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On the existence and uniqueness of positive solutions for a p-Laplacian fractional boundary value problem with an integral boundary condition with a parameter

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Abstract The aim of this work is to prove the existence and uniqueness of the positive solutions for a fractional boundary value problem by a parameterized integral boundary condition with p-Laplacian operator. By using iteration sequence, the existence of two solutions is proved. Also by applying a fixed point theorem on solid cone, the result for the uniqueness of a positive solution to the problem is obtained. Two examples are given to confirm our results.

Keywords. Fractional differential equations, P-Laplacian operator, Integral boundary condition.

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1. Introduction

Fractional derivatives as extensions of ordinary derivatives, are used in modeling of many physical and engineering phenomena such as viscoelasticity, rheology, fluid flow, diffusive transport, electrical networks, probability, electromagnetic theory, mechanics, chemistry, and control system(See [10, 21, 22, 23]). Therefore, investigation of the existence and uniqueness of the solutions of fractional initial and boundary value problems has great importance. There are several papers including fractional boundary value problem with various types of boundary conditions such as local, nonlocal, etc. (see [2, 3, 4, 8, 9, 11, 12, 13, 18, 24]).
In the last two decades, in spite of that noticeable number of papers have been studied about the existence and uniqueness of solutions for fractional boundary value problem with p-Laplacian operator (see [5, 7, 14, 15, 16, 17, 19, 26]), however, a few number of them concern about the p-Laplacian fractional boundary value problems with integral boundary condition.

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Zhang et al. [27] in 2014, discussed about a singular fractional eigenvalue problem with p-Laplacian operator including the Riemann-Stieltjes integral boundary conditions

$$-D_t^{\beta}\left(\varphi_p(D_t^{\alpha}u)\right)(t) = \lambda f(t,u(t)), \quad 0<t<1,$$
$$u(0)=0, \quad D_t^{\alpha}u(0)=0, \quad u(1)=\int_0^1 u(s)dB(s),$$

where D_t^{β} and D_t^{α} are the Riemann-Liouville fractional derivatives, $1<\alpha\leq 2$, $0<\beta\leq 1$, $\varphi_p(s)=|s|^{p-2}s$, B is a bounded variation function and $\int_0^1 u(s)dB(s)$ is the Riemann-Stieltjes integral of u with respect to B, $f(t;u):(0;1)\times(0;1)\to[0;1)$ is a continuous function or has singularity at $t=0,1$ and $x=0$. Authors have been derived their results, by using the Schauder fixed point theorem and the upper and lower solutions method.

Yunhong Li and Guogang Li [15] in 2016 investigated the existence and multiplicity of positive solutions for p-Laplacian fractional boundary value problem,

$$\begin{aligned}
&D_{0+}^{\beta}\left(\varphi_p(D_{0+}^{\alpha}x(t))\right)+\lambda f(t,x(t))=0, \quad 0<t<1,\\
&\varphi_p(D_{0+}^{\alpha}x(0))^{(j)}=0, \quad j=1,2,\ldots,m-1,\\
&\varphi_p(D_{0+}^{\alpha}x(1))=\int_0^1 h(t)\varphi_p(D_{0+}^{\alpha}(x(t))dt,\\
&x^{(i)}=0, \quad i=1,2,\ldots,n-1,\\
&x(0)=\int_0^1 k(t)x(t)dt,
\end{aligned}$$

where D_{0+}^{β} and D_{0+}^{α} are the Caputo derivatives, $m-1<\beta\leq m, n-1<\alpha\leq n, m\geq 1, n\geq 1, \varphi_p(s)=|s|^{p-2}s$ and $m+n-1<\alpha+\beta\leq m+n$. They used five functionals fixed point theorem and obtained their results.

Zhang, et al. [26] in 2020 investigated the boundary value problem,

$$\begin{aligned}
&D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}u(t))+f(t,u(t),D_{0+}^{\beta}u(t))=0, \quad 0<t<1,\\
&\varphi_p(D_{0+}^{\alpha}u(0))^{(j)}=\varphi_p(D_{0+}^{\alpha}u(1))=0, \quad j=1,2,\ldots,n-1,\\
&u(0)+u'(0)=\int_0^1 g_0(s)u(s)ds+a,\\
&u(1)+u'(1)=\int_0^1 g_1(s)u(s)ds+b,\\
&u^{(i)}(0)=0, \quad j=2,3,\ldots,m-1,
\end{aligned}$$

where $1<m-1<\alpha<m$, $1<n-1<\beta<n$, $\alpha-\beta>1$, D_{0+}^{β} is the Caputo fractional derivative, $g_0,g_1:[0,1]\to[0,+\infty)$, $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ are continuous functions, a,b are disturbance parameters and $\varphi_p(s)=|s|^{p-2}s$.

According to the above works, we consider the mixed fractional boundary value problem

$$\begin{aligned} &D_{0+}^{\beta}(\varphi_p({}^cD_{0+}^{\alpha}u(t))) = f(t,u(t)), \quad t\in[0,1], \quad 1<\alpha\leq 2, 2<\beta\leq 3,\\ &(\varphi_p({}^cD_{0+}^{\alpha}u(0)))' ={}^c D_{0+}^{\alpha}u(0) ={}^c D_{0+}^{\alpha}u(1) = u'(0) = 0, u(1) = \int_0^{\eta} u(t)dt, \end{aligned} \tag{1.1}$$

where $0\leq\eta<1$ is a parameter, D_{0+}^{β} is the standard Riemann-Liouville fractional derivative, ${}^cD_{0+}^{\alpha}$ is the Caputo fractional derivative, and $\varphi_p(s) = |s|^{p-2}s, p>1$. We present some necessary and sufficient conditions to prove existence and uniqueness results for the problem.

The rest of this paper is structured as below. In section 2, some necessary preliminaries from fractional calculus theory will be presented. Section 3 is devoted to present the Green function of the problem and its properties. In section 4, main results about the existence and uniqueness of positive solutions of the problem (1.1) will be discussed and finally in section 5, two examples are delivered to confirm the results.

2. Preliminaries

We begin by recalling some existing necessary facts in the literature of fractional calculus. For more details, one can see [10, 20, 21, 22, 23].

Definition 2.1. Let $s>0$ and $g:(a,\infty)\to\mathbb{R}$ be given continuous function, the Riemann-Liouville fractional integral of order s of g is

$$I_{a+}^{s}g(\tau) = \frac{1}{\Gamma(s)}\int_0^{\tau}(\tau-z)^{s-1}g(z)dz. \tag{2.1}$$

In the same manner, the Riemann-Liouville fractional derivative of order s of g is

$$D_{a+}^{s}g(\tau) = \frac{1}{\Gamma(1-s)}\int_0^{\tau}(\tau-z)^{-s}g(z)dz. \tag{2.2}$$

If $n\leq s<n+1$, we can extend this definition by

$$D_{a+}^{s}g(\tau) = \frac{d^n}{d\tau^n}D^{n-s+1}g(\tau) = \frac{d^{n+1}}{d\tau^{n+1}}I_{a^+}^{1-n}g(\tau).$$

In the same way, the Caputo fractional derivative of order s of function g is

$${}^cD_{a+}^{s}g(\tau) = D_{a^+}^{-(n+1-s)}\frac{d^{n+1}}{d\tau^{n+1}}g(\tau) = \frac{1}{\Gamma(n+1-s)}\int_a^{\tau}(\tau-z)^{n-s}g^{(n+1)}(z)dz.$$

Lemma 2.2. *Let $g\in C(0,1)\cap L^1(0,1)$ and $D_{0+}^{s}g(\tau)\in[C(0,1)\cap L^1(0,1)]$. Then*

$$I_{0+}^{s}D_{0+}^{s}g(\tau) = g(\tau) + C_1\tau^{s-1} + C_2\tau^{s-2} + \cdots + C_n\tau^{s-n},$$

where $n=[s]+1$.

Proof. See [23] □

Lemma 2.3. *Assume that $g \in C^n[0,1]$. Then*

$$I^s_{0+}\,{}^cD^s_{0+}g(\tau) = g(\tau) + C_1 + C_2\tau^2 + \cdots + C_n\tau^n,$$

where $n = [s] + 1$.

Proof. See [23] □

Definition 2.4. (See [6]) Let X be a real Banach Space and $P \subset X$ be a cone. P is solid if and only if $P^o \neq \emptyset$.

Definition 2.5. (See [6]) Let $0 \le \theta < 1$ be constant, the operator $T : P^o \to P^o$ is called θ-concave operator if for all $0 < k < 1$ and $u \in P^o$, we have

$$T(ku) \ge k^\theta Tu,$$

where P is a solid cone in a real Banach space X.

Theorem 2.6. *(See [6]) Let $0 \le \theta < 1$ be a constant and X be a real Banach space. If $P \subset X$ be solid cone and $T : P^o \to P^o$ be an increasing θ-concave operator, then T has a unique fixed point in P^o.*

3. Green Function

Lemma 3.1. *([1]) Assume that continuity holds for g in $[0,1]$ and $2 < \beta \le 3$. Then the fractional boundary value problem*

$$\begin{cases} D^\beta_{0+}y(t) = g(t), \quad 0 < t < 1, \\ y(0) = y'(0) = y(1) = 0, \end{cases} \tag{3.1}$$

has a solution explained by

$$y(t) = -\int_0^1 H(t,s)g(s)ds, \tag{3.2}$$

where

$$H(t,s) = \frac{1}{\Gamma(\beta)}\begin{cases} t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \le s \le t \le 1 \\ (1-s)^{\beta-1}, & 0 \le t \le s \le 1. \end{cases} \tag{3.3}$$

Lemma 3.2. *Let h be a continuous function in $[0,1]$, $1 < \alpha \le 2$, $0 \le \eta \le 1$. Then*

$$u(t) = -\int_0^1 G(t,s)h(s)ds, \tag{3.4}$$

is a solution of the fractional boundary value problem

$$\begin{cases} {}^cD^\alpha_{0+}u(t) = h(t), \quad t \in [0,1], \\ u'(0) = 0, \quad u(1) = \int_0^\eta u(t)dt, \end{cases} \tag{3.5}$$

where

$$G(t,s) = \begin{cases} \frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(1-s)^{\alpha-1}-\frac{1}{\alpha}(\eta-s)^{\alpha}}{(1-\eta)\Gamma(\alpha)}, & 0 \le s \le t \le \eta \le 1, \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(1-s)^{\alpha-1}-\frac{1}{\alpha}(\eta-s)^{\alpha}}{(1-\eta)\Gamma(\alpha)}, & 0 \le t \le s \le \eta \le 1, \\ \frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(1-s)^{\alpha-1}-\frac{1}{\alpha}(\eta-s)^{\alpha}}{(1-\eta)\Gamma(\alpha)}, & 0 \le s \le \eta \le t \le 1, \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(1-s)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)}, & 0 \le \eta \le t \le s \le 1. \end{cases}$$

$$\tag{3.6}$$

Proof. From Lemma 2.3 considering boundary value problem (3.5), we have

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + C_0 + C_1 t.$$

Since $u'(0) = 0$, we have $C_1 = 0$, and

$$u(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds + C_0 = \int_0^\eta u(t)dt,$$

so

$$C_0 = \int_0^\eta u(t)dt - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds,$$

and

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + \int_0^\eta u(t)dt - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds. \tag{3.7}$$

By integrating from 0 to η it follows that

$$\int_0^\eta u(t)dt = \frac{1}{\alpha\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha} h(s)ds - \frac{\eta}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds + \eta \int_0^\eta u(t)dt,$$

so

$$\int_0^\eta u(t)dt = \frac{1}{\alpha\Gamma(\alpha)(1-\eta)} \int_0^\eta (\eta-s)^{\alpha} h(s)ds - \frac{\eta}{\alpha\Gamma(\alpha)(1-\eta)} \int_0^1 (1-s)^{\alpha-1} h(s)ds.$$

Consequently

$$\begin{aligned} u(t) \quad = \quad & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\ & + \frac{1}{\alpha\Gamma(\alpha)(1-\eta)} \int_0^\eta (\eta-s)^{\alpha} h(s)ds - \frac{\eta}{\alpha\Gamma(\alpha)(1-\eta)} \int_0^1 (1-s)^{\alpha-1} h(s)ds. \end{aligned} \tag{3.8}$$

and this is precisely the assertion of the lemma. □

Lemma 3.3. *The fractional boundary value problem* (1.1) *has a following unique solution*

$$u(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) f(\tau, u(\tau))d\tau\right) ds \tag{3.9}$$

Proof. Let $h(t) = \varphi_q(y(t))$, $g(t) = -f(t, u(t))$, then from Lemma 3.1 we obtain

$$y(t) = \int_0^1 H(t,s) f(s, u(s))ds,$$

$$h(t) = \varphi_q(y(t)) = \varphi_q\left(\int_0^1 H(t,s) f(s, u(s))ds\right).$$

Hence from Lemma 3.2 we conclude that

$$u(t)=\int_0^1 G(t,s)h(s)ds=\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds.$$

□

Lemma 3.4. *Let $H(t,s)$ is a function that defined by (3.3), then*

(1) *$H:[0,1]\times[0,1]\to\mathbb{R}$ is a continuous function and $H(t,s)>0$ for all $0<t,s<1$;*

(2) $\max_{0\le t\le 1} H(t,s)\le \frac{s(1-s)^{\beta-1}}{\Gamma(\beta-1)}$

Proof. We first prove the statement (1). Clearly $H(t,s)$ is continuous on $[0,1]\times[0,1]$ and obviously $H(t,s)\ge 0$ for $s\ge t$.
Let $0\le s\le t\le 1$, so

$$t^{\beta-1}(1-s)^{\beta-1}-(t-s)^{\beta-1}=t^{\beta-1}(1-s)^{\beta-1}\left(1-\left(\frac{1-\frac{s}{t}}{1-s}\right)^{\beta-1}\right)\ge 0.$$

Hence for $0\le t,s\le 1$, $H(t,s)\ge 0$, and $H(t,s)>0$ for $0<t,s<1$.
Statement (2) is obtained directly from [25] that states:

$$\frac{t^{\beta-1}(1-t)s(1-s)^{\beta-1}}{\Gamma(\beta)}\le H(t,s)\le\frac{s(1-s)^{\beta-1}}{\Gamma(\beta-1)}.$$

□

Lemma 3.5. *Let $G(t,s)$ is defined by (3.6), then*

(1) *$G:[0,1]\times[0,1]\to\mathbb{R}$ is continuous and $G(t,s)>0$ for all $0<t,s<1$;*

(2) $\max_{0\le t\le 1} G(t,s)\le \frac{1}{1-\eta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}$

Proof. Note that, similar the proof of statement (1) of Lemma 3.4, it can be shown that $g(t,s)=\frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}>0$ for $t,s\in(0,1)$. One can see that $g(t,s)$ is increasing for $t\le s$ and decreasing for $s\le t$, with respect to t. Hence

$$\max_{0\le t\le 1} g(t,s)=g(s,s)=\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)},\quad s\in(0,1).$$

Now for proving the statement (1), let $0<s\le t\le\eta\le 1$, from the definition of $G(t,s)$ we have

$$\begin{aligned}
G(t,s) &= \frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\eta(1-s)^{\alpha-1}-\frac{1}{\alpha}(\eta-s)^{\alpha}}{(1-\eta)\Gamma(\alpha)}\\
&= \frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\eta(1-s)^{\alpha-1}-\frac{1}{\alpha}(\eta-s)(\eta-s)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)}\\
&\ge \frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\eta-s}{\alpha(1-\eta)}\frac{(1-s)^{\alpha-1}-(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\\
&= g(t,s)+\frac{\eta-s}{\alpha(1-\eta)}g(\eta,s)\\
&> 0.
\end{aligned}$$

By using an analogous arguments, we have $G(t,s) > 0$ for $0 \leq t \leq s \leq \eta \leq 1$ or $0 \leq s \leq \eta \leq t \leq 1$ or $0 \leq \eta \leq t \leq s \leq 1$.
Now we prove statement (2).
For $0 < s \leq t \leq \eta \leq 1$ or $0 \leq s \leq \eta \leq t \leq 1$ one has

$$\begin{aligned} \max_{0\leq t\leq 1} G(t,s) &= \max_{0\leq t\leq 1}\left(\frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\eta(1-s)^{\alpha-1}-\frac{1}{\alpha}(\eta-s)^{\alpha}}{(1-\eta)\Gamma(\alpha)}\right) \\ &\leq g(s,s)+\frac{\eta}{1-\eta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}=\frac{1}{1-\eta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \end{aligned}$$

For $0 \leq t \leq s \leq \eta \leq 1$, one has

$$\begin{aligned} \max_{0\leq t\leq 1} G(t,s) &= \max_{0\leq t\leq 1}\left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\eta(1-s)^{\alpha-1}-\frac{1}{\alpha}(\eta-s)^{\alpha}}{(1-\eta)\Gamma(\alpha)}\right) \\ &\leq g(s,s)+\frac{\eta}{1-\eta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}=\frac{1}{1-\eta}g(s,s)=\frac{1}{1-\eta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \end{aligned}$$

For the case $0 \leq \eta \leq t \leq s \leq 1$, there is nothing to prove. □

4. Main Result

Consider the Banach space $E = C[0,1]$ equipped with the maximum norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|$ and define the cone $P \subset E$ by $P = \{u \in E : u(t) \geq 0, 0 \leq t \leq 1\}$ with its interior $P^{o} = \{u \in E : u(t) > 0, 0 \leq t \leq 1\}$. We are in a position to prove the main results. For this aim, we notice the following assumptions

- $(H1)$ There exists a constant $\lambda > 0$ such that $f(t,u_1) \leq f(t,u_2)$ for any $0 \leq u_1 \leq u_2 \leq \lambda$;
- $(H2)$ $\max_{0\leq t\leq 1} f(t,\lambda) \leq \varphi_p(\lambda K)$;
- $(H3)$ $f(t,0) \neq 0$ for $0 \leq t \leq 1$.
- $(H4)$ $f(t,u)$ is a nondecreasing function with respect to the second variable;
- $(H5)$ there exists $0 \leq \theta < 1$ such that $f(t,ku) \geq (\varphi_p(k))^{\theta} f(t,u(t))$ for any $0 < k < 1, 0 < u < \infty$

In view of Lemma 3.3 we know the solutions of fractional boundary value problem (1.1) are the fixed points of the operator

$$(Tu)(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau\right)ds. \tag{4.1}$$

In order to prove the existence of the solutions for problem (1.1), the operator (4.1) must be completely continuous.

Theorem 4.1. *Operator $T : P \to E$ defined by (4.1) is completely continuous operator.*

Proof. Since functions $G(t,s), H(t,s), f(t,u)$ and $\varphi_q(s)$ are continuous in their domains, continuity of the operator T is concluded. Now by applying the Lebesgue dominated convergence and Arzela-Ascoli theorems one can see easily $T : P \to P$ is completely continuous.

□

Now let us for convenience introduce the notation

$$K = \left[\int_0^1 \frac{1}{1-\eta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^1 \frac{\tau(1-\tau)^{\beta-1}}{\Gamma(\beta-1)}d\tau\right)ds\right]^{-1}$$

Theorem 4.2. *Assume $(H1),(H2)$ and $(H3)$ hold. Then problem (1.1) has two positive solutions u^* and v^* with the following properties.*

- $0 \le \|v^*\| \le \lambda$ *and* $\lim_{n\to\infty} T^n v_0 = v^*$, *where* $v_0(t) = 0$
- $0 \le \|u^*\| \le \lambda$ *and* $\lim_{n\to\infty} T^n u_0 = u^*$, *where* $u_0(t) = \lambda$

Proof. Define $P_\lambda = \{u \in P : \|u\| \le \lambda\}$. We show that $TP_\lambda \subset P_\lambda$. Let $u \in P_\lambda$, then $0 \le u(t) \le \|u\| \le \lambda$. By assumptions (H1) and (H2), we have

$$0 \le f(t,u(t)) \le f(t,\lambda) \le \varphi_p(\lambda K).$$

Let $u \in P_\lambda$, in view of Theorem 4.1, we conclude that $Tu \in P$, and hence

$$\begin{aligned}
\|Tu\| &= \max \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau,u(\tau)d\tau\right)ds\\
&\le \int_0^1 \frac{1}{1-\eta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^1 \varphi_p(\lambda K)\frac{\tau(1-\tau)^{\beta-1}}{\Gamma(\beta-1)}d\tau\right)\\
&= \lambda K \int_0^1 \frac{1}{1-\eta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^1 \frac{\tau(1-\tau)^{\beta-1}}{\Gamma(\beta-1)}d\tau\right)ds\\
&= \lambda.
\end{aligned}$$

So $Tu \in P_\lambda$. Thus, we get $TP_\lambda \subset P_\lambda$.
Assume $u_0(t) = \lambda$ and $0 \le t \le 1$, then $\|u_0\| = \lambda$ and $u_0 \in P_\lambda$. Similarly if $u_1(t) = Tu_0(t)$, then $u_1 \in P_\lambda$. Now we define

$$u_{n+1} = Tu_n = T^{n+1}u_0, \quad n = 0,1,2,\ldots.$$

Since $TP_\lambda \subset P_\lambda$, One has $u_n \in P_\lambda (n = 0,1,2,\ldots)$. From Theorem 4.1, T is a compact operator; we show that $\{u_n\}_{n=1}^\infty$ has a subsequence like $\{u_{n_k}\}_{k=1}^\infty$ that is convergence, so there exists $u^* \in P_\lambda$ such that $u_{n_k} \to u^*$ as $k \to \infty$. In view of the definition of T and $(H1)$, for any $t \in [0,1]$, we have

$$\begin{aligned}
u_1(t) &= (Tu_0)(t)\\
&= \int_0^1 G(t,s)\varphi_q\left(\int_0^s H(s,\tau)f(\tau,u_0(\tau)d\tau\right)ds\\
&\le \int_0^1 \frac{1}{1-\eta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^1 \varphi_p(\lambda K)\frac{\tau(1-\tau)^{\beta-1}}{\Gamma(\beta-1)}d\tau\right)\\
&= \lambda K \int_0^1 \frac{1}{1-\eta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\left(\int_0^1 \frac{\tau(1-\tau)^{\beta-1}}{\Gamma(\beta-1)}d\tau\right)ds\\
&= \lambda.
\end{aligned}$$

So,

$$u_2(t) = Tu_1(t) \le Tu_0(t) = u_1(t), \quad 0 \le t \le 1.$$

One can see easily $u_{n+1} \leq u_n$ for $0 \leq t \leq 1, n = 0, 1, 2, \ldots$. Thus by use of induction for $n = 0, 1, 2, \ldots$, $0 \leq t \leq 1$, we have $u_{n+1} \leq u_n$. So there exists $u^* \in P_\lambda$ such that $u_n \to u^*$ as $n \to \infty$. Since the operator T is continuous and $u_{n+1} = Tu_n$, we get $Tu^* = u^*$.
Now assume $v_0 = 0$, $0 \leq t \leq 1$, then $v_0 \in P_\lambda$. Let $v_1 \in P_\lambda$. We define

$$v_{n+1} = Tv_n = T^{n+1} v_0, \quad n = 0, 1, 2, \ldots$$

We know $T : P_\lambda \to P_\lambda$, so for $n = 0, 1, 2, \ldots$ $v_n \subset P_\lambda$. Applying completely continuity of T, we show that $\{v\}_n^\infty$ is a sequentially compact set.
Since $v_1(t) = Tv_0(t) = (T0)(t) \geq 0, 0 \leq t \leq 1$, one has

$$v_2(t) = Tv_1(t) \geq (T0)(t) = v_1(t), \quad 0 \leq t \leq 1.$$

By using analogues argument about u_n, one can see

$$v_{n+1} \geq v_n, \quad 0 \leq t \leq 1, \quad n = 0, 1, 2, \ldots$$

Hence, there exists $v^* \in P_\lambda$ such that $v_n \to v^*$ as $n \to \infty$. Applying the continuity of T and $v_{n+1} = Tv_n$, we get $Tv^* = v^*$.
Now we show that zero function is not satisfying problem (1.1). This is concluded from the fact that $f(t, 0) \neq 0$, $0 \leq t \leq 1$. Thus $\|u^*\| > 0$ and $\|v^*\| > 0$. Consequently the fractional boundary value problem (1.1) has two positive solutions u^* and v^*. □

The next result is obtained by using the Theorem 2.6.

Theorem 4.3. *Assume* $(H4)$ *and* $(H5)$ *hold. Then there is only one positive solution for the fractional boundary value problem* (1.1).

Proof. Note that $P = \{u \in E : u(t) \geq 0, 0 \leq t \leq 1\}$ is a normal solid cone in $C[0, 1]$ with the interior

$$P^o = \{u \in C[0, 1] : u(t) > 0 \quad on \quad [0, 1]\}. \tag{4.2}$$

Also let T be the operator defined with (4.1). Then $T : P^o \to P^o$. It is obvious that T is an increasing operator, we show that T is a θ-concave operator. (H5) implies

$$\begin{aligned} T(ku)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau, ku(\tau))d\tau\right) ds \\ &\geq \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)\varphi_p(k^\theta)f(\tau, u(\tau))d\tau\right) ds \\ &= k^\theta \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f(\tau, u(\tau))d\tau\right) ds \\ &= k^\theta (Tu)(t). \end{aligned}$$

That is T is a θ-concave operator. In view of Theorem 2.6, T has a unique fixed point u^* in P^o, that it is the unique solution of fractional boundary value problem (1.1). □

5. Examples

Example 5.1. Let $p = \frac{3}{2}$, $\alpha = \beta = \frac{3}{2}$, $\eta = \frac{1}{2}$. Consider the following boundary value problem

$$\begin{aligned} &D_{0^+}^{\frac{5}{2}}(\varphi_{\frac{3}{2}}({}^cD_{0^+}^{\frac{3}{2}}u(t))) = f(t,u(t)), \quad 0 < t < 1, \\ &(\varphi_{\frac{3}{2}}({}^cD_{0^+}^{\frac{3}{2}}u(0)))' = u(0) = {}^c D_{0^+}^{\frac{3}{2}}u(0) = {}^c D_{0^+}^{\frac{3}{2}}u(1) = 0, \\ &u(1) = \int_0^{\frac{1}{2}} u(t)dt, \end{aligned} \tag{5.1}$$

where

$$f(t,u(t)) = \frac{1}{30}(1 + ue^t + u^{\frac{3}{2}}).$$

A simple computation shows that $K \approx 0.1773$. Now we choose $\lambda = 5$ and $f(t,u)$ satisfies

(1) $f(t,u_1) \leq f(t,u_2)$ for any $0 \leq t \leq 1$, $0 \leq u_1 \leq u_2 \leq 5$;
(2) $\max_{0\leq t\leq 1} f(t,\lambda) = f(1,5) \approx 0.858 < \varphi_{\frac{3}{2}}(\lambda K) \approx 0.9412$;
(3) $f(t,0) = \frac{1}{30} \neq 0$ for $0 \leq t \leq 1$.

Then in view of Theorem 5.1 we conclude that the problem (5.1) has two positive solutions u^* and v^* such that

$$\begin{aligned} &0 < \|u^*\|, \quad \lim_{n\to\infty} T^n u_0 = u^*, \quad \text{where} \quad u_0(t) = 5, \\ &0 < \|v^*\|, \quad \lim_{n\to\infty} T^n v_0 = v^*, \quad \text{where} \quad u_0(t) = 0. \end{aligned}$$

Example 5.2. Let $p = \frac{3}{2}$, $\alpha = \beta = \frac{3}{2}$, $\eta = \frac{1}{2}$. Consider fractional boundary value problem

$$\begin{aligned} &D_{0^+}^{\frac{5}{2}}(\varphi_{\frac{3}{2}}({}^cD_{0^+}^{\frac{3}{2}}u(t))) = f(t,u(t)), \quad 0 < t < 1, \\ &(\varphi_{\frac{3}{2}}({}^cD_{0^+}^{\frac{3}{2}}u(t)))' = u(0) = {}^c D_{0^+}^{\frac{3}{2}}u(0) = {}^c D_{0^+}^{\frac{3}{2}}u(1) = 0, \\ &u(1) = \int_0^{\frac{1}{2}} u(t)dt, \end{aligned} \tag{5.2}$$

where $f(t,u(t)) = t\sqrt{u(t)}$.
It can be seen that (5.1) satisfies condition $(H4)$. We show that $(H5)$ holds. Let $\theta = \frac{1}{2}$, then

$$f(t,ku(t)) = t\sqrt{ku(t)} = (k)^{\frac{1}{2}} f(t,u(t)) > \varphi_{\frac{3}{2}}(k)^{\frac{1}{2}} f(t,u(t))$$

So $(H5)$ satisfied. Thus by Theorem 4.3, fractional boundary value problem (5.2) has a unique solution.

6. Conclusion

In this work, despite the integral boundary condition with a parameter, we were able to prove the existence of positive solutions to the fractional boundary value problem. The presented examples showed how to apply proven theorems.

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