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Multiple solutions for a fourth-order elliptic equation involving singularity

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Abstract Here, we consider a fourth-order elliptic problem involving singularity and $p(x)$-biharmonic operator. Using Hardy's inequality, S_+-condition, and Palais-Smale condition, the existence of weak solutions in a bounded domain in $\mathbb{R}^N$ is proved. Finally, we percent some examples.

Keywords. Higher-order elliptic equations, Singular nonlinear boundary value problems, Critical point theory, Variational methods.

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1. Introduction

The area of partial differential equations (PDE's) has been growing steadily since middle of the 19th century. PDE's can be used to describe a wide variety of phenomena such as sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics (for example see [23–30]).

The boundary value problems with $p(x)$-biharmonic operator have been studied by many researchers [1–3, 14–20, 31, 33, 36].

In this paper we consider the following problem

$$\begin{cases} \Delta^2_{p(x)} u + \frac{|u|^{s-2}u}{|x|^{2s}} = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

- $\Omega \subset \mathbb{R}^N (N \geq 5)$ is a bounded domain with smooth boundary.
- $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$, denotes $p(x)$-biharmonic operator.
- $p(x) \in C(\bar{\Omega})$, $1 < s < p(x) < \infty$ and

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- $q(x) \in C(\bar{\Omega})$ with $1 < q(x) < p^*(x)$ where

  $$P^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & p(x) < N, \\ \infty & p(x) \geq N. \end{cases}$$

- λ is strictly positive real parameter and
- The Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

  $$|f(x,t)| \leq a_1 + a_2|t|^{q(x)-2}, \text{ for all } (x,t) \in \Omega \times \mathbb{R}, \tag{1.2}$$

  where a_1, a_2 are two positive constants.

Wang [34], considered the existence of solutions for the following biharmonic problem

$$\begin{cases} \Delta^2 u = \lambda \frac{|u|^{2^{**}-2}}{|x|^s} + \beta a(x)|u|^{r-2}u = f(x,u) & x \in \mathbb{R}^N, \\ \\ u \in D_0^{2,2}(R^N) & N \geq 5, \end{cases}$$

where $D_0^{2,2}(R^N)$ is the closure of $C^\infty(R^N)$, $2^{**}(s) = \frac{2(N-s)}{N-4}$, $0 \leq s < 4$ and $1 < r < 2^{**}$.

In 2013, Xie [35] studied the following problem

$$\begin{cases} \Delta_p^2 u - \lambda \frac{|u|^{p-2}u}{|x|^{2p}} = f(x,u) & \text{in } \Omega, \\ u = \frac{\partial u}{\partial x} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < \frac{N}{2}$ and $0 \leq \lambda < [\frac{N(p-1)(N-2p)}{p^2}]^p$.

In this work, we investigate the problem (1.1) and prove the existence of weak solutions, by applying Hardy's inequality, S_+-condition and Palais-Smale condition (or (PS) condition). Due to do this, we recall the following definitions.

Definition 1.1. [21] Let $1 < s < \frac{N}{2}$, for all $u \in X$

$$\int_\Omega \frac{|u(x)|^s}{|x|^{2s}} dx \leq \frac{1}{H} \int_\Omega |\Delta u(x)|^s dx, \tag{1.3}$$

is called the classical Hardy's inequality, where $H := (\frac{N(s-1)(N-2s)}{s^2})^s$.

Definition 1.2. [32] Let X be a reflexive real Banach space. If the assumptions $\limsup_{n\to+\infty}\langle T(u_n) - T(u_0) \;\; u_n - u_0\rangle \leq 0$ and $u_n \rightharpoonup u_0$ in X imply $u_n \to u_0$ in X, then the operator $T : X \to X^*$ is said to satisfy the (S_+) condition.

Definition 1.3. [4] Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1-functional. Φ is said to satisfy the Palais-Smale condition (denoted by (PS)), if any sequence u_n in X such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$ admits a convergent subsequence.

As before

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}; u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

and it is endowed with

$$\|\varphi\|_{L^{p(x)}} := inf\left\{\lambda > 0; \int_\Omega \left|\frac{\varphi(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

Also

$$W^{1,p(x)}(\Omega) := \left\{\varphi \in L^{p(x)}; |\nabla\varphi| \in L^{p(x)}\right\},$$

and its norm is defined by

$$\|\varphi\|_{W^{1,p(x)}} := \|\varphi\|_{L^{p(x)}} + \||\nabla\varphi|\|_{L^{p(x)}}.$$

Finally

$$W_0^{1,p(x)}(\Omega) := \left\{\varphi \in W^{1,p(x)};\ \varphi|_{\partial\Omega} = 0\right\}.$$

Set $p^- := \inf_{x\in\Omega} p(x)$ and $p^+ := \sup_{x\in\Omega} p(x)$. Let $X := W_0^{1,p(x)}(\Omega) \bigcap W^{2,p(x)}$ endowed with the norm

$$\|u\| = \||\Delta u|\|_{L^{p(x)}},$$

by the compact embedding $X \hookrightarrow L^{q(x)}(\Omega)$, there exists a $C_q > 0$ such that

$$\|u\|_{L^{q(x)}} \leq C_q\|u\|, \tag{1.4}$$

where $1 < q(x) < p^*(x)$ for all $x \in \Omega$ (see [11, Proposition 2.5]).
Suppose $\Phi : X \to \mathbb{R}$ is a functional defined by

$$\Phi(u) = \int_\Omega (\frac{1}{p(x)}|\Delta u|^{p(x)} + \frac{|u|^s}{s|x|^{2s}})dx, \tag{1.5}$$

where $1 < s < p^- \leq p(x) \leq p^+ < \infty$. By [22] and [11, Theorem 3.1],

- Φ is a continuously Gâteaux differentiable functional and for $u, v \in X$
$$\Phi'(u)(v) = \int_\Omega (|\Delta u|^{p(x)-2}|\Delta u||\Delta v| + \frac{|u|^{s-2}uv}{|x|^{2s}})dx. \tag{1.6}$$
- $\Phi' : X \to X^*$ is strictly monotone, homeomorphism and satisfies the (S_+) condition.

Proposition 1.4. *[10, Theorem 1.3] Assume $\varphi \in W_0^{1,p(x)}$ and $\rho_p(\varphi) := \int_\Omega |\varphi(x)|^{p(x)}dx$. Then*

(i) $\|\varphi\| < 1(= 1, > 1)$ *iff* $\rho_p(|\Delta\varphi|) < 1(= 1 :> 1)$.
(ii) $\|\varphi\| > 1$, *then* $\frac{1}{p^+}\|\varphi\|^{p^-} \leq \Phi(\varphi) \leq \frac{1}{p^-}\|\varphi\|^{p^+} + \int_\Omega \frac{|\varphi|^s}{s|x|^{2s}}dx.$
(iii) $\|\varphi\| < 1$, *then* $\frac{1}{p^+}\|\varphi\|^{p^+} \leq \Phi(\varphi) \leq \frac{1}{p^-}\|\varphi\|^{p^-} + \int_\Omega \frac{|\varphi|^s}{s|x|^{2s}}dx.$

Assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and for all $(x, \xi) \in X$, define

$$F(x, \xi) := \int_\Omega^\xi f(x, t)dt. \tag{1.7}$$

- For $u \in X$, define $\Psi : X \to \mathbb{R}$ by
$$\Psi(u) := \int_\Omega F(x, u(x))dx. \tag{1.8}$$

- Ψ is continuously Gâteaux differentiable functional, has compact derivative and

$$\Psi'(u)(v) := \int_{\Omega} f(x, u(x))v(x)dx, \tag{1.9}$$

for u, v in X (see [22]).

- Define $I := \Phi - \lambda\Psi$. Notice that $I'(u) = 0$ implies for $u, v \in X$,

$$\int_{\Omega}(|\Delta u|^{p(x)-2}|\Delta u||\Delta v| + \frac{|u|^{s-2}uv}{|x|^{2s}})dx = \lambda \int_{\Omega} f(x, u(x))v(x)dx. \tag{1.10}$$

So the weak solutions of the problem (1.1) are the critical points of I.

For $x \in \Omega$, set

$$\delta(x) := sup\left\{\delta > 0 : B(x, \delta) \subseteq \Omega\right\}, \tag{1.11}$$

$$D := \sup_{x\in\Omega} \delta(x). \tag{1.12}$$

Clearly, there exists $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$.

Also, for $a > 0$ and $q(x) \in C(\overline{\Omega})$ with

$$1 < q^- := \inf_{x\in\Omega} q(x) < q(x) < q^+ := \sup_{x\in\Omega} q(x) < 0,$$

we have:

$$[a]^{q(x)} := max\left\{a^{q^-}, a^{q^+}\right\},$$

$$[a]_{q(x)} := min\left\{a^{q^-}, a^{q^+}\right\},$$

where $x \in \Omega$. Let $r > 0$, set

$$\overline{\omega} := \frac{1}{r}\left\{a_1C_1(p^+)^{\frac{1}{p^-}}[r]^{\frac{1}{p}} + \frac{a_2}{q^-}[C_q]^q(p^+)^{\frac{q^+}{p^-}}[[r]^{\frac{1}{p}}]^q\right\} \tag{1.13}$$

where a_1, a_2 are positive numbers and C_1, C_q are ordinary embedding constants $X \hookrightarrow L^1(\Omega)$ and $X \hookrightarrow L^{q(x)}(\Omega)$.

2. Existence of weak solutions

Here the existence of multiple weak solutions of the problem (1.1) in different cases are proved and some examples are presented.

2.1. **A weak solutions.** In order to study the existence of weak solution of the problem (1.1), we recall the following theorem.

Theorem 2.1. *[7] Assume X is a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ are two continuously Gâteaux differentiable functional such that*

$$\inf_{x\in\Omega} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Suppose there exist $r > 0$ and $\overline{x} \in X$ with $0 < \Phi(\overline{x}) < r$ such that

- *(i)* $\frac{\sup_{\Phi(x)<r} \Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$.

- *(ii) for all* $\lambda \in \Lambda :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)<r}\Psi(x)}[$, $I_\lambda := \Phi - \lambda\Psi$ *satisfies* $(PS)^{[r]}$ *condition.*

For all $\lambda \in \Lambda$ *there exists* $x_0 \in \Phi^{-1}(]0, r[)$ *such that* $I'_\lambda(x_0) = 0$ *and* $I_\lambda(x_0) \leq I_\lambda(x)$, *for all* $x \in \Phi^{-1}(]0, r[)$.

Here we state the existence of at least one weak solution.

Theorem 2.2. *Suppose* f *and* F *are defined by* (1.7) *and*

$$limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{t^{p^-}} = +\infty. \tag{2.1}$$

For all $\lambda \in]0, \lambda^*[$, *where*

$$\lambda^* := \frac{1}{a_1 k_1 (p^+)^{\frac{1}{p^-}} + \frac{a_2}{q^-}[k_q]^q (p^+)^{\frac{q^+}{p^-}}}.$$

The problem (1.1) *has at least a nontrivial weak solution.*

Proof. Let $r = 1$ and apply Theorem 2.1 to get the result.
Assume $X, \|u\|, \Phi$ and Ψ are in the previous section. Fix $\lambda \in]0, \lambda^*[$, by (2.1) there exists

$$0 < \delta_\lambda < min\left\{1, \left(\frac{p^-}{[\frac{2}{D}]^p m(D^N - (\frac{D}{2})^N)}\right)^{\frac{1}{p^-}}\right\},$$

such that

$$\frac{sH\, inf_{x\in\Omega} F(x,\delta)}{(H+1)[\frac{2\delta}{D}]^p (2^N - 1)} > \frac{1}{\lambda}.$$

Suppose $\overline{u} \in X$ such that

$$\overline{u}(x) = \begin{cases} 0 & x \in \Omega \backslash B(x_0, D), \\ \delta_\lambda x & x \in B(x_0, \frac{D}{2}), \\ \frac{\delta_\lambda}{D}(D^2 - (x - x_0)^2) & x \in B(x_0, D)\backslash B(x_0, \frac{D}{2}), \end{cases}$$

Proposition 1.4 shows

$$\begin{aligned} \Phi(\overline{u}) = \ & \int_\Omega \left(\frac{1}{p(x)}|\Delta\overline{u}|^{p(x)} + \frac{|\overline{u}|^s}{s|x|^{2s}}\right)dx \\ < \ & \frac{1}{s}[\frac{2\delta}{D}]^p m(D^N - (\frac{D}{2})^N) + \frac{1}{sH}[\frac{2\delta}{D}]^s m(D^N - (\frac{D}{2})^N) \\ \leq \ & \frac{H+1}{sH}[\frac{2\delta}{D}]^p m(D^N - (\frac{D}{2})^N) \\ \leq \ & 1. \end{aligned}$$

So $0 < \Phi(\overline{u}) < 1$, therefor Proposition 1.4 for each $u \in \Phi^{-1}(\infty, 1[$ implies

$$\|u\| \leq [p^+\Phi(u)]^{\frac{1}{p^-}} \leq (p^+)^{\frac{1}{p^-}}. \tag{2.2}$$

Hence

$$\begin{aligned} \Psi(u) = \ & \int_\Omega F(x, u(x))dx \\ \leq \ & a_1 \int_\Omega |u(x)|dx + \frac{a_2}{q^-}\int_\Omega |u(x)|^{q(x)}dx \\ \leq \ & a_1 k_1 \|u\| + \frac{a_2}{q^-}[k_q]^q \|u\|^{\frac{q^+}{p^-}} \\ \leq \ & a_1 k_1 (p^+)^{\frac{1}{p^-}} + \frac{a_2}{q^-}[k_q]^q (p^+)^{\frac{q^+}{p^-}} \end{aligned}$$

for $u \in \Phi^{-1}(-\infty, 1[$, so

$$\sup_{\Phi(u)<1} \Psi(u) \leq a_1 k_1 (p^+)^{\frac{1}{p^-}} + \frac{a_2}{q^-}[k_q]^q (p^+)^{\frac{q^+}{p^-}} = \frac{1}{\lambda^*} < \frac{1}{\lambda}$$

therefor

$$\sup_{\Phi(u)<1} \Psi(u) < \frac{1}{\lambda} < \frac{\Psi(\overline{u})}{\Phi(\overline{u})}.$$

Thus I has a local minimum point $\overline{u}$ (see Theorem 2.1) and Theorem 2.2 is proved. □

The following example presents a function where satisfies the conditions of Theorem 2.2.

Example 2.3. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by $f(x,t) = e^{\|x\|} t^{k-1}$, where $0 < k < p^- < p < \infty$. We have $F(x,t) = e^{\|x\|} t^k$. So

$$\begin{aligned} limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{t^{p^-}} &= limsup_{t\to 0^+} \frac{\inf_{x\in\mathbb{R}^n} e^{\|x\|} t^k}{t^{p^-}} \\ &= limsup_{t\to 0^+} \frac{1}{t^{p^- - k}} \\ &= +\infty. \end{aligned}$$

2.2. **Two weak solutions.** By recalling another theorem, the existence of two weak solutions of the problem (1.1) can be proved.

Theorem 2.4. *[7] Suppose X is a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ are two continuously Gâteaux differentiable functionals and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and suppose for*

$$\lambda \in]0, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}[$$

the functional $I_\lambda := \Phi - \lambda\Psi$ is unbounded from below and satisfies (PS) condition. I_λ has two distinct critical points for

$$\lambda \in]0, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}[.$$

The existence of two weak solutions is presented by applying Theorem 2.4 in case $r = 1$.

Theorem 2.5. *Let f satisfies (1.2), F be in (1.7), and there exist $\theta > p^+$ and $r > 0$ such that*

$$0 < \theta F(x,t) \leq t f(x,t). \tag{2.3}$$

There exists two weak solutions of the problem (1.1) for $\lambda \in]0, \lambda^*[$, where

$$\lambda^* := \frac{1}{a_1 C_1 (p^+)^{\frac{1}{p^-}} + \frac{a_2}{q^-}[C_q]^q (p^+)^{\frac{q^+}{p^-}}}. \tag{2.4}$$

Proof. Assume Φ and Ψ are defined by (1.5) and (1.8), respectively. Theorem is proved by the following steps:
Step 1. $I := \Phi - \lambda\Psi$ satisfies (PS) condition.
Assume $\{u_n\}$ is a sequence in X such that

$$d := \sup_{n\to+\infty} I(u_n) < \infty, \quad \|I'(u_n)\|_{X^*} \to 0,$$

thus

$$\begin{aligned} d \quad &\geq I(u_n) \\ &\geq \int_\Omega \tfrac{1}{p(x)}|\Delta u_n|^{p(x)}dx + \tfrac{1}{s}\int_\Omega \tfrac{|u_n|^s}{|x|^{2s}}dx - \lambda\int_\Omega F(x,u_n)dx \\ &\geq \int_\Omega \tfrac{1}{p(x)}|\Delta u_n|^{p(x)}dx + \tfrac{1}{s}\int_\Omega \tfrac{|u_n|^s}{|x|^{2s}}dx - \tfrac{\lambda}{\theta}\int_\Omega f(x,u_n)u_n dx \\ &\geq \int_\Omega \tfrac{1}{p(x)}|\Delta u_n|^{p(x)}dx + \tfrac{1}{\theta}\int_\Omega \tfrac{|u_n|^s}{|x|^{2s}}dx - \tfrac{\lambda}{\theta}\int_\Omega f(x,u_n)u_n dx \\ &\geq (\tfrac{1}{p^+} - \tfrac{1}{\theta})\int_\Omega |\Delta u_n|^{p(x)}dx + \tfrac{1}{\theta}\|I'(u_n)\|\,\|u_n\|, \end{aligned}$$

so $\|u_n\|$ is bounded. Therefore, if $u_n \rightharpoonup u$ so $\Psi'(u_n) \to \Psi'(u)$, since $I'(u_n) = \Phi'(u_n) - \lambda\Psi'(u_n) = 0$ then $\Phi'(u_n) \to \lambda\Psi'(u_n)$, thus $u_n \to u$ (because Φ' is homeomorphism). So I satisfies the condition (PS).

Step 2. I is unbounded from below.
First we show, there exists $M \in \mathbb{R}^+$ such that for $x \in \Omega$ and $|t| > M$

$$F(x,\xi) \geq K|\xi|^\theta. \tag{2.5}$$

(2.3) implies

$$0 < \theta F(x,\xi t) \leq \xi t f(x,\xi t), \text{ for all } \xi > 0.$$

Let $m(x) := \min_{|\xi|=M} F(x,\xi)$ and $g_t(z) := F(x,zt)$ for all $z > 0$ thus

$$0 < \theta g_t(z) = \theta F(x,zt) \leq zt f(x,zt) = z g_t'(z)$$

for all $z > \frac{M}{|t|}$, so

$$\int_{\frac{M}{|t|}}^1 \frac{g_t'(z)}{g_t(z)}dz \geq \int_{\frac{M}{|t|}}^1 \frac{\theta}{z}dz,$$

then

$$Ln\left(\frac{g_t(1)}{g_t(\frac{M}{|t|})}\right) \geq Ln\left(\frac{|t|^\theta}{M^\theta}\right),$$

therefore

$$F(x,t) = g_t(1) > F(x, \frac{M}{|t|}t)\frac{|t|^\theta}{M^\theta} \geq m(x)\frac{|t|^\theta}{M^\theta} \geq K|t|^\theta$$

so (2.5) is established.

Fixed $v \in X - \{0\}$, for each $t > 1$ one has

$$\begin{aligned} I(tv) &= \int_\Omega \frac{1}{p(x)}|t\Delta v|^{p(x)}dx + \frac{1}{s}\int_\Omega \frac{|tv|^s}{|x|^{2s}}dx - \lambda \int_\Omega F(x,tv)dx \\ &\leq t^{p^+}\int_\Omega \frac{1}{p(x)}|\Delta v|^{p(x)}dx + \frac{t^s}{sH}\int_\Omega \frac{|\Delta v|^s}{|x|^{2s}}dx - \lambda K t^\theta \int_\Omega |v|^\theta dx - C_1 \\ &\leq t^{p^+}(\int_\Omega \frac{1}{p(x)}|\Delta v|^{p(x)}dx + \frac{1}{sH}\int_\Omega \frac{|\Delta v|^s}{|x|^{2s}}dx) - \lambda K t^\theta \int_\Omega |v|^\theta dx - C_1, \end{aligned}$$

since $p^+ < \theta$ if $t \to +\infty$ then $I \to -\infty$.

Fix $\lambda \in]0, \lambda^*[$ where λ^* is defined as (2.4). By Proposition 1.4

$$\|u\| \leq [p^+\Phi(u)]^{\frac{1}{p^-}} \leq [p^+]^{\frac{1}{p^-}} = (p^+)^{\frac{1}{p^-}}, \tag{2.6}$$

for each $u \in \Phi^{-1}(]-\infty, 1[)$ and by (1.4)

$$\int_\Omega |u(x)|^{q(x)}dx = \rho_q(u) \leq [\|u\|_{L^{q(x)}(\Omega)}]^q \leq [C_q\|u\|]^q \tag{2.7}$$

for $u \in X$. Also, the compact embedding $X \hookrightarrow L^1(\Omega)$, $X \hookrightarrow L^q(\Omega)$ there exist $C_1, C_q > 0$ and by (1.2), (2.3), (2.6) and (2.7)

$$\begin{aligned} \Psi(u) = &\int_\Omega F(x,u)dx \\ \leq\ & a_1\int_\Omega |u(x)|dx + \frac{a_2}{q^-}\int_\Omega |u(x)|^{q(x)}dx \\ \leq\ & a_1C_1\|u\| + \frac{a_2}{q^-}[C_q\|u\|]^q \\ \leq\ & a_1C_1[p^+]^{\frac{1}{p^-}} + \frac{a_2}{q^-}[C_q]^q(p^+)^{\frac{q^+}{p^-}} \\ =\ & \frac{1}{\lambda^*} \\ <\ & \frac{1}{\lambda}, \end{aligned}$$

therefore $\lambda < \frac{1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,1[)} \Psi(u)}$. Let $I := I_\lambda$, by Theorem 2.4, problem (1.1) has two weak solutions. □

Example 2.6. Assume $x \in \mathbb{R}$, $1 < p^+ < \theta < |t| < q(x) < \infty$, $F(x,t) = q(x)[cosht - 1]$. Consider

$$\begin{cases} \Delta^2_{p(x)}u + \frac{|u|^{s-2}u}{|x|^{2s}} = \lambda q(x)sinht & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Notice that $\theta F(x,t) < tf(x,t)$ or equivalently $\theta cosht - tsinht - \theta \leq 0$ for $x \in \mathbb{R}$ and $1 < \theta < |t|$. For this, we consider two different cases:

(i) If $t > \theta > 1$ then

$$\theta cosht - tsinht - \theta < \theta cosht - tsinht - \theta < \theta e^{-t} - \theta < 0.$$

(ii) If $t < -\theta < -1$ hence

$$\theta cosht - tsinht - \theta < \theta cosht + tsinht - \theta < \theta e^{t} - \theta < 0.$$

Therefore, the function f satisfies (2.3), so by Theorem 2.5 this problem has two weak solutions.

Remark 2.7. In Example 2.6, $q(x)$ can be replaced by all positive functions $\cosh x$, e^x, x^2.

2.3. **Three weak solutions.** Finally one can prove the existence of three weak solutions of the problem (1.1) by recalling the following theorem.

Theorem 2.8. *[7] Assume X is a reflexive real Banach space, $\Psi : X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, $\Phi : X \to \mathbb{R}$ is a continuously Gâteaux differentiable, coercive, sequentially weakly lower semi-continuous functional, whose Gâteaux derivative has a continuous inverse and*

$$\inf_{x\in\Omega} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Suppose there exist $r > 0$ and $\overline{x} \in X$, with $\Phi(\overline{x}) < r$, such that

(i) $\frac{\sup_{\Phi(x)<r} \Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$.

(ii) *$\Phi - \lambda\Psi$ is coercive for $\lambda \in \Lambda :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)<r} \Psi(x)}[$.*

$\Phi - \lambda\Psi$ has at least three distinct critical points in X for $\lambda \in \Lambda$.

Theorem 2.9. *Assume that f, F be in (1.7) and*

$$F(x,t) \geq 0, \tag{2.8}$$

for $(x,t) \in \Omega \times \mathbb{R}^+$, also there exists $C \in [0,\infty)$ such that

$$F(x,t) \leq C(1 + |t|^{q(x)}), \tag{2.9}$$

for $(x,t) \in \Omega \times \mathbb{R}$, $q(x) \in C(\overline{\Omega})$ and $1 < q^- < q(x) < q^+ < p^-$. Moreover, there exist $\delta, r > 0$ with $r < \frac{1}{p^+}[\frac{2\delta}{D}]_p m(D^N - (\frac{D}{2})^N)$ such that

$$\varpi < \frac{(H+1)[\frac{2\delta}{D}]^p(2^N - 1)}{sH \inf_{x\in\Omega} F(x,\delta)}.$$

The problem (1.1) implies at least three weak solutions for $\lambda \in \Lambda$, where

- *$\Lambda :=]\frac{(H+1)[\frac{2\delta}{D}]^p(2^N-1)}{sH \inf_{x\in\Omega} F(x,\delta)}, \frac{1}{\varpi}[$, ϖ is in (1.13) and D is in (1.12).*
- *$m := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$ is the measure of unit of $\mathbb{R}^{\mathbb{N}}$ where Γ is the Gamma function.*

Proof. Let X, Φ, Ψ and I be the same of as the last section. We investigate the conditions (i) and (ii) of Theorem 2.8.
Let $\overline{u} \in X$ such that

$$\overline{u}(x) = \begin{cases} 0 & x \in \Omega\backslash B(x_0, D), \\ \delta_\lambda x & x \in B(x_0, \frac{D}{2}), \\ \frac{\delta_\lambda}{D}(D^2 - (x - x_0)^2) & x \in B(x_0, D)\backslash B(x_0, \frac{D}{2}), \end{cases}$$

By Proposition 1.4 and hypothesis of theorem

$$\begin{aligned} r &< \tfrac{1}{p^+}[\tfrac{2\delta}{D}]_p m(D^N - (\tfrac{D}{2})^N) \\ &\le \textstyle\int_\Omega (\tfrac{1}{p(x)}|\Delta\overline{u}|^{p(x)} + \tfrac{|\overline{u}|^s}{s|x|^{2s}})dx = \Phi(\overline{u}) \\ &\le \textstyle\int_\Omega \tfrac{1}{s}|\Delta\overline{u}|^{p(x)}dx + \tfrac{1}{sH}\int_\Omega |\Delta\overline{u}|^s dx \\ &\le \tfrac{1}{s}[\tfrac{2\delta}{D}]^p m(D^N - (\tfrac{D}{2})^N) + \tfrac{1}{sH}[\tfrac{2\delta}{D}]^s m(D^N - (\tfrac{D}{2})^N) \\ &\le \tfrac{H+1}{sH}[\tfrac{2\delta}{D}]^p m(D^N - (\tfrac{D}{2})^N). \end{aligned}$$

therefore

$$\frac{\Psi(\overline{u})}{\Phi(\overline{u})} \ge \frac{sH \inf_{x\in\Omega} F(x,\delta)}{(H+1)[\frac{2\delta}{D}]^p(2^N-1)}, \tag{2.10}$$

because

$$\Psi(\overline{u}) \ge \int_{B(x_0,D)} F(x,\overline{u}(x))dx \ge \inf_{x\in\Omega} F(x,\delta)m(\frac{D}{2})^N.$$

Proposition 1.4 for $u \in \Phi^{-1}(-\infty, r]$ shows

$$\|u\| \le [p^+\Phi(u)]^{\frac{1}{p}} \le (p^+)^{\frac{1}{p^-}}[r]^{\frac{1}{p}}. \tag{2.11}$$

Now (2.11), (1.2), the compact embedding $X \hookrightarrow L^1(\Omega)$ and $X \hookrightarrow L^{q(x)}(\Omega)$ impliy

$$\begin{aligned} \Psi(u) &= \textstyle\int_\Omega F(x,u(x))dx \\ &\le a_1 \textstyle\int_\Omega |u(x)|dx + \tfrac{a_2}{q^-}\int_\Omega |u(x)|^{q(x)}dx \\ &\le a_1 C_1\|u\| + \tfrac{a_2}{q^-}[C_q]^q(p^+)^{\frac{q^+}{p^-}}[[r]^{\frac{1}{p}}] \\ &\le a_1 C_1 (p^+)^{\frac{1}{p^-}}[r]^{\frac{1}{p}} + \tfrac{a_2}{q^-}[C_q]^q(p^+)^{\frac{q^+}{p^-}}[[r]^{\frac{1}{p}}] \end{aligned}$$

for $u \in \Phi^{-1}(-\infty, r]$, therefore

$$\frac{1}{r}\sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u) \le \frac{1}{r}\left\{a_1C_1(p^+)^{\frac{1}{p^-}}[r]^{\frac{1}{p}} + \frac{a_2}{q^-}[C_q]^q(p^+)^{\frac{q^+}{p^-}}[[r]^{\frac{1}{p}}]\right\} < \frac{\Psi(\overline{u})}{\Phi(\overline{u})}$$

this implies part (i) of Theorem 2.8 is satisfied.

Notice that $I := \Phi - \lambda\Psi$ is coercive for each $\lambda > 0$. Let $u \in X$ with $\|u\| \ge \left\{1, \frac{1}{C_q}\right\}$, by (1.4) and (2.9), we have

$$\Psi(u) = \int_\Omega F(x,t)dx \le \int_\Omega (C(1+|t|^{q(x)}))dx \le C(|\Omega| + [C_q\|u\|]^{q^+}),$$

therefor

$$I(u) = \Phi(u) - \lambda\Psi(u) \ge \frac{1}{p^+}\|u\|^{p^-} - \lambda C(|\Omega| + C_q^{q^+}\|u\|^{q^+}),$$

hence, $q^+ < p^-$ implies I is coercive and (1.1)implies at least three weak solutions. □

Example 2.10. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by $f(x,t) = e^{-\|x\|}t^{k-1}$, where $1 < k < q^- < q(x) < \infty$. Thus $F(x,t) = e^{-\|x\|}t^k$. So, by $C > 1$, we have

$$F(x,t) = e^{-\|x\|}t^k < Ct^k < C(1 + |t|^{q(x)})$$

and the conditions of Theorem 2.8 are satisfied.

References

[1] F. Abdolrazaghi and A. Razani, *On the weak solutions of an overdetermined system of nonlinear fractional partial integro-differential equations*, Miskolc Math. Notes, *20*(1) (2019), 3–16, DOI: 10.18514/MMN.2019.2755.

[2] R. Ayazoghlu, G. Alisay, and I. Ekincioglu, *Existence of one weak solution for $p(x)-$biharmonic equations involving a concave-convex nonlinearity*, Matematicki Vesnik, *69* (2017), 296–307.

[3] F. Behboudi and A. Razani, *Two weak solutions for a singular (p, q)-Laplacian problem*, Filomat, *33*(11) (2019), 3399-3407, https://doi.org/10.2298/FIL1911399B.

[4] G. Bonanno, *Relation between the mountain pass theorem and local minima*, Adv. Nonlinear Anal., *1* (2012), 205–220.

[5] G. Bonanno and A. Chinni, *Discontinuous elliptic problems involving the $p(x)$-Laplacian*, Math. Nacher., *284* (2011), 639–652.

[6] G. Bonanno and A. Chinni, *Existence and multiplicity of weak solution for elliptic dirichlet problems with variable exponent*, J. Math. Anal. Appl., *418* (2014), 812–827.

[7] G. Bonanno and S. A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition*, Apple. Anal., *89* (2010), 1–10.

[8] C. C. Chen and C. S. Lin, *Local bihavior of singular positive solutions of semilinear elliptic equations with sobolev exponent*, Duke Math. J., *78* (1995), 315–334.

[9] P. Drábek, *Strongly nonlinear degenerated and singular elliptic problems*, Pitman Research Notes in Mathematics Series, *343* (1996), 112–146.

[10] X. L. Fan and D. Zhao, *On the space $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl., *263* (2001), 424–446.

[11] X. L. Fan and Q. Zhang, *Existence of solutions for $p(x)$-Laplacian Dirichlet problem*, Nonlinear Anal. *52* (2011), 1843–1852.

[12] N. Ghoussoub and C. Yuan, *Multiple solutions for Quashi-linear PDEs involving the critical Sobolev and Hardy exponents*, Amer. Math. Soc., *352* (2000), 5703–5743.

[13] M. Khodabakhshi, A. M. Aminpour, G. A. Afrouzi, and A. Hadjian, *Existence of two weak solutions for some singular elliptic problems*, Rev. R. Acad. Cienc. Ex. Fis. Nat. Ser. A. Mat. *110* (2016), 385–393.

[14] L. Kong, *Multiple solutions fourth order elliptic problems with $p(x)$-biharmonic operators*, Opuscula Math., *36* (2016), 253–264.

[15] L. Li and C. Tang, *Existence and multiplicity of solutions for a class of $p(x)$-biharmonic equations*, Acta Math. Sci. Ser. A Chin Ed, *33* (2013), 155-170.

[16] R. Mahdavi Khanghahi and A. Razani, *Existence of at least three weak solutions for a singular fourth-order elliptic problems*, J. Math. Anal., *8* (2017), 45–51.

[17] R. Mahdavi Khanghahi, and A. Razani, *Solutions for a singular elliptic problem involving the $p(x)$-Laplacian*, Filomat, *32*(14) (2018), 4841-4850.

[18] M. Makvand Chaharlang and A. Razani, *Infinitely many solutions for a fourth order singular elliptic problem*, Filomat, *32*(14) (2018), 5003-5010.

[19] M. Makvand Chaharlang and A. Razani, *A fourth order singular elliptic problem involving p-biharmonic operator*, Taiwanese J. Math., *23*(3) (2019), 589-599, doi:10.11650/tjm/180906, https://projecteuclid.org/euclid.twjm/1537927424.

[20] M. Makvand Chaharlang and A. Razani, *Existence of infinitely many solutions for a class of nonlocal problems with Dirichlet boundary condition*, Commun. Korean Math. Soc., *34*(1) (2019), 155–167, https://doi.org/10.4134/CKMS.c170456.

[21] E. Mitidier, *A simple approach to Hardy inequalities*, Math. Notes, *67* (2000), 479–486.

[22] E. Montefusco, *Lower semicontinuity of functional via concentration-compactness principle*, J. Math. Anal. Appl., *263* (2001), 264–276.

[23] A. Razani, *Weak and strong detonation profiles for a qualitative model*, J. Math. Anal. Appl., *276* (2002), 868–881, https://doi.org/10.1016/S0022-247X(02)00459-6.

[24] A. Razani, *Chapman-Jouguet detonation profile for a qualitative model*, Bull. Austral. Math. Soc., *66* (2002), 393–403, https://doi.org/10.1017/S0004972700040259.

[25] A. Razani, *Existence of Chapman-Jouguet detonation for a viscous combustion model*, J. Math. Anal. Appl., *293* (2004), 551–563, https://doi.org/10.1016/j.jmaa.2004.01.018.

[26] A. Razani, *On the existence of premixed laminar flames*, Bull. Austral. Math. Soc., *69* (2004), 415–427, https://doi.org/10.1017/S0004972700036194.

[27] A. Razani, *Shock waves in gas dynamics*, *Surv. Math. Appl.*, *2* (2007), 59-89.

[28] A. Razani, *An existence theorem for ordinary differential equation in Menger probabilistic metric space*, Miskolc Mathematical Notes, *15* (2014), No. 2, 711–716, DOI: 10.18514/MMN.2014.640.

[29] A. Razani, *Chapman-Jouguet travelling wave for a two-steps reaction scheme*, Ital. J. Pure Appl. Math., *39* (2018), 544–553.

[30] A. Razani, *Subsonic detonation waves in porous media*, Phys. Scr., *94*(8) (2019), 6 pages, https://doi.org/10.1088/1402-4896/ab029b.

[31] S. Shokooh and G. Afrouzi, *Existene results of infinity many solutions for a class of $p(x)$-biharmonic problems*, Computational Methods for Differential Equations, *5* (2017), 310–323.

[32] J. Simon, *Regularite de la solution d'une equation non lineaire dans $\mathbb{R}^{\mathbb{N}}$*, Jornees d'Analyse NonLineaire, *665* (1978), 205–227.

[33] S. Taarabti, Z. El Allali, and K. Hadddouch, *Eigenvalues of $p(x)$-biharmonic operator with indefinite weight under Neumann boundary condition*, Bol. Soc. Paran. Mat., *36* (2018), 195–213.

[34] Y. Wang and Y. Shen, *Multiple and sign-changing solutions for a class of semilinear biharmonic equation*, J. Differential Equations, *246* (2009), 3109–3125.

[35] H. Xie and J. Wang, *Infinitely many solutions for p-Harmonic equation with singular term*, J. Inequal. Appl., *2013*(9) (2013), https://doi.org/10.1186/1029-242X-2013-9.

[36] H. Yin and Y. Liu, *Existence of three solutions for a Navier boundary value problem involving the $p(x)$-biharmonic*, Bull. Korean Math. soc., *50* (2013), 1817-1826.

[37] E. Zeidler, *Nonlinear functional analysis and its application*, Springer Verlage, Berlin-Heidelberg-Newyork, 1986.