation with power law nonlinearity. Ekici et al. [22] found the exact soliton solutions to magneto-optic waveguides that appear with Kerr, power and log-law nonlinearities using the extended trial equation method. The ETEM approach possesses a number of advantages compared with other methods. For further information see references therein ([23, 26]).

Optical solitons form the essential fabric of communications network technology, which is modeled by the nonlinear Schrödinger's equation. One popular model across a few of the well known models that describes the propagation of solitons through a variety of waveguides is the Lakshmanan-Porsezian-Daniel model, which has gained a very significant place in the current research [5]. One form of the LPD model can be considered as a member of extended nonlinear Schrödinger equation family, which has been investigated by known authors [6]. One can state that the LPD equation is an NLS type equation by adding higher order nonlinear terms with NLS equation, which has the second-order dispersion, the fourth-order dispersion, the cubic and quintic nonlinearities. Furthermore, the Hirota-LPD model is as particular case of nonlinear Schrödinger equation with quintic terms. The Hirota-LPD model describes the propagation of solitons through a variety of waveguides.



The outline of this work is as: In section 2, an analysis of the model is given. We present the extended trial equation method in section 3 and its applications to Hirota-LPD equation with variable coefficients are discussed and derived exact solutions. Also, in section 4, the improved $\tan(\phi/2)$ -expansion method and its applications has been presented to solve Hirota-LPD equation. Finally, the conclusion is given in section 5.

2. ANALYSIS OF THE MODEL

We also consider a combined Hirota-LPD equation with variable coefficients and with higher order dispersion, full nonlinearity and spatio-temporal dispersion (STD) was investigated by [56, 57] where is given as

$$\begin{aligned} & i \frac{\partial \Psi}{\partial t} + d_1(t) \frac{\partial^2 \Psi}{\partial x^2} + a(t) |\Psi|^2 \Psi + i d_2(t) \frac{\partial^3 \Psi}{\partial x^3} + i b(t) \frac{\partial \Psi}{\partial x} |\Psi|^2 + \\ & d_3(t) \frac{\partial^4 \Psi}{\partial x^4} + c(t) \frac{\partial^2 \Psi}{\partial x^2} |\Psi|^2 + e(t) \Psi^2 \frac{\partial^2 \Psi^*}{\partial x^2} + f(t) \Psi \left| \frac{\partial \Psi}{\partial x} \right|^2 + \\ & g(t) \Psi^* \left(\frac{\partial \Psi}{\partial x} \right)^2 + h(t) |\Psi|^4 \Psi + i \gamma(t) \Psi = 0, \end{aligned} \quad (2.1)$$

where $\Psi = \Psi(x, t)$, x is the propagation variable and t is the transverse variable. $d_1(t), d_2(t)$ and $d_3(t)$ represent the group-velocity dispersion, third-order dispersion and fourth-order dispersion, respectively. The $a(t)$ is related to the self-steepening, $b(t), c(t), e(t), f(t), f(t)$ and $h(t)$ are the nonlinearity functions, and the gain/loss coefficient $\gamma(t)$ is real-valued function of time. The variable coefficients combined Hirota-LPD equation (2.1) has been extensively investigated, which is an important development and application of soliton dispersion management experiment in nonlinear optics and BECs. Take the following transformations

$$\Psi(x, t) = P(x, t) \exp(i\phi), \quad \phi(x, t) = -kx + wt + \theta, \quad (2.2)$$

where $P(x, t)$ represents the shape of the wave profile, and $\phi(x, t)$ is the phase component, while k is the soliton frequency, and w the wave number, while θ the phase constant and all of them are to be determined. By substituting the (2.2) into Eq. (2.1) and decomposing into real and imaginary parts, we obtain the following results including the real part as

$$\begin{aligned} & \lambda_1 P(x, t) + \lambda_3 P^3(x, t) + \lambda_5 P^5(x, t) + \lambda_6 P(x, t) \left(\frac{dP(x, t)}{dx} \right)^2 + \\ & \lambda_7 \frac{d^2 P(x, t)}{dx^2} + \lambda_8 P^2(x, t) \frac{d^2 P(x, t)}{dx^2} + d_3(t) \frac{d^4 P(x, t)}{dx^4} = 0. \end{aligned} \quad (2.3)$$

By utilizing the transformation $P(x, t) = P(\eta)$ and $\eta = x - \mu t$, Eq. (2.3) is reduced to

$$\lambda_1 P(\eta) + \lambda_3 P^3(\eta) + \lambda_5 P^5(\eta) + \lambda_6 P(\eta) \left(\frac{dP(\eta)}{d\eta} \right)^2 + \lambda_7 \frac{d^2 P(\eta)}{d\eta^2} + \quad (2.4)$$



$$\lambda_8 P^2(\eta) \frac{d^2 P(\eta)}{d\eta^2} + d_3(t) \frac{d^4 P(\eta)}{d\eta^4} = 0,$$

$$\lambda_1 = d_3(t)k^4 - (w + d_1(t)k^2 + d_2(t)k^3), \tag{2.5}$$

$$\lambda_3 = a(t) + b(t)k + (f(t) - e(t) - c(t))k^2, \tag{2.6}$$

$$\lambda_5 = h(t), \tag{2.7}$$

$$\lambda_6 = f(t) + g(t), \tag{2.8}$$

$$\lambda_7 = d_1(t) + 3d_2(t)k - 6d_3(t)k^2, \tag{2.9}$$

$$\lambda_8 = c(t) + e(t), \tag{2.10}$$

with the imaginary part as

$$\frac{\partial P(x, t)}{\partial t} + \gamma(t)P(x, t) + \Lambda_1 P^2(x, t) \frac{\partial P(x, t)}{\partial x} + \Lambda_2 \frac{\partial P(x, t)}{\partial x} + \Lambda_3 \frac{\partial^3 P(x, t)}{\partial x^3} = 0. \tag{2.11}$$

By utilizing the transformation $P(x, t) = P(\eta)$ and $\eta = x - \mu t$, Eq. (2.3) is reduced to

$$\gamma(t)P(\eta) + \Lambda_1 P^2(\eta) \frac{dP(\eta)}{d\eta} + \Lambda_2 \frac{dP(\eta)}{d\eta} + \Lambda_3 \frac{d^3 P(\eta)}{d\eta^3} = 0, \tag{2.12}$$

$$\Lambda_1 = b(t) + 2k(e(t) - g(t) - c(t)), \tag{2.13}$$

$$\Lambda_2 = 4d_3(t)k^3 - (\mu + 2d_1(t)k + 3d_2(t)k^2), \tag{2.14}$$



$$\Lambda_3 = d_2(t) - 4d_3(t)k. \quad (2.15)$$

Once again, we can write Eq. (2.12) as

$$P^2(\eta) \frac{dP(\eta)}{d\eta} = -\frac{1}{\Lambda_1} \left(\gamma(t)P(\eta) + \Lambda_2 \frac{dP(\eta)}{d\eta} + \Lambda_3 \frac{d^3P(\eta)}{d\eta^3} \right). \quad (2.16)$$

By differentiating of Eq. (2.16) we have

$$2P(\eta) \left(\frac{dP(\eta)}{d\eta} \right)^2 + P^2(\eta) \frac{d^2P(\eta)}{d\eta^2} = -\frac{1}{\Lambda_1} \left(\gamma(t) \frac{dP(\eta)}{d\eta} + \Lambda_2 \frac{d^2P(\eta)}{d\eta^2} + \Lambda_3 \frac{d^4P(\eta)}{d\eta^4} \right). \quad (2.17)$$

By supposing $\lambda_6 = 2\lambda_8$ and inserting Eq. (2.17) in Eq. (2.4), we get to

$$\lambda_1 P(\eta) + \lambda_3 P^3(\eta) + \lambda_5 P^5(\eta) - \frac{\lambda_8}{\Lambda_1} \left(\gamma(t) \frac{dP(\eta)}{d\eta} + \Lambda_2 \frac{d^2P(\eta)}{d\eta^2} + \Lambda_3 \frac{d^4P(\eta)}{d\eta^4} \right) + \lambda_7 \frac{d^2P(\eta)}{d\eta^2} + d_3(t) \frac{d^4P(\eta)}{d\eta^4} = 0. \quad (2.18)$$

3. EXTENDED TRIAL EQUATION METHOD

Principal steps of the present method are:

Step 1. Assume that we have the following nonlinear PDE

$$\mathcal{N}(u, u_t, u_x, u_{xx}, \dots) = 0, \quad (3.1)$$

where $u = u(x, t)$ is an indeterminate function and can be converted to an ODE

$$\mathcal{Q}(u, -\mu u', u', u'', \dots) = 0, \quad (3.2)$$

by the transformation $\eta = x - \mu t$, in which μ is constant to be determined later. Putting (3.2) into Eq. (3.1) yields a NODE as,

$$\mathcal{Q}(u, v, u', v', \dots) = 0, \quad (3.3)$$

where prime shows the derivation with respect to ξ .

Step 2. Use the wave transformation

$$u(\eta) = \sum_{i=0}^{\delta} \tau_i \Gamma^i, \quad (3.4)$$



where

$$(\Gamma')^2 = \Omega(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + \dots + \xi_1 \Gamma + \xi_0}{\zeta_\epsilon \Gamma^\epsilon + \dots + \zeta_1 \Gamma + \zeta_0}. \tag{3.5}$$

By employing the above relations (3.4) and (3.5), we obtain

$$(u')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i \tau_i \Gamma^{i-1} \right)^2, \tag{3.6}$$

$$u'' = \frac{\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma)}{2\Psi^2(\Gamma)} \left(\sum_{i=0}^{\delta} i \tau_i \Gamma^{i-1} \right) + \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i(i-1) \tau_i \Gamma^{i-2} \right), \tag{3.7}$$

where $\Phi(\Gamma)$ and $\Psi(\Gamma)$ are polynomials of Γ function. Plugging the above terms into Eq. (3.3) gains an equation of polynomial $\Lambda(\Gamma)$ of Γ :

$$\Lambda(\Gamma) = \varrho_s \Gamma^s + \dots + \varrho_1 \Gamma + \varrho_0 = 0. \tag{3.8}$$

Based on the balance principle on (3.8), we can find a relation of θ, ϵ and δ .

Step 3. Setting each coefficient of polynomial $\Lambda(\Gamma)$ to zero to derive a system of algebraic equations:

$$\varrho_i = 0, \quad i = 1, 2, \dots, s. \tag{3.9}$$

Solving the system (3.9), we will acquire the values of $\xi_0, \xi_1, \dots, \xi_\theta, \zeta_0, \zeta_1, \dots, \zeta_\sigma$ and $\tau_0, \tau_1, \dots, \tau_\delta$.

Step 4. In the continue, we earn the elementary form of the integral by reduction of Eq. (3.5), as follows

$$\pm(\eta - \eta_0) = \int \frac{d\Gamma}{\sqrt{\Omega(\Gamma)}} = \int \sqrt{\frac{\Psi(\Gamma)}{\Phi(\Gamma)}} d\Gamma, \tag{3.10}$$

where η_0 is an free constant.

3.1. Application of ETEM for Hirota-LPD model. By utilizing the balance principle technique, between P^5 and P'''' , we obtain the following relationship for δ, θ , and ϵ :

$$2\delta = \theta - \epsilon - 2. \tag{3.11}$$

For different values of δ, θ , and ϵ , we have the following cases:

Case I: $\delta = 1, \theta = 4$, and $\epsilon = 0$.

If we take $\delta = 1, \theta = 4$, and $\epsilon = 0$ for Eqs. (3.4) and (3.5), then we obtain

$$P(\eta) = \tau_0 + \tau_1 \Gamma,$$



(3.12)

$$(P'(\eta))^2 = \frac{\tau_1^2(\xi_4\Gamma^4 + \xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0)}{\zeta_0}, \quad (3.13)$$

where $\xi_4 \neq 0$ and $\zeta_0 \neq 0$. Solving the algebraic equation system (3.9) yields

• **First set of parameters:**

$$\tau_0 = \frac{1}{4} \frac{\tau_1 \xi_3}{\xi_4}, \quad \tau_1 = \tau_1, \quad \xi_0 = \xi_0, \quad \xi_1 = \frac{\xi_3(4\xi_2\xi_4 - \xi_3^2)}{8\xi_4^2}, \quad (3.14)$$

$$\xi_2 = \xi_2, \quad \xi_3 = \xi_3, \quad \xi_4 = \xi_4, \quad \zeta_0 = \zeta_0,$$

$$k = \frac{24\xi_4^2(d_3b - d_2c - d_2e) + h\zeta_0^2\tau_1^4b}{2h\zeta_0^2\tau_1^4(c + g - e) + 48\xi_4^2d_3(g - c - 3e)}, \quad (3.15)$$

$$\mu = \frac{\Xi}{4\xi_4[h\tau_1^4\zeta_0^2(g + c - e) - 24d_3\xi_4^2(3e - g + c)]^3},$$

$$\begin{aligned} \Xi = & -13824d_3\xi_4^7\Theta_1 - 2880hd_3^2\tau_1^4\xi_4^4\zeta_0\Theta_2 + 6912\zeta_0\tau_1^2\xi_4^6\Theta_3 - 576h\xi_4^5\zeta_0^2\tau_1^4\Theta_4 + \\ & 240h^2d_3\tau_1^8\xi_4^2\zeta_0^3\Theta_5 - 576h\tau_1^6\xi_4^4\zeta_0^3\Theta_6 - 24h^2\xi_4^3\zeta_0^4\tau_1^8\Theta_7 - 5h^3\tau_1^{12}\zeta_0^5\Theta_8 + \\ & 12h^2\tau_1^{10}\zeta_0^5\xi_4^2\Theta_9 + h^3\tau_1^12\zeta_0^6b\xi_4\Theta_{10}, \end{aligned}$$

in which

$$\begin{aligned} \Theta_1 = & -6d_2^3g^2e - 6cd_2^3g^2 - 24bcd_2^2d_3e + 18bcd_2^2d_3g - 19d_2^3e^3c^3d_2^3 \\ & - 15b^2d_2d_3^2g - 15b^2cd_3^2d_2 - 36bd_3^2d_1e^2 - 4bd_3^2d_1g^2 - 6bd_2^2d_3e^2 \\ & + 18cd_2^3eg + 6d_2^2bd_3g^2 - 4d_2^2c^2d_1b + 4d_3d_1g^3d_2 - 72d_3d_1d_2e^3 \\ & + 6bc^2d_3d_2^2 + 15b^2d_2d_3^2e + 10b^3d_3^3 + 4d_3c^2d_1gd_2 - 6bd_2^2d_3eg \\ & - 15cd_2^3e^2 + 3c^2d_2^3e + 21d_2^3e^2g - 3c^2d_2^3g - 24bd_3^2cd_1e - 48d_3cd_1d_2e^2 \\ & - 8d_3cd_1d_2g^2 - 8d_3c^2ed_1d_2 + 84d_3d_1d_2e^2g - 32d_3d_1eg^2d_2 + 24bd_3^2d_1eg \\ & + 8bd_3^2cd_1g + 40d_3cd_1d_2eg, \end{aligned}$$

$$\Theta_2 = (c - g + 3e)^2(-3\xi_3^2 + 8\xi_2\xi_4)(-cd_2 + 2bd_3 - gd_2 + ed_2),$$

$$\begin{aligned} \Theta_3 = & (-cd_2 + 2bd_3 - gd_2 + ed_2)(-d_2^2c^3 + 4c^2ad_3^2 + 4c^2d_2bd_2 - 3c^2d_2^2e \\ & + c^2d_2^2f + 24cd_3^2ae - 8cd_3^2ag - 3b^2cd_3^2 - 2d_3cbd_2g + 12d_3cbd_2e - 2d_3cd_2fb \\ & - 3ce^2d_2^2 + 2cfd_2^2e + 2d_3^2b^2g - 24d_3^2aeg + 36d_3^2ae^2 - 7d_3^2b^2e + d_3^2fb^2 \\ & + 4d_3^2ag^2 - 2bd_3fd_2e + 8bd_3e^2d_2 - 2bd_3d_2eg - e^3d_2^2 + fd_2^2e^2), \end{aligned}$$



$$\begin{aligned} \Theta_4 = & -16ced_1d_2d_3g - 12bcd_2^2d_3e + 24bcd_2^2d_3g - 8d_3c^2gd_1d_2 \\ & - 24bd_2^2d_3eg + 40bd_3^2cd_1e + 32cd_1d_2d_3e^2 + 16d_3c^2ed_1d_2 \\ & + 88d_1d_2d_3e^2g - 48d_1d_3eg^2d_2 + 8bd_3^2d_1eg - 8bd_3^2cd_1g - 3c^3d_2^3 \\ & - 9d_2^3e^3 + 18b^3d_3^3 + 9c^2d_2^3e - 6d_2^3g^2e - 6cd_2^3g^2 + 3cd_2^3e^2 \\ & + 15d_2^3e^2g - 9c^2d_2^3g - 27b^2d_2d_3^2g - 27b^2cd_3^2d_2 + 12bd_3^2d_1e^2 \\ & - 4bd_3^2d_1g^2 + 18bd_2^2d_3e^2 + 6cd_2^3eg + 12bd_2^2d_3g^2 + 12d_3^2c^2d_1b \\ & + 8d_1d_3g^3d_2 - 48d_1d_2d_3e^3 + 18bc^2d_3d_2^2 + 27b^2d_2d_3^2e, \end{aligned}$$

$$\Theta_5 = (c - g + 3e)(c + g - e)(-3\xi_3^2 + 8\xi_2\xi_4)(-cd_2 + 2bd_3 - gd_2 + ed_2),$$

$$\begin{aligned} \Theta_6 = & (-cd_2 + 2bd_3 - gd_2 + ed_2)(-2b^2d_3g + 5b^2ed_3 + b^2d_3c - b^2d_3f \\ & + bcf d_2 - 2be^2d_2 + bcd_2g + bgd_2e + bfd_2e - 2bcd_2e - 4ad_3g^2 - 12ad_3e^2 \\ & + 4ad_3c^2 + 16aed_3g + 8acd_3e), \end{aligned}$$

$$\begin{aligned} \Theta_7 = & -16eg^2d_1d_2 + 4g^3d_1d_2 - 8e^3d_1d_2 + 4c^2gd_1d_2 + 6b^3d_3^2 - 12c^2bd_1d_3 \\ & + 8cg^2d_1d_2 - 8bcgd_1d_3 + 9eb^2d_2d_3 + 9cb^2d_2d_3 + 16ce^2d_1d_2 - 20ge^2d_1d_2 \\ & + 4bg^2d_1d_3 - 12bcd_2^2 + 6d_2^2bg^2 - 8bcd_1d_3 - 24cged_1d_2 - 9b^2gd_2d_3 \\ & + 6bcgd_2^2 - 8ec^2d_1d_2 + 20be^2d_1d_3 - 24begd_1d_3 - 18begd_2^2 + 12bd_2^2e^2, \end{aligned}$$

$$\Theta_8 = (c + g - e)^2(-3\xi_3^2 + 8\xi_2\xi_4)(-cd_2 + 2bd_3 - gd_2 + ed_2),$$

$$\begin{aligned} \Theta_9 = & (-cd_2 + 2bd_3 - gd_2 + ed_2)(4c^2a - 8cae + b^2c + 8cag - 8aeg \\ & + 2b^2g + 4ag^2 + 4ae^2 - 3b^2e + fb^2), \end{aligned}$$

$$\begin{aligned} \Theta_{10} = & -4d_1g^2 - 3bgd_2 - 3bcd_2 - 8cd_1g + 8cd_1e + 8d_1eg \\ & + 2b^2d_3 + 3bed_2 - 4d_1e^2 - 4c^2d_1, \end{aligned}$$

$$w = -\frac{\Pi}{256\xi_4^4(-24\xi_4^2cd_3 - 72\xi_4^2ed_3 + 24\xi_4^2d_3g - h\zeta_0^2\tau_1^4e + h\zeta_0^2\tau_1^4c + h\zeta_0^2\tau_1^4g)^4}, \tag{3.16}$$



$$\begin{aligned}
\Pi = & 10616832hd_3^4\tau_1^4\xi_4^9\Omega_1 + 10616832d_3^2\tau_1^2\xi_2\xi_4^{11}\Omega_2 \\
& - 3649536hd_3^4\tau_1^4\xi_3^4\xi_4^8\Omega_3 - 3981312d_3^2\tau_1^2\xi_3^2\xi_4^{10}\Omega_4 \\
& - 5308416d_3\xi_4^{12}\Omega_5 - 1769472h^2d_3^3\tau_1^8\xi_0^2\xi_4^7\Omega_6 \\
& - 884736d_3\xi_0^2\tau_1^6h\xi_2\xi_4^9\Omega_7 + 608256h^2d_3^3\tau_1^8\xi_3^4\xi_4^6\xi_0^2\Omega_8 \\
& + 331776d_3\xi_0^2\tau_1^6h\xi_3^2\xi_4^8\Omega_9 - [442368h\tau_1^4\xi_4^{10}\xi_0^2\Omega_{10} \\
& + 110592h^3d_3^2\tau_1^{12}\xi_0^4\xi_4^5\Omega_{11} + 18432\xi_0^4\tau_1^{10}h^2\xi_2\xi_4^7\Omega_{12} \\
& - 38016h^3d_3^2\tau_1^{12}\xi_3^4\xi_4^4\xi_0^4\Omega_{13} - 6912\xi_0^4\tau_1^{10}h^2\xi_3^2\xi_4^6\Omega_{14} \\
& - 6912\xi_0^4\tau_1^{10}h^2\xi_3^2\xi_4^6\Omega_{15} - 3072h^4d_3\tau_1^{16}\xi_0^6\xi_4^3\Omega_{16} \\
& - 1536\xi_0^6\tau_1^{14}h^3\xi_2\xi_4^5\Omega_{17} + 1056h^4d_3\tau_1^{16}\xi_3^4\xi_4^2\xi_0^6\Omega_{18} \\
& + 576\xi_0^6\tau_1^{14}h^3\xi_3^2\xi_4^4\Omega_{19} - 768bh^3\tau_1^{12}\xi_4^6\xi_0^6\Omega_{20} \\
& + 128h^5\tau_1^{20}\xi_0^3\xi_4^8\Omega_{21} - 32h^5\tau_1^{20}\xi_2\xi_0^8\xi_4\Omega_{22} \\
& - 11h^5\tau_1^{20}\xi_3^4\xi_0^8\Omega_{23} - 12\xi_0^8\tau_1^{18}h^4\xi_3^2\xi_4^2\Omega_{24} \\
& - 16b^2h^4\xi_0^8\tau_1^{16}\xi_4^4\Omega_{25},
\end{aligned}$$

in which

$$\begin{aligned}
\Omega_1 = & (c - g + 3e)^4(4\xi_0\xi_4^2 - 3\xi_2^2\xi_4 + 2\xi_3^2\xi_2), \\
\Omega_2 = & (c - g + 3e)^2(-d_2^2c^3 + 4c^2ad_3^2 + 4c^2bd_2d_3 - 3c^2d_2^2e \\
& + c^2d_2^2f + 24cd_3^2ae - 8cd_3^2ag - 3cd_3^2b^2 - 2bcgd_2d_3 + 12bcd_2d_3 \\
& - 2bcfd_2d_3 - 3ce^2d_2^2 + 2cef d_2^2 + 2d_3^2b^2g - 24d_3^2aeg + 36d_3^2ae^2 \\
& - 7d_3^2b^2e + d_3^2fb^2 + 4d_3^2ag^2 - 2bfed_2d_3 + 8be^2d_2d_3 - 2begd_2d_3 \\
& - e^3d_2^2 + fd_2^2e^2), \\
\Omega_3 = & (c - g + 3e)^4, \\
\Omega_4 = & (c - g + 3e)^2(-d_2^2c^3 + 4c^2ad_3^2 + 4c^2bd_2d_3 - 3c^2d_2^2e \\
& + c^2d_2^2f + 24cd_3^2ae - 8cd_3^2ag - 3cd_3^2b^2 - 2bcgd_2d_3 + 12bcd_2d_3 \\
& - 2bcfd_2d_3 - 3ce^2d_2^2 + 2cef d_2^2 + 2d_3^2b^2g - 24d_3^2aeg + 36d_3^2ae^2 \\
& - 7d_3^2b^2e + d_3^2fb^2 + 4d_3^2ag^2 - 2bfed_2d_3 + 8be^2d_2d_3 - 2bed_2d_3 \\
& - e^3d_2^2 + fd_2^2e^2), \\
\Omega_5 = & (d_3b - cd_2 - ed_2)^2(d_3^2b^2 - 2bgd_2d_3 + 4bed_2d_3 - 24ced_1d_3 \\
& - 4c^2d_1d_3 - 36d_1d_3e^2 + 24egd_1d_3 + 8cgd_1d_3 - 4d_1d_3g^2 + 2d_2^2eg \\
& - 5d_2^2e^2 - c^2d_2^2 + 2cd_2^2g - 6ced_2^2), \\
\Omega_6 = & (c + g - e)(c - g + 3e)^3(4\xi_0\xi_4^2 - 3\xi_2^2\xi_4 + 2\xi_3^2\xi_2),
\end{aligned}$$



$$\begin{aligned}
 \Omega_7 &= (c - g + 3e)(-d_2^2c^4 + e^4d_2^2 + 2d_3^2b^2fg - 4d_3^2b^2fe \\
 &\quad - 20d_3^2b^2eg - 56d_3^2aeg^2 + 120d_3^2ae^2g - 2c^2d_3^2b^2 - e^3d_2^2g \\
 &\quad - fd_2^2e^3 + 22d_3^2b^2e^2 + 4d_3^2b^2g^2 + 8d_3^2ag^3 - 72d_3^2ae^3 \\
 &\quad - c^3d_2^2g - 2c^3d_2^2e - 14d_3be^3d_2 - 8c^2d_3^2ag + 40c^2d_3^2ae - 3c^2d_2^2eg \\
 &\quad + c^2fd_2^2e + c^2d_2^2fg + 24cd_3^2ae^2 + 4cd_3^2b^2e - 4cd_3^2b^2g + 4c^3bd_2d_3 \\
 &\quad + fd_2^2e^2g + 16cd_3^2aeg - 12bcd_2d_3e^2 - 3cbd_2d_3g^2 + 2cefgd_2^2g \\
 &\quad + 15d_3be^2d_2g + 5d_3bfd_2e^2 - 3bed_2d_3g^2 + 3c^2bgd_2d_3 + 6c^2bed_2d_3 \\
 &\quad - c^2bfd_2d_3 + 16cd_3^2aeg - 12cd_3bd_2e^2 - 3cd_3bd_2g^2 + 2cfed_2^2g), \\
 \Omega_8 &= (c + g - e)(c - g + 3e)^3, \\
 \Omega_9 &= (c - g + 3e)(e^4d_2^2 - 3cd_3bd_2g^2 + 2cfed_2^2g - d_2^2c^4 + 15d_3be^2d_2g \\
 &\quad - 72d_3^2ae^3 - c^3d_2^2g + 16cd_3^2aeg - 12cd_3bd_2e^2 - 3d_3bfd_2g - 3c^2d_2^2eg \\
 &\quad + c^2fd_2^2e - 3d_3bd_2eg^2 + c^2d_2^2fg + 5d_3bfd_2e^2 - 2c^3d_2^2e - 8c^2d_3^2ag \\
 &\quad + 40c^2d_3^2ae + 4c^3d_2bd_3 + fd_2^2e^2g + 2d_3^2b^2fg - 4d_3^2b^2fe \\
 &\quad - 20d_3^2b^2eg - 56d_3^2aeg^2 + 120d_3^2ae^2g - 14d_3be^3d_2 + 2ce^3d_2^2 \\
 &\quad + 8c^3ad_3^2 + c^3d_2^2f - 8cd_3^2ag^2 - cfd_2^2e^2 + 24cd_3^2ae^2 - 3ce^2d_2^2g \\
 &\quad - fd_2^2e^3 - 2c^2d_3^2b^2 - e^3d_2^2g + 4cd_3befd_2 + 4d_3^2b^2g^2 \\
 &\quad + 22d_3^2b^2e^2 + 4cd_3^2b^2g - 8d_3^2ag^3 + 18cd_3bd_2eg - 3cd_3bfd_2g \\
 &\quad + 3c^2d_3bd_2g + 6c^2d_3bd_2e - c^2d_3d_2fb + 4cd_3^2b^2e), \\
 \Omega_{10} &= (d_3b - ed_2 - cd_2)(-4d_1d_2d_3(e + c)(3e + c - g)(c - e + g) \\
 &\quad - 8bd_1d_3^2(-g + 2e)(3e + c - g) - d_2^3(e + c)^2(c - e + g) \\
 &\quad + bd_2^2d_3(e + c)(c - 9e + 5g) + 6b^2ed_2d_3^2 + 2b^3d_3^3 - 2b^2d_2d_2^2(c + 2g)), \\
 \Omega_{11} &= (c - g + 3e)^2(c + g - e)^2(4\xi_0\xi_4^2 - 3\xi_2^2\xi_4 + 2\xi_3^2\xi_2), \\
 \Omega_{12} &= 2bd_2d_3(c + e)(c + g - e)(2c^2 + 3cg + cf - 2ce + 15eg - 3fg \\
 &\quad - 16e^2 + 7fe - 3g^2) - d_2^2(c + e)^2(c + g - e)^2(c - f + e) + 2d_3^2(-96aeg^3 \\
 &\quad + 3b^2fg^2 - b^2c^2f + 11b^2fe^2 + 264ae^2g^2 - 144ace^3 - 288ae^3g \\
 &\quad - 39b^2eg^2 + 15b^2ce^2 + 15b^2c^2e - 3b^2cg^2 - 6b^2c^2g - 24c^2ag^2 \\
 &\quad + 78b^2e^2g + 48c^3ae - 24c^2ae^2 + 108ae^4 - 47b^2e^3 + 12c^4a + 6b^2g^3 \\
 &\quad + 12ag^4 - 2b^2cfe - 12b^2efg + 192ace^2g - 48aceg^2 + b^2c^3 + 96c^2aeg), \\
 \Omega_{13} &= (c - g + 3e)^2(c + g - e)^2,
 \end{aligned}$$



$$\begin{aligned}
\Omega_{14} = \Omega_{15} &= 2bd_2d_3(c+e)(c+g-e)(2c^2+3cg+cf-2ce \\
&+ 15eg-3fg-16e^2+7fe-3g^2) - d_2^2(c+e)^2(c+g-e)^2(c-f+e) \\
&+ 2d_3^2(-96aeg^3+3b^2fg^2-b^2c^2f+11b^2fe^2+264ae^2g^2-144ace^3 \\
&- 288ae^3g-39b^2eg^2+15b^2ce^2+15b^2c^2e-3b^2cg^2-6b^2c^2g \\
&- 24c^2ag^2+78b^2e^2g+48c^3ae-24c^2ae^2+108ae^4-47b^2e^3+12c^4a \\
&+ 6b^2g^3+12ag^4-2b^2cfe-12b^2efg+192ace^2g-48aceg^2+b^2c^3+96c^2aeg), \\
\Omega_{16} &= (c-g+3e)(c+g-e)^3(4\xi_0\xi_4^2-3\xi_2^2\xi_4+2\xi_3^2\xi_2), \\
\Omega_{17} = \Omega_{19} &= (c+g-e)(-bd_2(c+e)(c+g-e)(2e-g-f) \\
&+ 2d_3(-4ag^3+4ac^3-7b^2e^2-b^2fg+c^2b^2+4c^2ag+24aceg \\
&- 2b^2g^2-20ace^2-4acg^2-28ae^2g+20aeg^2+2b^2fe+8b^2eg \\
&+ 12ae^3+2cb^2e+4c^2ae), \\
\Omega_{18} &= (c-g+3e)(c+g-e)^3, \\
\Omega_{20} &= 4d_1d_2(c+e)(c+g-e)^2 + (c+g-e)(8bd_1d_3(2e-g) \\
&+ 3bd_2^2(c+e)-4b^2d_2d_3)+2b^3d_3^2, \\
\Omega_{21} = \Omega_{23} &= (c+g-e)^4, \\
\Omega_{22} &= (c+g-e)^4(3\xi_2\xi_4-2\xi_3^2), \\
\Omega_{24} &= (c+g-e)^2(4ac^2-8ace+cb^2+8acg-8aeg+2b^2g+4ag^2 \\
&+ 4ae^2-3b^2e+fb^2), \\
\Omega_{25} &= -2bcd_2-4d_1g^2-2bd_2g+2bed_2-4c^2d_1+8cd_1e \\
&- 8cd_1g+b^2d_3+8ed_1g-4d_1e^2.
\end{aligned}$$

In the above relations the variable coefficients of Hirota-LPD equation are considered $a = a(t)$, $b = b(t)$, $c = c(t)$, $d_1 = d_1(t)$, $d_2 = d_2(t)$, $d_3 = d_3(t)$, $e = e(t)$, $f = f(t)$, $g = g(t)$, $h = h(t)$, $\gamma = \gamma(t)$. Inserting these findings into Eqs. (3.5) and (3.10), we conclude

$$\pm(\eta - \eta_0) = \int \frac{\sqrt{\frac{\xi_0}{\xi_4}} d\Gamma}{\sqrt{\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_3(4\xi_2\xi_4 - \xi_3^2)}{8\xi_4^3}\Gamma + \frac{\xi_0}{\xi_4}}}. \quad (3.17)$$

Integrating Eq. (3.17), we acquire the solutions for Eq. Hirota-LPD equation in the following forms:

First solution:

$$\pm(\eta - \eta_0) = -\frac{\Pi_0}{\Gamma - \alpha_1}, \quad \Pi_0 = \sqrt{\frac{\xi_0}{\xi_4}}, \quad (3.18)$$



where

$$\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_3(4\xi_2\xi_4 - \xi_3^2)}{8\xi_4^3}\Gamma + \frac{\xi_0}{\xi_4} = (\Gamma - \alpha_1)^4, \tag{3.19}$$

then by solving (3.19) we obtain the following results

$$\alpha_1 = \alpha_1, \quad \xi_0 = \alpha_1^4 \xi_4, \quad \xi_2 = 6\alpha_1^2 \xi_4, \quad \xi_3 = -4\alpha_1 \xi_4, \quad \xi_4 = \xi_4. \tag{3.20}$$

Using (3.20), we can find the exact solution for the Hirota-LPD model (2.1) as:

$$\Gamma = \alpha_1 - \frac{\Pi_0}{\eta - \eta_0} \Rightarrow P_1(x, t) = -\frac{\alpha_1 \Pi_0}{x - \frac{\Xi}{4\xi_4[h\tau_1^4 \zeta_0^2(g+c-e) - 24d_3 \xi_4^2(3e-g+c)]^3} t - \eta_0}, \tag{3.21}$$

$$\Psi_1(x, t) = e^{i\left(-\frac{24\xi_4^2(d_3b-d_2c-d_2e)+h\zeta_0^2\tau_1^4b}{2h\zeta_0^2\tau_1^4(c+g-e)+48\xi_4^2d_3(g-c-3e)}x - \frac{\Pi_1}{\Lambda_1}t + \theta\right)} \tag{3.22}$$

$$\times \left\{ -\frac{\alpha_1 \Pi_0}{x - \frac{\Xi}{4\xi_4[h\tau_1^4 \zeta_0^2(g+c-e) - 24d_3 \xi_4^2(3e-g+c)]^3} t - \eta_0} \right\},$$

$$\Lambda_1 = 256\xi_4^4(-24\xi_4^2cd_3 - 72\xi_4^2ed_3 + 24\xi_4^2d_3g - h\zeta_0^2\tau_1^4e + h\zeta_0^2\tau_1^4c + h\zeta_0^2\tau_1^4g)^4.$$

Second solution:

$$\pm(\eta - \eta_0) = \frac{\Pi_0}{\alpha_1 - \alpha_2} \ln \left| \frac{\Gamma - \alpha_1}{\Gamma - \alpha_2} \right|, \tag{3.23}$$

where

$$\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_3(4\xi_2\xi_4 - \xi_3^2)}{8\xi_4^3}\Gamma + \frac{\xi_0}{\xi_4} = (\Gamma - \alpha_1)^2(\Gamma - \alpha_2)^2, \tag{3.24}$$

then by solving (3.24) we obtain the following results

$$\alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \quad \xi_0 = \alpha_1^2 \alpha_2^2 \xi_4, \quad \xi_2 = (\alpha_2^2 + 4\alpha_1 \alpha_2 + \alpha_1^2) \xi_4, \tag{3.25}$$

$$\xi_3 = -2(\alpha_1 + \alpha_2) \xi_4, \quad \xi_4 = \xi_4.$$

Using (3.25), we can find the exact solution for the Hirota-LPD model as:

$$P_2(x, t) = \frac{1}{2}(\alpha_2 - \alpha_1)\tau_1 + \frac{\tau_1(\alpha_2 - \alpha_1)}{\exp\left(\frac{\alpha_1 - \alpha_2}{\Pi_0} \left[x - \frac{\Xi}{4\xi_4[h\tau_1^4 \zeta_0^2(g+c-e) - 24d_3 \xi_4^2(3e-g+c)]^3} t - \eta_0 \right] \right) - 1}, \tag{3.26}$$



$$\begin{aligned} \Psi_2(x, t) &= e^{i\left(-\frac{24\xi_4^2(d_3b-d_2c-d_2e)+h\xi_0^2\tau_1^4b}{2h\xi_0^2\tau_1^4(c+g-e)+48\xi_4^2d_3(g-c-3e)}x-\frac{\Pi}{\Lambda_2}t+\theta\right)} \\ &\times \left\{ \frac{1}{2}(\alpha_2 - \alpha_1)\tau_1 + \frac{\tau_1(\alpha_2 - \alpha_1)}{\exp\left(\frac{\alpha_1 - \alpha_2}{\Pi_0} \left[x - \frac{\Xi}{4\xi_4[h\tau_1^4\xi_0^2(g+c-e)-24d_3\xi_4^2(3e-g+c)]^3}t - \eta_0 \right] \right) - 1} \right\}, \\ \Lambda_2 &= 256\xi_4^4(-24\xi_4^2cd_3 - 72\xi_4^2ed_3 + 24\xi_4^2d_3g - h\xi_0^2\tau_1^4e + h\xi_0^2\tau_1^4c + h\xi_0^2\tau_1^4g)^4. \end{aligned} \quad (3.27)$$

Third solution:

$$\pm(\eta - \eta_0) = \frac{2\Pi_0}{\alpha_1 - \alpha_2} \sqrt{\frac{\Gamma - \alpha_2}{\Gamma - \alpha_1}}, \quad \alpha_2 > \alpha_1, \quad (3.28)$$

where

$$\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_3(4\xi_2\xi_4 - \xi_3^2)}{8\xi_4^3}\Gamma + \frac{\xi_0}{\xi_4} = (\Gamma - \alpha_1)^2(\Gamma - \alpha_2)^2, \quad (3.29)$$

then by solving (3.29) we obtain the following results

$$\begin{aligned} \alpha_1 &= \alpha_1, \quad \alpha_2 = \alpha_2, \quad \xi_0 = \alpha_1^2\alpha_2^2\xi_4, \quad \xi_2 = (\alpha_2^2 + 4\alpha_1\alpha_2 + \alpha_1^2)\xi_4, \\ \xi_3 &= -2(\alpha_1 + \alpha_2)\xi_4, \quad \xi_4 = \xi_4. \end{aligned} \quad (3.30)$$

Using (3.30), we can find the exact solution for the Hirota-LPD model as:

$$\begin{aligned} P_3(x, t) &= \frac{1}{2}(\alpha_2 - \alpha_1)\tau_1 \\ &+ \frac{4\Pi_0^2\tau_1(\alpha_2 - \alpha_1)}{4\Pi_0^2 - \left((\alpha_1 - \alpha_2) \left[x - \frac{\Xi}{4\xi_4[h\tau_1^4\xi_0^2(g+c-e)-24d_3\xi_4^2(3e-g+c)]^3}t - \eta_0 \right] \right)^2}, \end{aligned} \quad (3.31)$$

$$\begin{aligned} \Psi_3(x, t) &= e^{i\left(-\frac{24\xi_4^2(d_3b-d_2c-d_2e)+h\xi_0^2\tau_1^4b}{2h\xi_0^2\tau_1^4(c+g-e)+48\xi_4^2d_3(g-c-3e)}x-\frac{\Pi}{\Lambda_3}t+\theta\right)} \\ &\times \left\{ \frac{1}{2}(\alpha_2 - \alpha_1)\tau_1 + \frac{4\Pi_0^2\tau_1(\alpha_2 - \alpha_1)}{4\Pi_0^2 - \left((\alpha_1 - \alpha_2) \left[x - \frac{\Xi}{4\xi_4[h\tau_1^4\xi_0^2(g+c-e)-24d_3\xi_4^2(3e-g+c)]^3}t - \eta_0 \right] \right)^2} \right\}, \\ \Lambda_3 &= 256\xi_4^4(-24\xi_4^2cd_3 - 72\xi_4^2ed_3 + 24\xi_4^2d_3g - h\xi_0^2\tau_1^4e + h\xi_0^2\tau_1^4c + h\xi_0^2\tau_1^4g)^4. \end{aligned} \quad (3.32)$$

Fourth solution:

$$\pm(\eta - \eta_0) = \frac{\Pi}{\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}} \ln \left| \frac{\sqrt{(\alpha_1 - \alpha_3)(\Gamma - \alpha_2)} - \sqrt{(\alpha_1 - \alpha_2)(\Gamma - \alpha_3)}}{\sqrt{(\alpha_1 - \alpha_3)(\Gamma - \alpha_2)} + \sqrt{(\alpha_1 - \alpha_2)(\Gamma - \alpha_3)}} \right|, \quad (3.33)$$



where

$$\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_3(4\xi_2\xi_4 - \xi_3^2)}{8\xi_4^3}\Gamma + \frac{\xi_0}{\xi_4} = (\Gamma - \alpha_1)^2(\Gamma - \alpha_2)(\Gamma - \alpha_3), \tag{3.34}$$

then by solving (3.34) we obtain the following results

$$\alpha_1 = \frac{1}{2}(\alpha_2 + \alpha_3), \quad \alpha_2 = \alpha_2, \quad \alpha_3 = \alpha_3, \quad \xi_0 = \frac{1}{4}(\alpha_2 + \alpha_3)^2\alpha_2\alpha_3\xi_4, \tag{3.35}$$

$$\xi_2 = \frac{1}{4}(5\alpha_2^2 + 14\alpha_2\alpha_3 + 5\alpha_3^2)\xi_4, \quad \xi_3 = -2(\alpha_2 + \alpha_3)\xi_4, \quad \xi_4 = \xi_4.$$

Using (3.35), we can find the exact solution for the Hirota-LPD model as:

$$P_4(x, t) = \frac{1}{4}(\alpha_3 - \alpha_2)\tau_1 \sec\left(\frac{(\alpha_3 - \alpha_2)}{2\Pi_0}\left[x - \frac{\Xi}{\Delta_1}t - \eta_0\right]\right), \tag{3.36}$$

$$\Delta_1 = 4\xi_4[h\tau_1^4\xi_0^2(g + c - e) - 24d_3\xi_4^2(3e - g + c)]^3,$$

$$\Psi_4(x, t) = e^{i\left(-\frac{24\xi_4^2(d_3b - d_2c - d_2e) + h\xi_0^2\tau_1^4b}{2h\xi_0^2\tau_1^4(c + g - e) + 48\xi_4^2d_3(g - c - 3e)}x - \frac{\Pi}{\Lambda_4}t + \theta\right)} \tag{3.37}$$

$$\times \left\{ \frac{1}{4}(\alpha_3 - \alpha_2)\tau_1 \sec\left(\frac{(\alpha_3 - \alpha_2)}{2\Pi_0}\left[x - \frac{\Xi}{\Delta_1}t - \eta_0\right]\right) \right\}.$$

$$\Lambda_4 = 256\xi_4^4(-24\xi_4^2cd_3 - 72\xi_4^2ed_3 + 24\xi_4^2d_3g - h\xi_0^2\tau_1^4e + h\xi_0^2\tau_1^4c + h\xi_0^2\tau_1^4g)^4.$$

Fifth solution:

$$\pm(\eta - \eta_0) = \frac{2\Pi_0}{\sqrt{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}F(\varphi, l), \quad \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4, \tag{3.38}$$

where

$$F(\varphi, l) = \int_0^\varphi \frac{d\psi}{\sqrt{1 - l^2 \sin^2 \psi}}, \quad \varphi = \arcsin \sqrt{\frac{(\alpha_2 - \alpha_4)(\Gamma - \alpha_1)}{(\alpha_1 - \alpha_4)(\Gamma - \alpha_2)}}, \tag{3.39}$$

$$l^2 = \frac{(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_4)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)},$$

and

$$\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_3(4\xi_2\xi_4 - \xi_3^2)}{8\xi_4^3}\Gamma + \frac{\xi_0}{\xi_4} = (\Gamma - \alpha_1)(\Gamma - \alpha_2)(\Gamma - \alpha_3)(\Gamma - \alpha_4), \tag{3.40}$$

then by solving (3.40) we obtain the following results

$$\alpha_1 = \alpha_2 - \alpha_3 + \alpha_4, \quad \alpha_2 = \alpha_2, \quad \alpha_3 = \alpha_3, \quad \alpha_4 = \alpha_4, \quad \xi_0 = \alpha_1\alpha_2\alpha_3\alpha_4\xi_4, \tag{3.41}$$

$$\xi_2 = (\alpha_3\alpha_4 + 3\alpha_2\alpha_4 + \alpha_2\alpha_3 + \alpha_4^2 - \alpha_3^2 + \alpha_2^2)\xi_4, \quad \xi_3 = -2(\alpha_2 + \alpha_3)\xi_4, \quad \xi_4 = \xi_4,$$



or

$$\alpha_1 = -\alpha_2 + \alpha_3 + \alpha_4, \quad \alpha_2 = \alpha_2, \quad \alpha_3 = \alpha_3, \quad \alpha_4 = \alpha_4, \quad \xi_0 = \alpha_1 \alpha_2 \alpha_3 \alpha_4 \xi_4, \quad (3.42)$$

$$\xi_2 = (3\alpha_3 \alpha_4 + \alpha_2 \alpha_4 + \alpha_2 \alpha_3 + \alpha_4^2 + \alpha_3^2 - \alpha_2^2) \xi_4, \quad \xi_3 = -2(\alpha_3 + \alpha_4) \xi_4, \quad \xi_4 = \xi_4,$$

or

$$\alpha_1 = \alpha_2 + \alpha_3 - \alpha_4, \quad \alpha_2 = \alpha_2, \quad \alpha_3 = \alpha_3, \quad \alpha_4 = \alpha_4, \quad \xi_0 = \alpha_1 \alpha_2 \alpha_3 \alpha_4 \xi_4, \quad (3.43)$$

$$\xi_2 = (\alpha_3 \alpha_4 + \alpha_2 \alpha_4 + 3\alpha_2 \alpha_3 - \alpha_4^2 + \alpha_3^2 + \alpha_2^2) \xi_4, \quad \xi_3 = -2(\alpha_2 + \alpha_3) \xi_4, \quad \xi_4 = \xi_4.$$

Using (3.41), (3.42), and (3.43), we can find the exact solutions for the Hirota-LPD model as:

$$P_5(x, t) = \frac{1}{2}(\alpha_2 - \alpha_3)\tau_1 + \frac{(\alpha_4 - \alpha_3)(\alpha_4 - \alpha_2)\tau_1}{\alpha_4 - \alpha_2 + (\alpha_2 - \alpha_3)sn^2\Delta_1(x, t)}, \quad (3.44)$$

$$\Delta_1(x, t) = \left[\mp \frac{\sqrt{(\alpha_2 + \alpha_4 - 2\alpha_3)(\alpha_2 - \alpha_4)}}{2\Pi_0} (\eta - \eta_0), \frac{(\alpha_2 - \alpha_3)^2}{(\alpha_2 + \alpha_4 - 2\alpha_3)(\alpha_2 - \alpha_4)} \right],$$

$$\Psi_5(x, t) = e^{i\left(-\frac{24\xi_4^2(d_3b-d_2c-d_2e)+h\zeta_0^2\tau_1^4b}{2h\zeta_0^2\tau_1^4(c+g-e)+48\xi_4^2d_3(g-c-3e)}x - \frac{\Pi}{\Lambda_5}t + \theta\right)} \quad (3.45)$$

$$\times \left\{ \frac{1}{2}(\alpha_2 - \alpha_3)\tau_1 + \frac{(\alpha_4 - \alpha_3)(\alpha_4 - \alpha_2)\tau_1}{\alpha_4 - \alpha_2 + (\alpha_2 - \alpha_3)sn^2\Delta_1(x, t)} \right\},$$

$$\Lambda_5 = 256\xi_4^4(-24\xi_4^2cd_3 - 72\xi_4^2ed_3 + 24\xi_4^2d_3g - h\zeta_0^2\tau_1^4e + h\zeta_0^2\tau_1^4c + h\zeta_0^2\tau_1^4g)^4,$$

$$P_6(x, t) = \frac{1}{2}(2\alpha_2 - \alpha_3 - \alpha_4)\tau_1 + \frac{(\alpha_4 - \alpha_2)(\alpha_3 + \alpha_4 - 2\alpha_2)\tau_1}{\alpha_4 - \alpha_2 + (\alpha_3 - \alpha_2)sn^2\Delta_2(x, t)}, \quad (3.46)$$

$$\Psi_6(x, t) = e^{i\left(-\frac{24\xi_4^2(d_3b-d_2c-d_2e)+h\zeta_0^2\tau_1^4b}{2h\zeta_0^2\tau_1^4(c+g-e)+48\xi_4^2d_3(g-c-3e)}x - \frac{\Pi}{\Lambda_5}t + \theta\right)} \quad (3.47)$$

$$\times \left\{ \frac{1}{2}(2\alpha_2 - \alpha_3 - \alpha_4)\tau_1 + \frac{(\alpha_4 - \alpha_2)(\alpha_3 + \alpha_4 - 2\alpha_2)\tau_1}{\alpha_4 - \alpha_2 + (\alpha_3 - \alpha_2)sn^2\Delta_2(x, t)} \right\},$$

$$\Delta_2(x, t) = \left[\mp \frac{\sqrt{-(\alpha_2 - \alpha_4)^2}}{2\Pi_0} (\eta - \eta_0), \frac{-(\alpha_2 - \alpha_3)^2}{(\alpha_2 - \alpha_4)^2} \right],$$

and

$$P_7(x, t) = \frac{1}{2}(\alpha_2 - \alpha_3)\tau_1 + \frac{(\alpha_3 - \alpha_4)(\alpha_4 - \alpha_2)\tau_1}{\alpha_4 - \alpha_2 + (\alpha_2 + \alpha_3 - 2\alpha_4)sn^2\Delta_3(x, t)}, \quad (3.48)$$

$$\Psi_7(x, t) = e^{i\left(-\frac{24\xi_4^2(d_3b-d_2c-d_2e)+h\zeta_0^2\tau_1^4b}{2h\zeta_0^2\tau_1^4(c+g-e)+48\xi_4^2d_3(g-c-3e)}x - \frac{\Pi}{\Lambda_5}t + \theta\right)} \quad (3.49)$$



$$\begin{aligned} & \times \left\{ \frac{1}{2}(\alpha_2 - \alpha_3)\tau_1 + \frac{(\alpha_3 - \alpha_4)(\alpha_4 - \alpha_2)\tau_1}{\alpha_4 - \alpha_2 + (\alpha_2 + \alpha_3 - 2\alpha_4)s\eta^2\Delta_3(x, t)} \right\}, \\ \Delta_3(x, t) &= \left[\mp \frac{\sqrt{(\alpha_2 - \alpha_4)^2}}{2\Pi_0} (\eta - \eta_0), \frac{(\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_4)}{(\alpha_2 - \alpha_4)^2} \right]. \end{aligned}$$

Remark 1. If the modulus $l \rightarrow 0$, then the solution (3.49), can be converted into the compacton solution

$$\begin{aligned} \Psi_8(x, t) &= e^{i\left(-\frac{24\xi_4^2(d_3b-d_2c-d_2e)+h\zeta_0^2\tau_1^4b}{2h\zeta_0^2\tau_1^4(c+g-e)+48\xi_4^2d_3(g-c-3e)}x - \frac{\Pi}{\Lambda_5}t + \theta\right)} \\ & \times \left\{ \frac{(\alpha_3 - \alpha_4)(\alpha_4 - \alpha_3)\tau_1}{\alpha_4 - \alpha_3 + 2(\alpha_3 - \alpha_4)\sin^2\left[\mp \frac{\sqrt{(\alpha_3 - \alpha_4)^2}}{2\Pi_0} (\eta - \eta_0)\right]} \right\}, \end{aligned} \tag{3.50}$$

where $\alpha_2 = \alpha_3$.

4. DESCRIPTION OF THE IMPROVED $\tan(\phi(\eta)/2)$ -EXPANSION METHOD

Principal steps of the present method are:

Step 1. Assume that we have the following nonlinear PDE

$$\mathcal{N}(u, u_t, u_x, u_{xx}, \dots) = 0, \tag{4.1}$$

where $u = u(x, t)$ is an indeterminate function and can be converted to an ODE

$$\mathcal{Q}(u, -\mu u', u', u'', \dots) = 0, \tag{4.2}$$

by the transformation $\eta = x - \mu t$, in which μ is constant to be determined later.

Step 2. Assume that the traveling wave solution of Eq. (4.2) has the following form:

$$u(\eta) = \sum_{k=0}^M \xi_k \tan^k\left(\frac{\phi(\eta)}{2}\right) + \sum_{k=1}^M \zeta_k \cot^k\left(\frac{\phi(\eta)}{2}\right), \tag{4.3}$$

where $\xi_k (0 \leq k \leq M), \zeta_k (1 \leq k \leq M)$ are constants to be determined, such that $\xi_M \neq 0, \zeta_M \neq 0$, and $\phi = \phi(\eta)$ satisfies the following ordinary differential equation:

$$\phi'(\eta) = A \sin(\phi(\eta)) + B \cos(\phi(\eta)) + C. \tag{4.4}$$

Step 3. To find the value of M , the balancing principle is used. Moreover, precisely, we define the degree of $u(\eta)$ as $D(u(\eta)) = M$, which gives rise to degree of another expression as follows:

$$D\left(\frac{d^q u}{d\eta^q}\right) = M + q, \quad D\left(u^p \left(\frac{d^q u}{d\eta^q}\right)^s\right) = Mp + s(M + q). \tag{4.5}$$



Step 4. Using (4.3) into Eq. (4.2) and comparing each coefficients of $\tan(\phi/2)^k$, $\cot(\phi/2)^k$ ($k = 0, 1, 2, \dots, M$), to zero, we attain a system of algebraic equations in terms of ξ_0, ξ_k, ζ_k ($k = 1, 2, \dots, M$) A, B , and C with the aid of symbolic computation Maple. Solving the algebraic equations, then substituting $\xi_0, \xi_1, \dots, \xi_M, \zeta_1, \dots, \zeta_M, \mu$ in (4.3).

Consider the following special solutions of equation (4.4):

Family 1: When $\Delta = A^2 + B^2 - C^2 < 0$ and $B - C \neq 0$, then

$$\phi(\eta) = 2 \arctan \left[\frac{A}{B-C} - \frac{\sqrt{-\Delta}}{B-C} \tan \left(\frac{\sqrt{-\Delta}}{2} \tilde{\eta} \right) \right].$$

Family 2: When $\Delta = A^2 + B^2 - C^2 > 0$ and $B - C \neq 0$, then

$$\phi(\eta) = 2 \arctan \left[\frac{A}{B-C} + \frac{\sqrt{\Delta}}{B-C} \tanh \left(\frac{\sqrt{\Delta}}{2} \tilde{\eta} \right) \right].$$

Family 3: When $A^2 + B^2 - C^2 > 0$, $B \neq 0$ and $C = 0$, then

$$\phi(\eta) = 2 \arctan \left[\frac{A}{B} + \frac{\sqrt{B^2+A^2}}{B} \tanh \left(\frac{\sqrt{B^2+A^2}}{2} \tilde{\eta} \right) \right].$$

Family 4: When $A^2 + B^2 - C^2 < 0$, $C \neq 0$ and $B = 0$, then

$$\phi(\eta) = 2 \arctan \left[-\frac{A}{C} + \frac{\sqrt{C^2-A^2}}{C} \tan \left(\frac{\sqrt{C^2-A^2}}{2} \tilde{\eta} \right) \right].$$

Family 5: When $A^2 + B^2 - C^2 > 0$, $B - C \neq 0$ and $A = 0$, then $\phi(\eta) = 2 \arctan \left[\sqrt{\frac{B+C}{B-C}} \tanh \left(\frac{\sqrt{B^2-C^2}}{2} \tilde{\eta} \right) \right]$.

Family 6: When $A = 0$ and $C = 0$, then $\phi(\eta) = \arctan \left[\frac{e^{2B\tilde{\eta}}-1}{e^{2B\tilde{\eta}}+1}, \frac{2e^{B\tilde{\eta}}}{e^{2B\tilde{\eta}}+1} \right]$.

Family 7: When $B = 0$ and $C = 0$, then $\phi(\eta) = \arctan \left[\frac{2e^{A\tilde{\eta}}}{e^{2A\tilde{\eta}}+1}, \frac{e^{2A\tilde{\eta}}-1}{e^{2A\tilde{\eta}}+1} \right]$.

Family 8: When $A^2+B^2 = C^2$, then $\phi(\eta) = -2 \arctan \left[\frac{(B+C)(A\tilde{\eta}+2)}{A^2\tilde{\eta}} \right]$.

Family 9: When $C = A$, then $\phi(\eta) = -2 \arctan \left[\frac{(A+B)e^{B\tilde{\eta}}-1}{(A-B)e^{B\tilde{\eta}}-1} \right]$.

Family 10: When $C = -A$, then $\phi(\eta) = 2 \arctan \left[\frac{e^{B\tilde{\eta}}+B-A}{e^{B\tilde{\eta}}-B-A} \right]$.

Family 11: When $B = -C$, then $\phi(\eta) = -2 \arctan \left[\frac{Ae^{A\tilde{\eta}}}{Ce^{A\tilde{\eta}}-1} \right]$.

Family 12: When $B = 0$ and $A = C$, then $\phi(\eta) = -2 \arctan \left[\frac{C\tilde{\eta}+2}{C\tilde{\eta}} \right]$.

Family 13: When $A = 0$ and $B = C$, then $\phi(\eta) = 2 \arctan [C\tilde{\eta}]$.

Family 14: When $A = 0$ and $B = -C$, then $\phi(\eta) = -2 \arctan \left[\frac{1}{C\tilde{\eta}} \right]$.

Family 15: When $A = 0$ and $B = 0$, then $\phi(\eta) = C\tilde{\eta}$.

Family 16: When $B = C$ then $\phi(\eta) = 2 \arctan \left[\frac{e^{A\tilde{\eta}}-C}{A} \right]$,

in which $\tilde{\eta} = \eta + C_0$ and C_0 is a free constant.

4.1. Application of ITEM for Hirota-LPD model. By employing the ITEM for Eq. (2.18) and by balancing P^5 and P'''' in Eq. (2.18) we get the balance number $M = 1$, and the exact solution yields

$$u(\eta) = \sum_{j=0}^1 \xi_j \tan^j(\phi/2) + \sum_{j=1}^1 \zeta_j \cot^j(\phi/2).$$

(4.6)



Plugging (4.6) into Eq. (2.18) and comparing the terms, we will reach a system of nonlinear algebraic equations, and by solving system of the nonlinear equations, reads

Case 1:

$$A = 0, B = C, C = C, k = \frac{b}{2(g+c-e)}, \mu = \mu, \omega = \omega, \xi_0 = \xi_0, \xi_1 = \xi_1, \zeta_1 = 0. \tag{4.7}$$

By using (4.6) and (4.7), the exact solution for the Hirota-LPD equation becomes,

$$\Psi_1(x, t) = e^{i\left(-\frac{b}{2(g+c-e)}x + \omega t + \theta\right)} (\xi_0 + \xi_1 C(x - \mu t)), \tag{4.8}$$

when $B = C$ (**Family 13**).

Case 2:

$$A = A, B = C, C = C, k = \frac{b}{2(g+c-e)}, \omega = \omega, \xi_0 = \xi_0, \xi_1 = \xi_1, \zeta_1 = 0, \tag{4.9}$$

$$\mu = -\frac{4bd_1\beta_1^2 - d_2\beta_1(4A^2\beta_1^2 - 3b^2) + 2bd_3[4A^2\beta_1^2 - b^2]}{8bd_3[4A^2\beta_1^2 - b^2]},$$

$$\beta_1 = g - e + c.$$

By using (4.6) and (4.9), the exact solution for the Hirota-LPD equation gets,

$$\Psi_2(x, t) = e^{i\left(-\frac{b}{2(g+c-e)}x + \omega t + \theta\right)} \left\{ \xi_0 + \xi_1 \tan\left(\frac{1}{2} \arctan\left[\frac{2e^{A\tilde{\eta}}}{e^{2A\tilde{\eta}} + 1}, \frac{e^{2A\tilde{\eta}} - 1}{e^{2A\tilde{\eta}} + 1}\right]\right) \right\}, \tag{4.10}$$

in which

$$\tilde{\eta} = x + \frac{4bd_1\beta_1^2 - d_2\beta_1(4A^2\beta_1^2 - 3b^2) + 2bd_3[4A^2\beta_1^2 - b^2]}{8bd_3[4A^2\beta_1^2 - b^2]}t + C_0,$$

where is obtained by **Family 7**.

By using (4.6) and (4.9), the exact solution for the Hirota-LPD equation gets,

$$\Psi_3(x, t) = e^{i\left(-\frac{b}{2(g+c-e)}x + \omega t + \theta\right)} \left\{ \xi_0 + \xi_1 \left[\frac{e^{-A\tilde{\eta}} - 2A}{e^{-A\tilde{\eta}}}\right] \right\}, \tag{4.11}$$

in which

$$\tilde{\eta} = x + \frac{4bd_1\beta_1^2 - d_2\beta_1(4A^2\beta_1^2 - 3b^2) + 2bd_3[4A^2\beta_1^2 - b^2]}{8bd_3[4A^2\beta_1^2 - b^2]}t + C_0,$$

where is obtained by **Family 10**.

By using (4.6) and (4.9), the exact solution for the Hirota-LPD equation gets,

$$\Psi_4(x, t) = e^{i\left(-\frac{b}{2\beta_1}x + \omega t + \theta\right)} \left\{ \xi_0 + A\xi_1 e^{-A\tilde{\eta}} \right\},$$



$$(4.12)$$

in which

$$\tilde{\eta} = x + \frac{4bd_1\beta_1^2 - d_2\beta_1(4A^2\beta_1^2 - 3b^2) + 2bd_3[4A^2\beta_1^2 - b^2]}{8bd_3[4A^2\beta_1^2 - b^2]}t + C_0,$$

where is obtained by **Family 11**.

By using (4.6) and (4.9), the exact solution for the Hirota-LPD equation gets,

$$\Psi_5(x, t) = e^{i\left(-\frac{b}{2\beta_1}x + \omega t + \theta\right)} \left\{ \xi_0 + C\xi_1 \left(x - \frac{4bd_1\beta_1^2 + 3b^2d_2\beta_1 - 2bd_3b^2}{8bd_3b^2}t + C_0 \right) \right\}, \quad (4.13)$$

where is obtained by **Family 13**.

Here, we can also obtain the nonautonomous rogue wave solutions of Eq. (2.1). Meanwhile, type solution of equation (3.22) is singular wave solution, type solution of equation (3.27) is kink-singular wave solution, type solution of equation (3.32) is singular wave solution, type solution of equation (3.37) is bright soliton wave solution, type solution of equation (3.50) is compacton wave solution, type solution of equations (4.8) and (4.13) are polynomial wave solutions, type solution of equation (4.10) is periodic wave solution and type solution of the equation (4.11) is exponential wave solution.

The applied methods in this work are the ITEM and ETEM which are analytical methods and give solutions the more including rational function, hyperbolic function, trigonometric function and exponential function solutions. Usually, such methods can acquire four types of solutions as examples

First:

$$u(x, t) = \frac{f(x, t)}{a + bf(x, t)}, \quad f(x, t) \text{ is polynomial of one - order.}$$

Second:

$$u(x, t) = a \tanh(\mu(x, t)) + b \coth(\mu(x, t)),$$

Third:

$$u(x, t) = a \tan(\mu(x, t)) + b \cot(\mu(x, t)),$$

Fourth:

$$u(x, t) = \frac{\exp(\mu(x, t))}{a + b \exp(\mu(x, t))},$$

We can be improved these methods by using the following formulas

$$u(x, t) = \frac{a_1 \tanh(\mu(x, t)) + a_2 \coth(\mu(x, t))}{a_1 \coth(\mu(x, t)) + a_2 \tanh(\mu(x, t))},$$



or

$$u(x, t) = \frac{a_1 \tan(\mu(x, t)) + a_2 \cot(\mu(x, t))}{a_1 \cot(\mu(x, t)) + a_2 \tan(\mu(x, t))}.$$

Therefore, searching such solutions can be done by using the methods is expressed above. But, in this paper by two methods new soliton wave solutions can be found for the combined Hirota-LPD model where is considerable. Because we removed any parameters in Eq. (2.1) for sake of simplicity of solving it. Thus, solutions were obtained by two methods including all parameters existing in Eq. (2.1). One can see that the obtained solutions are completely exact solutions. All the presented solutions, including traveling wave solutions, soliton wave solutions and rational polynomial solutions, show the remarkable richness of the solution space of the combined Hirota-LPD model. From the presented figures, it is understood that there are rogue waves that appear in wave interactions and wave breaking. In addition, the tendency for dynamical rogue wave formation is likely to be very dependent on the directional spread of waves. As our work, in [51] Song and Xue investigated the rogue waves of the nonlinear Schrödinger equation with time-dependent linear potential function and by using the similarity transformation obtained the first-order and second-order solutions of the rogue waves and discussed the nonlinear dynamic behaviors of these solutions in detail. At last, Yu [60] by using similarity reduction of the variable coefficients combined Hirota-LPD equation to the high-order NLSE found the first and second order multi-rogue wave solutions with arbitrary constants and also obtained the nonautonomous rogue wave solutions of Eq. (2.1). Comparing our results with the results of [60] show that some solutions in both works offer the same results. The numerical and analytical results obtained in this paper are useful to study the traveling and rogue waves in the fields of nonlinear science.

5. CONCLUSION

In this work, we successfully applied the extended trial equation method and the improved $\tan(\phi/2)$ -expansion method on the variable coefficients combined Hirota-LPD equation and obtained abundant exact families of traveling wave solutions comprising periodic wave, singular-wave, singular-periodic wave, singular-solitary wave and shock wave solutions. Meanwhile, the mechanical features of some families are explained through the physical descriptions. The results derived in terms of Jacobi elliptic functions transform into trigonometric functions when $l \rightarrow 0$ as described by the ETEM. We conclude that improved $\tan(\phi/2)$ -expansion method is a resourceful algorithm for constructing the exact families of traveling wave solutions for the equations of nonlinear PDEs.

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