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An efficient technique based on the HAM with Green's function for a class of nonlocal elliptic boundary value problems

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Abstract In this paper, we propose an efficient technique-based optimal homotopy analysis method with Green's function technique for the approximate solutions of nonlocal elliptic boundary value problems. We first transform the nonlocal boundary value problems into the equivalent integral equations. We then apply the optimal homotopy analysis method for the approximate solution of the considered problems. Several examples are considered to compare the results with the existing technique. The numerical results confirm the reliability of the present method as it tackles such nonlocal problems without any limiting assumptions. We also provide the convergence and the error estimation of the proposed method.

Keywords. Optimal homotopy analysis method, Nonlinear nonlocal elliptic BVPs, Convergence analysis, Integral equations.

2010 Mathematics Subject Classification. 33F05, 65D20, 65L10, 65L80, 34B05, 34B15, 34B18, 34B27.

1. Introduction

The aim of this article is to use the homotopy analysis method with Green's function technique for obtaining the accurate numerical solutions for a class of one-dimensional nonlocal elliptic boundary value problems. We first consider the following class of linear nonlocal elliptic boundary value problems [2, 8] as

$$\begin{cases} -\alpha\left(\int_0^1 y(s)ds\right)y''(x) = h(x), \ x \in (0,1), \\ y(0) = a, \ y(1) = b, \ a, b \in [0, \infty). \end{cases} \tag{1.1}$$

We also consider the following class of nonlinear nonlocal elliptic nonlinear boundary value problems [3, 8, 20] as

$$\begin{cases} -\alpha\left(\int_0^1 y(s)ds\right)y''(x) + y^{2n+1}(x) = 0, \ x \in (0,1), \\ y(0) = a, \ y(1) = b, \ a, b \in [0, \infty), \ n \in 0 \cup \mathbb{Z}^+. \end{cases} \tag{1.2}$$

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Such nonlocal boundary value problems arise in modeling various fields such as the aero-elastic behavior of suspended flexible cables subjected to icing conditions and wind action [11, 12, 20]. For details on such physical applications of nonlocal boundary value problems see [3, 8, 19] and the references therein.

In [2], the existence and uniqueness of solution of (1.1) was discussed by using a fixed point theorem and then the numerical solutions were obtained via a finite difference scheme. In [3], authors discussed a numerical method for (1.2), where they found a priori estimates and the existence and uniqueness of the solution for the nonlinear auxiliary problem. In [8], Khuri and Wazwaz applied the variational iteration method for the approximate solutions of (1.1) and (1.2) and established uniform convergence of the scheme. In [20], Themistoclakis and Vecchio provided the sufficient conditions for the unique solution and a general convergence for (1.2) and used the iterative techniques to tackle the nonlocal nonlinearity of the problem. In [13], the authors presented the stability and numerical approximation for a spacial class of semi-linear parabolic equations on the Lipschitz bounded regions. The least square homotopy perturbation method for boundary value problems was studied in [1]. In [14] authors used the multi-wavelets Galerkin method to solve delay differential equations with vanishing delay known as Pantograph equation. A numerical technique based on the wavelet Galerkin method was applied for solving the nonlinear Benjamin-Bona-Mahony equation in [15]. The semi-analytic methods such as the homotopy perturbation method, the variational iteration method and the Adomian decomposition method were applied to solve the Fitzhugh-Nagumo equation [5]. The homotopy analysis method was applied to solve nonlinear fractional partial differential equations [6, 7].

In this paper, our goal is to use the homotopy analysis method with Green's function technique for obtaining the accurate numerical solutions for a class of one-dimensional nonlocal elliptic boundary value problems (1.1) and (1.2). Firstly, we will transform the problems (1.1) and (1.2) into the equivalent integral equations. We then apply the OHAM [16] to obtain the accurate numerical solutions.

2. Homotopy analysis method with Green's function

2.1. Integral form of the problem. Consider a general nonlocal elliptic BVPs form of (1.1) and (1.2) as

$$\begin{cases} \alpha(p)y''(x) = f(y(x)), \quad x \in (0,1), \\ y(0) = a, \quad y(1) = b, \end{cases} \tag{2.1}$$

where $p = \int_0^1 y(s)ds$ and $\alpha(p)$ is a continuous function. It should be noted that the nonlocal elliptic BVPs (1.1) with $f = -h(x)$ and (1.2) with $f = y^{2n+1}(x)$ are particular cases of (2.1).

Following Singh et al. [17, 18], the integral form of (2.1) is given by

$$y(x) = a + (b-a)x + \frac{1}{\alpha(p)} \int_0^1 G(x,s) f(y(s)) ds, \tag{2.2}$$

where $G(x,s)$ is

$$G(x,s) = \begin{cases} x(s-1), & x \leq s, \\ s(x-1), & s \leq x. \end{cases} \tag{2.3}$$

2.2. **Homotopy analysis method.** In view of HAM [9, 10], the zero-order deformation equation for (2.2) is constructed as

$$(1-q)[\phi(x,q) - y_0(x)] = q\, c_0\, T[\phi(x,q)],\ \ q \in [0,1], \tag{2.4}$$

where q is an embedding parameter, $y_0(x)$ is an initial guess, $c_0 \neq 0$ is convergence control parameter, $\phi(x,q)$ is an unknown function, and $T[\phi(x,q)]$ is given by

$$T[\phi(x,q)] = \phi(x,q) - \big(a + (b-a)x\big) - \frac{1}{\alpha(p[\phi(x,q)])} \int_0^1 G(x,s) f(\phi(s,q)) ds = 0. \tag{2.5}$$

Substituting $q = 0$, the zero-order deformation equation (2.4) reduces to $\phi(x,0) = y_0(x)$. If we substitute $q = 1$, equation (2.4) becomes $T[\phi(x,1)] = 0$. Therefore, we have $\phi(x,1) = y(x)$, where $y(x)$ is desired solution of (2.2).

Expanding $\phi(x,q)$ by Taylor expansion with respect to q, we find

$$\phi(x,q) = y_0(x) + \sum_{k=1}^{\infty} y_k(x) q^k, \tag{2.6}$$

where $y_k(x)$ is given by

$$y_k(x) = \frac{1}{k!} \frac{\partial^k}{\partial q^k} [\phi(x,q)] \bigg|_{q=0}. \tag{2.7}$$

The series (2.6) converges for $q = 1$ if $c_0 \neq 0$ is chosen properly and it reduces to

$$\phi(x,1) \equiv y(x) = \sum_{k=0}^{\infty} y_k(x), \tag{2.8}$$

which would be one of the solutions of (2.2). For further analysis we define the vector

$$\overrightarrow{y_k} = \{y_0(x), y_1(x), \ldots, y_k(x)\}.$$

In order to determine the function y_k, we differentiate (2.4) k-times with respect to q. The result is then divided by $k!$, and we substitute $q = 0$, which gives the so-called kth-order deformation equation as

$$y_k(x) - \chi_k\, y_{k-1}(x) = c_0\, \mathbf{R}_k(\overrightarrow{y}_{k-1}, x), \tag{2.9}$$

where χ_k is defined as

$$\chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k > 1, \end{cases} \tag{2.10}$$

and

$$\begin{aligned}\mathbf{R}_k(\overrightarrow{y}_{k-1},x) &= \frac{1}{(k-1)!}\left[\frac{\partial^{k-1}T[\phi(x,q)]}{\partial q^{k-1}}\right]_{q=0}\\ &= \frac{1}{(k-1)!}\left[\frac{\partial^{k-1}}{\partial q^{k-1}}T\left(\sum_{m=0}^{\infty}y_m(x)q^m\right)\right]_{q=0}\\ &= y_{k-1}(x)-(1-\chi_k)\big(a+(b-a)x\big)-\frac{1}{\alpha(p_{k-1})}\int_0^1 G(x,s)\mathcal{H}_{k-1}ds.\end{aligned}$$

Thus we have

$$\mathbf{R}_k(\overrightarrow{y}_{k-1},x) = y_{k-1}(x)-(1-\chi_k)\big(a+(b-a)x\big)-\frac{1}{\alpha(p_{k-1})}\int_0^1 G(x,s)\mathcal{H}_{k-1}ds, \quad (2.11)$$

where p_k and $\mathcal{H}_k$ are given by

$$p_k = \int_0^1\left(\sum_{m=0}^{k}y_m(x)q^m\right)ds, \quad (2.12)$$

$$\mathcal{H}_k = \frac{1}{k!}\frac{\partial^k}{\partial q^k}\left[f\left(\sum_{m=0}^{k}y_m(x)q^m\right)\right]_{q=0}. \quad (2.13)$$

Using equations (2.9)- (2.13), we find the mth-order deformation equation as

$$y_k-\chi_k y_{k-1} = c_0\left[y_{k-1}-(1-\chi_k)\big(a+(b-a)x\big)-\frac{1}{\alpha(p_{k-1})}\int_0^1 G(x,s)\mathcal{H}_{k-1}ds\right]. \quad (2.14)$$

By taking an initial guess $y_0 = a+(b-a)x$, the components y_k are successively obtained. Hence, the nth-order approximate solution of (2.2) is obtained as

$$\phi_n(x,c_0) = \sum_{k=0}^{n}y_k(x,c_0). \quad (2.15)$$

To find the optimal value of c_0, we minimize the following squared residual equation

$$E_n(c_0) = \int_0^1\big(N[\phi_n(\xi,c_0)]\big)^2 d\xi. \quad (2.16)$$

But in some cases, to compute the exact squared residual error (2.16) is very complicated for large n. To avoid this difficulty, we use the following discrete averaged residual error

$$E_n(c_0) \approx \frac{1}{M}\sum_{k=1}^{M}\big(N[\phi_n(\xi_k,c_0)]\big)^2,\ \xi_k = kh,\ k=1,2,...M. \quad (2.17)$$

Finally, we find the optimal value of c_0 by solving

$$\frac{dE_n}{dc_0} = 0.$$

Remark 2.1. By setting $c_0 = -1$, the scheme (2.14) reduces to the Adomian decomposition method (ADM) with Green's function [17, 18].

3. Convergence analysis

In this section, we establish the convergence of the proposed method. Let $\mathbb{X} = \big(C[0,1], \|y\|\big)$ be a Banach space with norm defined as

$$\|y\| = \max_{x\in[0,1]} |y(x)|, \; y \in \mathbb{X}.$$

Theorem 3.1. If the solution components $y_0, y_1, y_2, \dots$ given in (2.15) satisfy the condition $\|y_{k+1}\| \leq \delta\|y_k\|$, $\forall\ k \geq k_0$. Then the series solution $\sum_{k=0}^{\infty} y_k$ is convergent whenever $0 < \delta < 1$.

Proof. Define the sequence $\{\phi_n\}_{n=0}^{\infty}$ as,

$$\phi_0 = y_0, \; \phi_1 = y_0 + y_1, \; \phi_2 = y_0 + y_1 + y_2, \dots, \; \phi_n = y_0 + y_1 + y_2 + \cdots + y_n. \tag{3.1}$$

We next show that $\{\phi_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $\mathbb{X}$.

Consider

$$\|\phi_{n+1} - \phi_n\| = \|y_{n+1}\| \leq \delta\|y_n\| \leq \delta^2\|y_{n-1}\| \leq \dots \leq \delta^{n-k_0+1}\|y_{k_0}\|.$$

For $n, m \in \mathbb{N}$, $n \geq m > k_0$, we get

$$\begin{aligned} \|\phi_n - \phi_m\| &= \|(\phi_n - \phi_{n-1}) + \cdots + (\phi_{m+1} - \phi_m)\| \\ &\leq \|\phi_n - \phi_{n-1}\| + \cdots + \|\phi_{m+1} - \phi_m\| \\ &\leq (\delta^{n-k_0} + \delta^{n-k_0-1} + \cdots + \delta^{m-k_0+1})\|y_{k_0}\| \\ &= \frac{1-\delta^{n-m}}{1-\delta}\delta^{m-k_0+1}\|y_{k_0}\|. \end{aligned} \tag{3.2}$$

Since $0 < \delta < 1$, and above inequality becomes $\lim_{n,m\to\infty} \|\phi_n - \phi_m\| = 0$. Therefore, $\{\phi_n\}_{n=0}^{\infty}$ is a Cauchy sequence. ☐

Theorem 3.2. If $\sum_{k=0}^{\infty} y_k \to y$ and $\phi_m = \sum_{k=0}^{m} y_k$ is an approximation to the solution $y(x)$, then the absolute truncated error is given by

$$|y - \phi_m| \leq \frac{\delta^{m-k_0+1}}{1-\delta}\|y_{k_0}\|. \tag{3.3}$$

Proof. From (3.2), we see

$$\|\phi_n - \phi_m\| \leq \frac{1-\delta^{n-m}}{1-\delta}\delta^{m-k_0+1}\|y_{k_0}\|,$$

for $n \geq m$. Now, as $n \to \infty$ then $\phi_n \to y$ and $\delta^{n-m} \to 0$. So,

$$\|y - \phi_m\| \leq \frac{\delta^{m-k_0+1}}{1-\delta}\|y_{k_0}\|. \tag{3.4}$$

☐

Theorem 3.3. If $\sum_{k=0}^{\infty} y_k \to y$ then it must be a solution of equation (2.2).

Proof. Since the series $\sum_{k=0}^{\infty} y_k$ is convergent, then

$$\lim_{n\to\infty} y_n = 0, \ \forall\ x \in [0,1]. \tag{3.5}$$

By summing up the left hand-side of (2.9), we get

$$\sum_{k=1}^{n}(y_k - \chi_k y_{k-1}) = y_1 + \ldots + (y_n - y_{n-1}) = y_n. \tag{3.6}$$

Letting $n \to \infty$ and using (3.5), equation (3.6) reduces to

$$\sum_{k=1}^{\infty}(y_k - \chi_k\ y_{k-1}) = 0. \tag{3.7}$$

Using (3.7) and right hand-side of the relation (2.9), we obtain

$$\sum_{k=1}^{\infty} c_0\ \mathbf{R}_k(\overrightarrow{y}_{k-1}, x) = \sum_{k=1}^{\infty}(y_k - \chi_k\ y_{k-1}) = 0. \tag{3.8}$$

Since $c_0 \neq 0$, then equation (3.8) reduces to

$$\sum_{k=1}^{\infty}\ \mathbf{R}_k(\overrightarrow{y}_{k-1}, x) = 0. \tag{3.9}$$

Using (3.9) and (2.9), we have

$$\begin{aligned}
0 &= \sum_{k=1}^{\infty}\ \mathbf{R}_k(\overrightarrow{y}_{k-1}, x)\\
&= \sum_{k=1}^{\infty}\left(y_{k-1} - (1-\chi_k)\big(a + (b-a)x\big) - \frac{1}{\alpha(p_{k-1})}\int_0^1 G(x,s)\mathcal{H}_{k-1} ds\right)\\
&= \sum_{k=1}^{\infty} y_{k-1} - \big(a + (b-a)x\big) - \frac{1}{\sum_{k=1}^{\infty}\alpha(p_{k-1})}\int_0^1 G(x,s)\sum_{k=1}^{\infty}\mathcal{H}_{k-1} ds.
\end{aligned}$$

Since $\sum_{k=0}^{\infty} y_k \to y$, then $\sum_{k=0}^{\infty}\mathcal{H}_k \to f(y)$ and $\sum_{k=1}^{\infty}\alpha(p_{k-1}) \to \alpha(p)$ [4], we obtain

$$y = a + (b-a)x + \frac{1}{\alpha(p)}\int_0^1 G(x,s) f(y(s)) ds.$$

Hence, y is the exact solution of (2.2). □

4. Numerical results

In this section, we consider four examples to the compare numerical results with the existing exact solutions. For this purpose, we define the absolute errors as

$$E_n(x) = |y(x) - \phi_n(x)|,\ \ e_n(x) = |y(x) - \psi_n(x)|,$$

where $\phi_n(x)$ and $\psi_n(x)$ are the homotopy analysis and Adomian decomposition solutions, respectively.

Example 4.1. Consider the special linear case of the problem (1.1) with $\alpha(p) = p^{1/3}$ as

$$\begin{cases} p^{1/3} y''(x) = \dfrac{6}{\sqrt[3]{4}} x, \ x \in (0,1) \\ y(0) = 0, \ y(1) = 1, \ p = \left(\int\limits_0^1 y(s) ds \right). \end{cases} \tag{4.1}$$

The analytical solution is $y(x) = x^3$.

Applying the OHAM (2.14) to the problem (4.1), we have

$$y_k - \chi_k y_{k-1} = c_0 \left(y_{k-1} - (1 - \chi_k) y_0 - \frac{1}{\alpha(p_{k-1})} \int\limits_0^1 G(x,s) \frac{6s}{\sqrt[3]{4}} ds \right). \tag{4.2}$$

Using (4.2) with an initial approximation $y_0 = x$, we obtain $\phi_2(x, c_0)$. With the help of (2.17), the optimal value of the parameter c_0 is computed as $c_0 = -0.505595$. Hence the OHAM solution is given by

$$\phi_2(x) = 7.7 \times 10^{-16} x + x^3. \tag{4.3}$$

A comparison among the numerical results obtained by the OHAM solution $\phi_2(x)$, the ADM solution $\psi_2(x)$ and the exact solution $y(x)$ is depicted in Table 1 and Figure 1. Also, the absolute errors are plotted in Figure 4.

TABLE 1. Numerical results of Example 4.1

x	y	$\psi_2(x)$	$\phi_2(x)$	$\lvert y - \psi_2 \rvert$	$\lvert y - \phi_2 \rvert$
0.0	0.000	0.0000000000	0.000	0.000000000	0.00000000
0.1	0.001	-0.062559671	0.001	0.063559671	7.67615E-17
0.2	0.008	-0.115267240	0.008	0.123267240	1.49186E-16
0.3	0.027	-0.148270607	0.027	0.175270607	2.11636E-16
0.4	0.064	-0.151717670	0.064	0.215717670	2.49800E-16
0.5	0.125	-0.115756328	0.125	0.240756328	2.77556E-16
0.6	0.216	-0.030534480	0.216	0.246534480	3.05311E-16
0.7	0.343	0.1137999760	0.343	0.229200024	2.77556E-16
0.8	0.512	0.3270991400	0.512	0.184900860	2.22045E-16
0.9	0.729	0.6192151140	0.729	0.109784886	1.11022E-16
1.0	1.000	1.0000000000	1.000	2.22045E-16	0.000000000

Example 4.2. Consider the special nonlinear case of the problem (1.2) with $\alpha(p) = \frac{1}{p}$ as

$$\begin{cases} -\dfrac{1}{p} y''(x) + \dfrac{3}{4(2\sqrt{2} - 2)} y^5(x) = 0, \ x \in (0,1), \\ y(0) = 1, \ y(1) = \frac{\sqrt{2}}{2}, \ p = \left(\int\limits_0^1 y(s) ds \right). \end{cases} \tag{4.4}$$

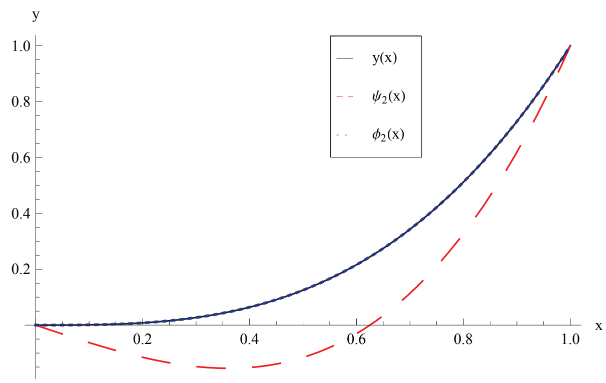

FIGURE 1. Comparison of the numerical solutions of Example 4.1

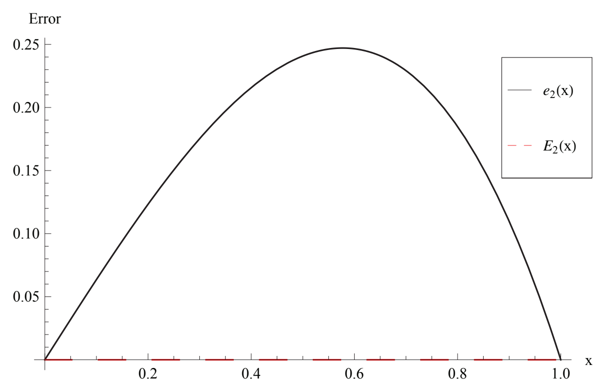

FIGURE 2. Comparison of the numerical absolute error of Example 4.1

The analytical solution is $y(x) = \frac{1}{\sqrt{1+x}}$.

Applying the OHAM (2.14) to the problem (4.2), we have

$$y_k - \chi_k y_{k-1} = c_0 \left(y_{k-1} - (1-\chi_k) y_0 - \frac{1}{\alpha(p_{k-1})} \int_0^1 G(x,s) \mathcal{H}_{k-1} ds \right). \tag{4.5}$$

Using (4.5) with an initial guess $y_0 = 1 + (\frac{\sqrt{2}}{2} - 1)x$ and (2.17) with $c_0 = -0.819014$, we obtain the homotopy optimal solution as

$$\begin{aligned}\phi_2(x) &= 1 - 0.4973x + 0.3737x^2 - 0.3060x^3 + 0.2235x^4 - 0.1258x^5 + 0.05148x^6 \\ &\quad - 0.0153x^7 + 0.0033x^8 - 0.00055x^9 + 0.0000646x^{10} + \cdots\end{aligned}$$

A comparison among the numerical solution obtained by the OHAM solution $\phi_2(x)$, the ADM solution $\psi_2(x)$ and the exact solution is depicted in Table 2 and Figure 3. Also, the absolute errors are plotted in Figure 4.

TABLE 2. Numerical results of Example 4.2

x	y	ψ_2	ϕ_2	$\lvert y-\psi_2\rvert$	$\lvert y-\phi_2\rvert$
0.0	1.000000000	1.000000000	1.000000000	0.000000000	0.000000000
0.1	0.953462589	0.954516555	0.953715758	0.001053966	0.000253169
0.2	0.912870929	0.914909713	0.913348055	0.002038784	0.000477126
0.3	0.877058019	0.879819116	0.877702187	0.002761097	0.000644168
0.4	0.845154255	0.848286083	0.845886767	0.003131828	0.000732512
0.5	0.816496581	0.819638873	0.817233417	0.003142293	0.000736836
0.6	0.790569415	0.793406404	0.791235825	0.002836989	0.000666410
0.7	0.766964989	0.769254375	0.767504100	0.002289386	0.000539111
0.8	0.745355992	0.746938889	0.745731089	0.001582896	0.000375097
0.9	0.725476250	0.726273545	0.725667948	0.000797295	0.000191698
1.0	0.707106781	0.707106781	0.707106781	0.000000000	2.22045E-16

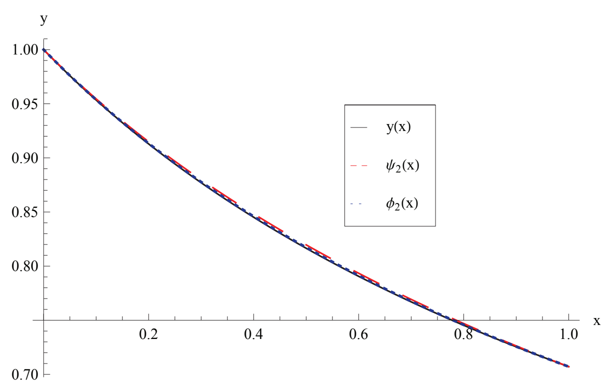

FIGURE 3. Comparison of the numerical solutions of Example 4.2

Example 4.3. Consider the special nonlinear case of the problem (1.2) with $\alpha(p) = p$ as

$$\begin{cases} -p\, y''(x) + \dfrac{3(2\sqrt{2}-2)}{4} y^5(x) = 0, \ x \in (0,1) \\ y(0) = 1, \ y(1) = \frac{\sqrt{2}}{2}, \ p = \left(\int\limits_0^1 y(s)ds \right). \end{cases} \tag{4.6}$$

The analytical solution is $y(x) = \frac{1}{\sqrt{1+x}}$.

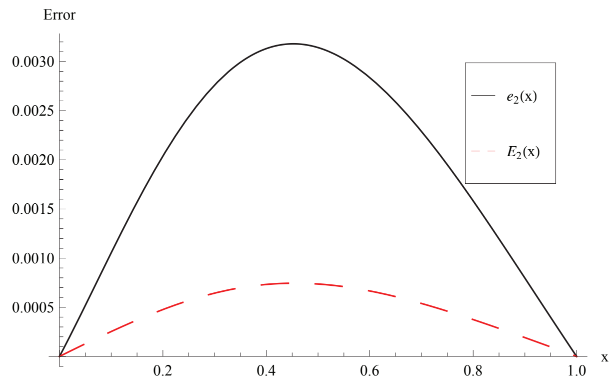

FIGURE 4. Comparison of the numerical absolute error of Example 4.2

Applying the OHAM (2.14) with $y_0 = 1 + (\frac{\sqrt{2}}{2} - 1)x$, we have

$$y_k(x) - \chi_k y_{k-1}(x) = c_0 \left(y_{k-1}(x) - (1-\chi_k) y_0(x) - \frac{1}{\alpha(p_{k-1})} \int_0^1 G(x,s)\mathcal{H}_{k-1} ds \right). \quad (4.7)$$

Using (4.7) and (2.17) with $c_0 = -0.933697$, we obtain the homotopy optimal solution as

$$\begin{aligned}\phi_2(x) &= 1 - 0.495211x + 0.362361x^2 - 0.280911x^3 + 0.195035x^4 \\ &\quad - 0.107115x^5 + 0.043453x^6 - 0.01294x^7 + 0.002854x^8 \\ &\quad - 0.000464x^9 + 0.000054x^{10} - \cdots\end{aligned}$$

A comparison among the numerical solution obtained by the OHAM solution $\phi_2(x)$, the ADM $\psi_2(x)$ and the exact solution is depicted in Table 3 and Figure 5. Also, the absolute errors are plotted in Figure 6.

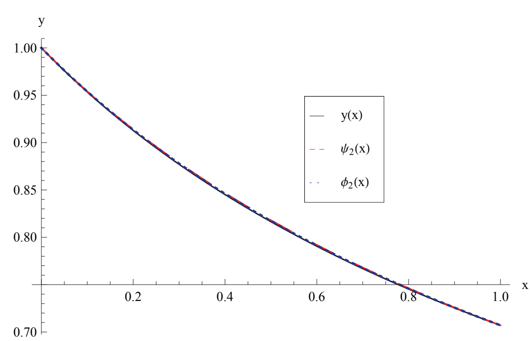

FIGURE 5. Comparison of the numerical solutions of Example 4.3

TABLE 3. Numerical results of Example 4.3

x	y	ψ_2	ϕ_2	$\lvert y-\psi_2\rvert$	$\lvert y-\phi_2\rvert$
0.0	1.000000000	1.000000000	1.000000000	0.000000000	0.000000000
0.1	0.953462589	0.954139200	0.953840089	0.000676611	0.000377500
0.2	0.912870929	0.914044009	0.913485384	0.001173080	0.000614454
0.3	0.877058019	0.878552757	0.877813154	0.001494738	0.000755135
0.4	0.845154255	0.846797874	0.845969674	0.001643619	0.000815419
0.5	0.816496581	0.818128147	0.817301384	0.001631566	0.000804803
0.6	0.790569415	0.792049712	0.791302438	0.001480297	0.000733023
0.7	0.766964989	0.768181846	0.767575192	0.001216857	0.000610203
0.8	0.745355992	0.746224331	0.745800843	0.000868338	0.000444851
0.9	0.725476250	0.725933776	0.725717917	0.000457526	0.000241667
1.0	0.707106781	0.707106781	0.707106781	0.000000000	2.22045E-16

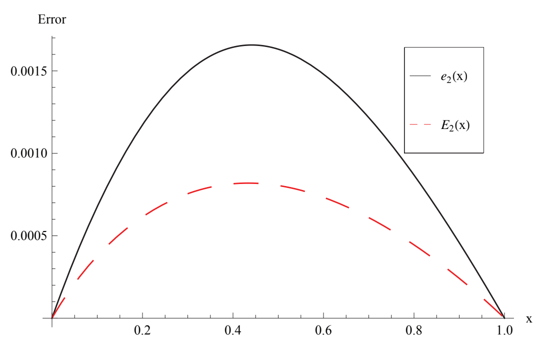

FIGURE 6. Comparison of the numerical absolute error of Example 4.3

Example 4.4. Consider the special case of the problem (1.2) with $\alpha(p) = \left(\frac{1}{p}\right)^2$ as

$$\begin{cases} -\left(\dfrac{1}{p}\right)^2 y''(x) + \dfrac{2}{(\ln 2)^2} y^3(x) = 0, \ x \in (0,1), \\ y(0) = 1, \ y(1) = \frac{1}{2}, \ p = \left(\int\limits_0^1 y(s)ds\right). \end{cases} \tag{4.8}$$

The analytical solution is $y(x) = \frac{1}{1+x}$.

Applying the OHAM (2.14) with $y_0 = 1 + (\frac{1}{2} - 1)x$, we have

$$y_k(x) - \chi_k y_{k-1}(x) = c_0 \left(y_{k-1}(x) - (1-\chi_k) y_0(x) - \frac{1}{\alpha(p_{k-1})} \int\limits_0^1 G(x,s)\mathcal{H}_{k-1} ds \right). \tag{4.9}$$

Using (4.9) and (2.17) with $c_0 = -0.612671$, we obtain the homotopy optimal solution as

$$\begin{aligned}\phi_2(x) = 1 &- 1.00399x + 0.995127x^2 - 0.90086x^3 + 0.655261x^4 \\ &- 0.338986x^5 + 0.115228x^6 - 0.0246916x^7 + 0.00308645x^8 \\ &- 0.00017147x^9 + \cdots\end{aligned}$$

A comparison among the numerical solution obtained by the OHAM solution $\phi_2(x)$, the ADM solution $\psi_2(x)$ and the exact solution is depicted in Table 4 and Figure 7. Also, the absolute errors are plotted in Figure 8.

TABLE 4. Numerical results of Example 4.4

x	y	ψ_2	ϕ_2	$\lvert y(x) - \psi_2 \rvert$	$\lvert y - \phi_2 \rvert$
0.0	1.000000000	1.000000000	1.000000000	0.000000000	0.000000000
0.1	0.909090909	0.914550054	0.908713383	0.005459145	0.000377526
0.2	0.833333333	0.844849352	0.832746652	0.011516019	0.000586682
0.3	0.769230769	0.785573674	0.768603045	0.016342905	0.000627724
0.4	0.714285714	0.733317575	0.713705107	0.019031861	0.000580607
0.5	0.666666667	0.686029523	0.666157571	0.019362857	0.000509096
0.6	0.625000000	0.642573190	0.624561470	0.017573190	0.000438530
0.7	0.588235294	0.602393994	0.587870965	0.014158699	0.000364329
0.8	0.555555556	0.565272431	0.555285414	0.009716875	0.000270141
0.9	0.526315789	0.531148042	0.526170109	0.004832252	0.000145681
1.0	0.500000000	0.500000000	0.500000000	6.66134E-16	1.66533E-16

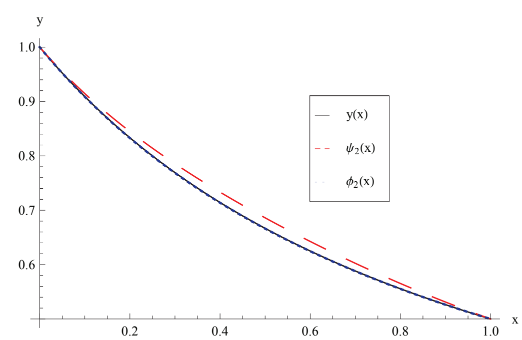

FIGURE 7. Comparison of the numerical solutions of Example 4.4

5. CONCLUSION

In this work, we have presented the optimal homotopy analysis method for solving a class of nonlocal linear and non-linear BVPs. The nonlocal BVPs were transformed

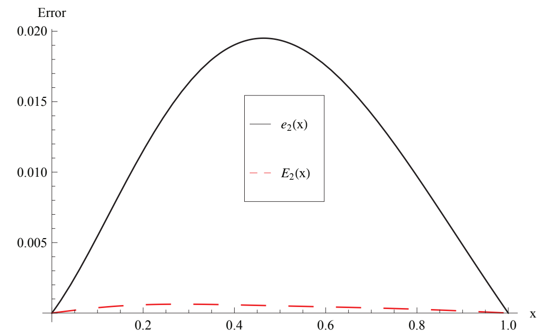

FIGURE 8. Comparison of the numerical absolute error of Example 4.4

into the equivalent integral equations. Then the OHAM was applied to obtain the approximate solutions of the problems. Unlike the ADM, the OHAM always provides better approximate series solution as shown in Tables. We have also discussed the convergence and error analysis of the proposed method.

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