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# A Laguerre approach for solving of the systems of linear differential equations and residual improvement

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**Abstract** In this study, a collocation method based on Laguerre polynomials is presented to numerically solve systems of linear differential equations with variable coefficients of high order. The method contains the following steps. Firstly, we write the Laguerre polynomials, their derivatives, and the solutions in matrix form. Secondly, the system of linear differential equations is reduced to a system of linear algebraic equations by means of matrix relations and collocation points. Then, the conditions in the problem are also written in the form of matrix of Laguerre polynomials. Hence, by using the obtained algebraic system and the matrix form of the conditions, a new system of linear algebraic equations is obtained. By solving the system of the obtained new algebraic equation, the coefficients of the approximate solution of the problem are determined. For the problem, the residual error estimation technique is offered and approximate solutions are improved. Finally, the presented method and error estimation technique are demonstrated with the help of numerical examples. The results of the proposed method are compared with the results of other methods.

**Keywords.** Collocation method, Collocation points, Laguerre collocation method, Laguerre polynomials, Systems of linear differential equations.

**2010 Mathematics Subject Classification.** 34A30, 34K28, 65L05, 65L60, 65L80.

## 1. Introduction

Differential equations and their systems are an important part of many fields of engineering such as electricity, hydraulics, mechanics, and science such as mathematics, physics, and dynamics. It is sometimes not possible to calculate exact solutions of these equations or their systems, in which case approximate solutions of the equations or their systems play an important role for us. It can be used several methods to calculate approximate solutions. Recently, the systems of differential equations have been solved by the numerical methods such as the Chebyshev polynomial approach [1], the Adomian decomposition method [17], the Taylor polynomial approach [24],

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the differential transformation method [16], the improvement of He's variational iteration method [25], the differential transform and Laplace transform methods [27], the Bessel polynomial approach [36], and the exponential Chebyshev collocation method [20]. Additionally, systems of rational differential equations [14, 28], systems of multi-pantograph equations [15, 30, 33], systems of integral equations [2, 12, 13, 21] have been solved by many numerical methods.

Furthermore, the Lane-Emden equation [18], the unsteady gas equation [19], the initial-boundary value problems [26], the flow and heat transfer [23], the Lane-Emden type functional differential equations [8], the pantograph-type Volterra integro-differential equations [34], the Fredholm integro-differential equations with functional arguments [5], the linear Fredholm integro-differential equation [29], the parabolic-type Volterra partial integro-differential equation [7], the nonlinear partial integro-differential equation [6], the delay partial functional differential equation [9], the singularly-perturbed differential equation [31], the system of linear Fredholm integro-differential equations [37], the fractional Volterra integro-differential equations [10], the infinite boundary integro-differential equations [22], the systems of first order linear differential equations [32] and the fractional Fredholm integro-differential equations [3] have been solved by Laguerre polynomial approach.

The first aim of this study is to present the Laguerre collocation method to solve the systems of linear differential equations with variable coefficient in form

$$\sum_{n=0}^{m}\sum_{j=1}^{k} P_{i,j}^{n}(x)y_j^{(n)}(x) = g_i(x), i = 1, 2, ..., k \tag{1.1}$$

with the mixed conditions

$$\sum_{j=0}^{m-1}\left(a_{i,j}^{n}y_n^{(j)}(a) + b_{i,j}^{n}y_n^{(j)}(b)\right) = \lambda_{n,i} \tag{1.2}$$

$i = 0, 1, 2, ..., m-1,\ n = 1, 2, ..., k$. Here, $y_j^{(0)}(x) = y_j(x)$ is an unknown function, $P_{i,j}^{n}(x)$ and $g_i(x)$ are the known functions defined on interval $0 \le a \le x \le b$ and $a_{i,j}^{n}$, $b_{i,j}^{n}$ and $\lambda_{n,i}$ are appropriate real constants.

The second aim of the paper is to improve the solutions by means of the residual error estimation technique.

By using the Laguerre collotion method in this study, we will obtain an approximate solution under conditions (1.2) of system (1.1) expressed in the truncated Laguerre series form

$$y_{i,N}(x) = \sum_{n=0}^{N} a_{i,n}L_n(x), \quad i = 1, 2, ..., k. \tag{1.3}$$

Here, $N$ is any chosen positive integer and $a_{i,n}, (n = 0, 1, 2, ..., N)$ are Laguerre coefficients to be determined.

Also, the recurrence relation for the derivative of Laguerre polynomials is [11]

$$L_n^{'}(x) = L_{n-1}^{'}(x) - L_{n-1}(x), \quad L_0^{'}(x) = 0. \tag{1.4}$$

## 2. Method of Solution

2.1. **Fundamental matrix relations.** In this section, approximate solutions given in relation (1.3) can be written in the matrix form as

$$[y_j(x)] = \mathbf{L}(x)\mathbf{A}_j, \quad j = 1, 2, ..., k, \tag{2.1}$$

where

$$\mathbf{A}_j = \begin{bmatrix} a_{j,0} & a_{j,1} & \cdots & a_{j,N} \end{bmatrix}^T, \quad j = 1, 2, ..., k$$

and

$$\mathbf{L}(x) = \begin{bmatrix} L_0(x) & L_1(x) & \cdots & L_N(x) \end{bmatrix}.$$

Let us find the matrix form of the relation between the matrix $[y_j(x)]$ and its $n$-th derivative $[y_j^{(n)}(x)]$. The relation between the matrix $[y_j(x)]$ and its 1-th derivative $[y_j^{'}(x)]$ is

$$[y_j^{'}(x)] = \mathbf{L}^{'}(x)\mathbf{A}_j, \quad j = 1, 2, ..., k, \tag{2.2}$$

where

$$\mathbf{L}^{'}(x) = \begin{bmatrix} L_0^{'}(x) & L_1^{'}(x) & \cdots & L_N^{'}(x) \end{bmatrix}.$$

With the help of the expression (1.4), the derivative of $\mathbf{L}(x)$ is [31]

$$\mathbf{L}^{'}(x) = \mathbf{L}(x)\mathbf{M}^T, \tag{2.3}$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 \end{bmatrix}_{(N+1)\times(N+1)}.$$

Similarly, the relation between the matrix $[y_j(x)]$ and its $n$-th derivative $[y_j^{(n)}(x)]$ is

$$[y_j^{(n)}(x)] = \mathbf{L}^{(n)}(x)\mathbf{A}_j, \quad j = 1, 2, ..., k \tag{2.4}$$

where

$$\mathbf{L}^{(n)}(x) = \begin{bmatrix} L_0^{(n)}(x) & L_1^{(n)}(x) & \cdots & L_N^{(n)}(x) \end{bmatrix}.$$

Thus, the matrix form of the relation between the matrix $\mathbf{L}(x)$ and its $n$-th derivative $\mathbf{L}^{(n)}(x)$, based on Laguerre polynomials, is

$$\mathbf{L}^{(n)}(x) = \mathbf{L}(x)(\mathbf{M}^T)^n. \tag{2.5}$$

By using the relations (2.4) and $\mathbf{L}^{(n)}(x)$ that $n$-th derivative of $\mathbf{L}(x)$ given in relation (2.5), we have recurrence relations

$$y_j^{(n)}(x) = \mathbf{L}(x)(\mathbf{M}^T)^n\mathbf{A}_j.$$

Hence, the matrices $y^{(n)}(x)$ can be written as

$$\mathbf{y}^{(n)}(x) = \bar{\mathbf{L}}(x)(\bar{\mathbf{M}})^n\mathbf{A}, \quad n = 0, 1, 2, ..., m, \tag{2.6}$$

where

$$\mathbf{y}^{(n)}(x) = \begin{bmatrix} y_1^{(n)}(x) \\ y_2^{(n)}(x) \\ \vdots \\ y_k^{(n)}(x) \end{bmatrix}_{k\times 1}, \bar{\mathbf{L}}(x) = \begin{bmatrix} \mathbf{L}(x) & 0 & \cdots & 0 \\ 0 & \mathbf{L}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{L}(x) \end{bmatrix}_{k\times k}, \bar{\mathbf{M}} = \begin{bmatrix} \mathbf{M}^T & 0 & \cdots & 0 \\ 0 & \mathbf{M}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{M}^T \end{bmatrix}_{k\times k}, \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_k \end{bmatrix}_{k\times 1}.$$

Finally, in this section, let's demonstrate system given in (1.1) in matrix form as

$$\sum_{n=0}^{m} \mathbf{P}_n(x)\mathbf{y}^{(n)}(x) = \mathbf{g}(x),$$

where

$$\mathbf{P}_n(x) = \begin{bmatrix} P_{1,1}^n(x) & P_{1,2}^n(x) & \cdots & P_{1,k}^n(x) \\ P_{2,1}^n(x) & P_{2,2}^n(x) & \cdots & P_{2,k}^n(x) \\ \vdots & \vdots & \ddots & \vdots \\ P_{k,1}^n(x) & P_{k,2}^n(x) & \cdots & P_{k,k}^n(x) \end{bmatrix}_{k\times k}, \mathbf{y}^{(n)}(x) = \begin{bmatrix} y_1^{(n)}(x) \\ y_2^{(n)}(x) \\ \vdots \\ y_k^{(n)}(x) \end{bmatrix}_{k\times 1} \text{ and } \mathbf{g}(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_k(x) \end{bmatrix}_{k\times 1}.$$

2.2. **Fundamental Matrix Relation Based on Collocation Points.** Let us demonstrate the collocation points $x_s$ by

$$x_s = a + \frac{b-a}{N}s, \quad s = 0, 1, 2, ..., N,$$

where

$$0 \leq a \leq x \leq b \quad \text{and} \quad a = x_0 < x_1 < ... < x_n = b.$$

Now, by substituting these collocation points into the system given in (1.1) we obtain the system of matrix equation

$$\sum_{n=0}^{m} \mathbf{P}_n(x_s)\mathbf{y}^{(n)}(x_s) = \mathbf{g}(x_s), \quad s = 0, 1, 2, ..., N. \tag{2.7}$$

The obtained system in (2.7) can be written briefly in the matrix form

$$\sum_{n=0}^{m} \mathbf{P}_n\mathbf{Y}^{(n)} = \mathbf{G}, \tag{2.8}$$

where

$$\mathbf{P}_n = \begin{bmatrix} \mathbf{P}_n(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{P}_n(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_n(x_N) \end{bmatrix}, \mathbf{Y}^{(n)} = \begin{bmatrix} \mathbf{y}^{(n)}(x_0) \\ \mathbf{y}^{(n)}(x_1) \\ \vdots \\ \mathbf{y}^{(n)}(x_N) \end{bmatrix}, \mathbf{G} = \begin{bmatrix} \mathbf{g}(x_0) \\ \mathbf{g}(x_1) \\ \vdots \\ \mathbf{g}(x_N) \end{bmatrix}.$$

Then, by substituting collocation points into system given in (2.6) we obtain the system of matrix equation

$$\mathbf{y}^{(n)}(x_s) = \bar{\mathbf{L}}(x_s)(\bar{\mathbf{M}})^n\mathbf{A}, \quad s = 0, 1, 2, ..., N.$$

This system can also be written as

$$\mathbf{Y}^{(n)} = \mathbf{L}(\bar{\mathbf{M}})^n\mathbf{A}, \tag{2.9}$$

where

$$\mathbf{L}=\begin{bmatrix}\bar{\mathbf{L}}(x_0)\\ \bar{\mathbf{L}}(x_1)\\ \vdots\\ \bar{\mathbf{L}}(x_N)\end{bmatrix},\quad \bar{\mathbf{L}}(x_s)=\begin{bmatrix}\mathbf{L}(x_s) & 0 & \cdots & 0\\ 0 & \mathbf{L}(x_s) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{L}(x_s)\end{bmatrix}_{k\times k},\quad s=0,1,2,...,N.$$

2.3. **The Collocation Method.** When the relation (2.9) is substituted into the system given in (2.8), the system of matrix equation is obtained as

$$\left\{\sum_{n=0}^{m}\mathbf{P}_n\mathbf{L}(\bar{\mathbf{M}})^n\right\}\mathbf{A}=\mathbf{G}. \tag{2.10}$$

Remark that full dimensions of the matrices $\mathbf{P}_n$, $\mathbf{L}$, $\bar{\mathbf{M}}$, $\mathbf{A}$ and $\mathbf{G}$ in the system given in (2.10) are respectively, $k(N+1)\times k(N+1)$, $k(N+1)\times k(N+1)$, $k(N+1)\times k(N+1)$, $(N+1)\times 1$ and $k(N+1)\times 1$.

Briefly, we can also write this system in the form

$$\mathbf{WA}=\mathbf{G}\quad \text{or}\quad [\mathbf{W};\mathbf{G}], \tag{2.11}$$

where

$$\mathbf{W}=\sum_{n=0}^{m}\mathbf{P}_n\mathbf{L}(\bar{\mathbf{M}})^n=[\omega_{p,q}],\quad p,q=1,2,...,k(N+1)$$

which corresponds to a linear system of $k(N+1)$ algebraic equations in $k(N+1)$ unknown Laguerre coefficients.

Now, with the help of the conditions (1.2) and the relations (2.6), we can obtain the matrix form for the mixed conditions (1.2) as

$$\sum_{j=0}^{m-1}\left[a_j\bar{\mathbf{L}}(a)+b_j\bar{\mathbf{L}}(b)\right](\bar{\mathbf{M}})^j\mathbf{A}=\lambda,$$

where

$$a_j=\begin{bmatrix}a_j^1 & 0 & \cdots & 0\\ 0 & a_j^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a_j^k\end{bmatrix},\quad b_j=\begin{bmatrix}b_j^1 & 0 & \cdots & 0\\ 0 & b_j^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & b_j^k\end{bmatrix},\quad \lambda=\begin{bmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_k\end{bmatrix},$$

$$a_j^i=\begin{bmatrix}a_{0,j}^i\\ a_{1,j}^i\\ \vdots\\ a_{m-1,j}^i\end{bmatrix},\quad b_j^i=\begin{bmatrix}b_{0,j}^i\\ b_{1,j}^i\\ \vdots\\ b_{m-1,j}^i\end{bmatrix},\quad \lambda_i=\begin{bmatrix}\lambda_{i,0}\\ \lambda_{i,1}\\ \vdots\\ \lambda_{i,m-1}\end{bmatrix},\quad i=1,2,...,k$$

or briefly, we can write as

$$\mathbf{UA}=\lambda\quad \text{or}\quad [\mathbf{U};\lambda], \tag{2.12}$$

where

$$\mathbf{U} = \sum_{j=0}^{m-1} \left[ a_j \bar{\mathbf{L}}(a) + b_j \bar{\mathbf{L}}(b) \right] (\bar{\mathbf{M}})^j.$$

Consequently, by replacing the rows of the matrices $\mathbf{U}$ and $\lambda$, respectively, by the last $mk$ rows of the matrices $\mathbf{W}$ and $\mathbf{G}$, we obtain the new augmented matrix as

$$[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,k(N+1)} & ; & g_1(x_0) \\ w_{2,1} & w_{2,2} & \cdots & w_{2,k(N+1)} & ; & g_2(x_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{k,1} & w_{k,2} & \cdots & w_{k,k(N+1)} & ; & g_k(x_0) \\ w_{k+1,1} & w_{k+1,2} & \cdots & w_{k+1,k(N+1)} & ; & g_1(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{k(N-m+1),1} & w_{k(N-m+1),2} & \cdots & w_{k(N-m+1),k(N+1)} & ; & g_k(x_{N-m}) \\ u_{1,1} & u_{1,2} & \cdots & u_{1,k(N+1)} & ; & \lambda_{1,0} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,k(N+1)} & ; & \lambda_{1,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m,1} & u_{m,2} & \cdots & u_{m,k(N+1)} & ; & \lambda_{1,m-1} \\ u_{m+1,1} & u_{m+1,2} & \cdots & u_{m+1,k(N+1)} & ; & \lambda_{2,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{mk,1} & u_{mk,2} & \cdots & u_{mk,k(N+1)} & ; & \lambda_{k,m-1} \end{bmatrix}$$

or briefly, we can write as

$$\widetilde{\mathbf{W}}\mathbf{A} = \widetilde{\mathbf{G}}. \tag{2.13}$$

When the matrix $\mathbf{W}$ is singular, the rows that have the same factor or all zero are replaced. Namely, we don't have to replace the last rows.

If $rank\widetilde{\mathbf{W}} = rank[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = N+1$, then we can write the system of matrix equation (2.13) as

$$\mathbf{A} = (\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}}.$$

Thus, by solving this linear system and by substituting $a_{i,0}, a_{i,1}, ..., a_{i,N}$ $(i = 1, 2, ..., k)$ in (1.3), the Laguerre coefficients matrix $\mathbf{A}$ is determined. In other words, we can find the Laguerre polynomial solutions

$$y_{i,N}(x) = \sum_{n=0}^{N} a_{i,N} L_n(x), \quad i = 1, 2, ..., k.$$

Furthermore, when $det(\widetilde{\mathbf{W}}) = 0$, if $rank\widetilde{\mathbf{W}} = rank[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] < N+1$, then we can find a particular solution. Otherwise if $rank\widetilde{\mathbf{W}} \neq rank[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] < N+1$, then it is not a solution.

## 3. Residual error estimation and improvement of solutions

In this section, $y_i$ and $y_{i,N}$ represent, respectively, the exact solutions, the approximate solutions and the actual error functions are defined as

$$e_{i,N}(x) = y_i(x) - y_{i,N}(x), \quad i = 1, 2, ..., k.$$

Firstly, the Laguerre polynomial solutions are written in system (1.1) and the residual function is defined as

$$R_{i,N}(x) = \sum_{n=0}^{m}\sum_{j=1}^{k} P_{i,j}^{n}(x) y_{i,N}^{(n)}(x) - g_i(x), \quad i = 1, 2, ..., k$$

or briefly, we can write as

$$\sum_{n=0}^{m}\sum_{j=1}^{k} P_{i,j}^{n}(x) y_{i,N}^{(n)}(x) = R_{i,N}(x) + g_i(x). \tag{3.1}$$

If (3.1) is subtracted from (1.1), then the system of error differential equation is obtained as

$$\sum_{n=0}^{m}\sum_{j=1}^{k} P_{i,j}^{n}(x) \big[y_j^{(n)}(x) - y_{i,N}^{(n)}(x)\big] = -R_{i,N}(x)$$

or briefly, we can write as

$$\sum_{n=0}^{m}\sum_{j=1}^{k} P_{i,j}^{n}(x) \big[e_{j,N}^{(n)}(x)\big] = -R_{i,N}(x). \tag{3.2}$$

Afterward likewise, due to the fact that the Laguerre polynomial solutions (1.3) provides conditions (1.2), we can write as

$$\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \big(a_{i,j}^{n} y_{n,N}^{(j)}(a) + b_{i,j}^{n} y_{n,N}^{(j)}(b)\big) = \lambda_{n,i}, \quad n = 1, ..., k. \tag{3.3}$$

If (3.3) is subtracted from (1.2), then we obtain also as

$$\sum_{j=0}^{m-1} \Big[ a_{i,j}^{n} \big[y_n^{(j)}(a) - y_{n,N}^{(j)}(a)\big] + b_{i,j}^{n} \big[y_n^{(j)}(b) - y_{n,N}^{(j)}(b)\big] \Big] = 0 \tag{3.4}$$

or briefly, we can write as

$$\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \Big( a_{i,j}^{n} e_{j,N}^{(j)}(a) + b_{i,j}^{n} e_{j,N}^{(j)}(b) \Big) = 0, \quad n = 1, ..., k. \tag{3.5}$$

In conclusion, we obtain the approximation

$$e_{i,N,M}(x) = \sum_{n=0}^{M} \widetilde{a_{i,n},} L_n(x), \quad i = 1, 2, ..., k$$

to the error function $e_{i,N}(x)$. Hence, the approximate solution and the estimated error function are added and the improved approximate solution is obtained as

$$y_{i,N,M} = y_{i,N} + e_{i,N,M}.$$

Finally, the error function of the improved approximate solution is obtained as

$$E_{i,N,M} = y_i - y_{i,N,M}.$$

## 4. Numerical Results

In this section, we present the application of the method. We demonstrate the numerical results in tables and figures. The obtained results were carried out by writing a program in Matlab. Additionally, the presented method is compared with other methods available in the literature. $y_i(x)$, $y_{i,N}(x)$, $y_{i,N,M}(x)$, $e_{i,N}(x)$, $e_{i,N,M}(x)$ and $E_{i,N,M}(x)$ in tables and figures for various $N$, $M$ represent respectively, the values of the exact solutions, the approximate solutions, the corrected approximate solutions, the absolute error functions, the estimated error function, and the corrected error function at the selected points of the given interval.

**Example 4.1.** Firstly, let's consider the system of second-order variable-coefficient linear differential equations

$$\begin{cases} y_1^{''}(x) + xy_1(x) + xy_2(x) = 2 \\ y_2^{''}(x) + 2xy_2(x) + 2xy_1(x) = -2 \end{cases} ,0 \leq x \leq 1 \tag{4.1}$$

with the conditions

$$y_1(0) = y_1(1) = 0, \quad y_2(0) = y_2(1) = 0. \tag{4.2}$$

Here,

$$\begin{aligned} &k = 2, \quad m = 2, \quad g_1(x) = 2, \quad g_2(x) = -2, \\ &P^0_{1,1}(x) = P^0_{1,2}(x) = x, \quad P^0_{2,1}(x) = P^0_{2,2}(x) = 2x, \\ &P^1_{1,1}(x) = P^1_{1,2}(x) = P^1_{2,1}(x) = P^1_{2,2}(x) = 0, \\ &P^2_{1,2}(x) = P^2_{2,1}(x) = 0, \quad P^2_{1,1}(x) = 1, \quad P^2_{2,2}(x) = 1. \end{aligned}$$

Now, let's find solutions truncated Laguerre series expansions in the form

$$y_{i,N}(x) = \sum_{n=0}^{2} a_{i,n} L_n(x), \quad i = 1, 2$$

by Laguerre polynomials for $N = 2$.

For this purpose, we find the set of the collocation points for $N = 2$. Since $0 \leq x \leq 1$ and $N = 2$, $a = 0$, $b = 1$, $i = 0, 1, 2$ and

$$\begin{aligned} x_0 &= 0 + \frac{1-0}{2}0 = 0 \\ x_1 &= 0 + \frac{1-0}{2}1 = \frac{1}{2} \\ x_2 &= 0 + \frac{1-0}{2}2 = 1. \end{aligned}$$

Thus, the set of the collocation points for $N = 2$ is computed as

$$\{x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1\}.$$

Hence, the system of fundamental matrix equation of the problem from Eq. (2.11) is written as

$$\Big\{\mathbf{P}_0\mathbf{L} + \mathbf{P}_1\mathbf{L}\bar{\mathbf{M}} + \mathbf{P}_2\mathbf{L}(\bar{\mathbf{L}})^2\Big\}\mathbf{A} = \mathbf{G},$$

where

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{P}_0(0) & 0 & 0 \\ 0 & \mathbf{P}_0(\frac{1}{2}) & 0 \\ 0 & 0 & \mathbf{P}_0(1) \end{bmatrix}, \mathbf{P}_1 = \begin{bmatrix} \mathbf{P}_1(0) & 0 & 0 \\ 0 & \mathbf{P}_1(\frac{1}{2}) & 0 \\ 0 & 0 & \mathbf{P}_1(1) \end{bmatrix}, \mathbf{P}_2 = \begin{bmatrix} \mathbf{P}_2(0) & 0 & 0 \\ 0 & \mathbf{P}_2(\frac{1}{2}) & 0 \\ 0 & 0 & \mathbf{P}_2(1) \end{bmatrix},$$

$$\mathbf{L} = \begin{bmatrix} \bar{\mathbf{L}}(0) \\ \bar{\mathbf{L}}(\frac{1}{2}) \\ \bar{\mathbf{L}}(1) \end{bmatrix}, \bar{\mathbf{L}}(0) = \begin{bmatrix} \mathbf{L}(0) & 0 \\ 0 & \mathbf{L}(0) \end{bmatrix}, \bar{\mathbf{L}}(\tfrac{1}{2}) = \begin{bmatrix} \mathbf{L}(\frac{1}{2}) & 0 \\ 0 & \mathbf{L}(\frac{1}{2}) \end{bmatrix}, \bar{\mathbf{L}}(1) = \begin{bmatrix} \mathbf{L}(1) & 0 \\ 0 & \mathbf{L}(1) \end{bmatrix},$$

$$\bar{\mathbf{M}} = \begin{bmatrix} \mathbf{M}^T & 0 \\ 0 & \mathbf{M}^T \end{bmatrix}, \mathbf{M}^T = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}, \mathbf{g}(0) = \mathbf{g}(\tfrac{1}{2}) = \mathbf{g}(1) = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}(0) \\ \mathbf{g}(\frac{1}{2}) \\ \mathbf{g}(1) \end{bmatrix}, \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}, \mathbf{A}_1 = \begin{bmatrix} a_{1,0} \\ a_{1,1} \\ a_{1,2} \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} a_{2,0} \\ a_{2,1} \\ a_{2,2} \end{bmatrix}.$$

Thus, the augmented matrix is calculated as

$$[\mathbf{W}; \mathbf{G}] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & ; & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & ; & -2 \\ 1/2 & 1/4 & 17/16 & 1/2 & 1/4 & 1/16 & ; & 2 \\ 1 & 1/2 & 1/8 & 1 & 1/2 & 9/8 & ; & -2 \\ 1 & 0 & 1/2 & 1 & 0 & -1/2 & ; & 2 \\ 2 & 0 & -1 & 2 & 0 & 0 & ; & -2 \end{bmatrix}.$$

Then, we find the matrix form of conditions from Eq. (2.12) as

$$[\mathbf{U}; \lambda] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & ; & 0 \\ 1 & 0 & -1/2 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & ; & 0 \\ 0 & 0 & 0 & 1 & 0 & -1/2 & ; & 0 \end{bmatrix}.$$

Hence, the new augmented matrix based on the conditions $[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}]$ is computed as

$$[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & ; & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & ; & -2 \\ 1 & 1 & 1 & 0 & 0 & 0 & ; & 0 \\ 1 & 0 & -1/2 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & ; & 0 \\ 0 & 0 & 0 & 1 & 0 & -1/2 & ; & 0 \end{bmatrix}$$

by writing $[\mathbf{U}; \lambda]$ instead of the last four lines in the $[\mathbf{W}; \mathbf{G}]$.

As a consequence, this system is solved and the Laguerre coefficient matrix is obtained as

$$\mathbf{A} = \left(\widetilde{\mathbf{W}}\right)^{-1}\widetilde{\mathbf{G}} = \begin{bmatrix} 1 & -3 & 2 & -1 & 3 & -2 \end{bmatrix}^T .$$

Finally, the matrix $\mathbf{A}$ is substituted into the (2.1)

$$y_j(x) = \mathbf{L}(x)\mathbf{A}_j \quad j = 1, 2$$

and thus we obtain approximate solutions as

$$y_1(x) = x^2 - x \quad \text{and} \quad y_2(x) = -x^2 + x,$$

which are the exact solutions of system of equations (4.1) according to the conditions (4.2).

**Example 4.2.** Secondly, let's consider the system of first-order variable-coefficient linear differential equations

$$\begin{cases} y_1^{'}(x) + y_2^{'}(x) + y_2(x) = x - e^{-x} \\ y_1^{'}(x) + 4y_2^{'}(x) + y_1(x) = 1 + 2e^{-x} \end{cases}, 0 \le x \le 1 \tag{4.3}$$

with the conditions

$$y_1(0) = 1, \quad y_2(0) = 0 \tag{4.4}$$

and the exact solutions

$$y_1(x) = e^{-x} + 3e^{-x/3} - 3 \quad \text{and} \quad y_2(x) = -\frac{e^{-x}}{2} + \frac{3e^{-x/3}}{2} - 1 + x.$$

Here,

$$\begin{aligned} &k = 2, \quad m = 1, \quad g_1(x) = x - e^{-x}, \quad g_2(x) = 1 + 2e^{-x}, \\ &P^0_{1,1}(x) = 0, \quad P^0_{1,2}(x) = 1, \quad P^0_{2,1}(x) = 1, \quad P^0_{2,2}(x) = 0, \\ &P^1_{1,1}(x) = 1, \quad P^1_{1,2}(x) = 1, \quad P^1_{2,1}(x) = 1, \quad P^1_{2,2}(x) = 4. \end{aligned}$$

Now, let's find solutions truncated Laguerre series expansions in the form

$$y_{i,N}(x) = \sum_{n=0}^{2} a_{i,n} L_n(x), \quad i = 1, 2$$

by Laguerre polynomials for $N = 2$. Here, the set of the collocation points for $N = 2$ is computed as

$$\{x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1.\}$$

Hence, the system of fundamental matrix equation of the problem is calculated from

$$\left\{\mathbf{P}_0\mathbf{L} + \mathbf{P}_1\mathbf{L}\bar{\mathbf{M}}\right\}\mathbf{A} = \mathbf{G}.$$

The augmented matrix for this system is calculated from Matlab as

$$[\mathbf{W};\mathbf{G}]=\begin{bmatrix} 0 & -1 & -2 & 1 & 0 & -1 & ; & -1\\ 1 & 0 & -1 & 0 & -4 & -8 & ; & 3\\ 0 & -1 & -3/2 & 1 & -1/2 & -11/8 & ; & -261/2450\\ 1 & -1/2 & -11/8 & 0 & -4 & -6 & ; & 2711/1225\\ 0 & -1 & -1 & 1 & -1 & -3/2 & ; & 921/1457\\ 1 & -1 & -3/2 & 0 & -4 & -4 & ; & 1097/632 \end{bmatrix}$$

and the matrix form of the conditions is also calculated as

$$[\mathbf{U};\lambda]=\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & ; & 1\\ 0 & 0 & 0 & 1 & 1 & 1 & ; & 0 \end{bmatrix}$$

From here, to find the new augmented matrix $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$, instead of the last two lines of the augmented matrix, the matrix form of the conditions is written. Therefore, the new augmented matrix is obtained as

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=\begin{bmatrix} 0 & -1 & -2 & 1 & 0 & -1 & ; & -1\\ 1 & 0 & -1 & 0 & -4 & -8 & ; & 3\\ 0 & -1 & -3/2 & 1 & -1/2 & -11/8 & ; & -261/2450\\ 1 & -1/2 & -11/8 & 0 & -4 & -6 & ; & 2711/1225\\ 1 & 1 & 1 & 0 & 0 & 0 & ; & 1\\ 0 & 0 & 0 & 1 & 1 & 1 & ; & 0 \end{bmatrix}.$$

Now, we solve this system to obtain the Laguerre coefficient matrix $\mathbf{A}$ and thus the matrix $\mathbf{A}$ is obtained as

$$\mathbf{A}=\begin{bmatrix} 167/2292 & -167/1146 & 1016/947 & 263/341 & -185/341 & -2695/11782 \end{bmatrix}^T.$$

Finally, when the matrix $\mathbf{A}$ is substituted, approximate solutions are obtained as

$$y_{1,2}(x)=0.536431061279x^2-2x+1 \quad \text{and} \quad y_{2,2}(x)=-0.114369376793x^2+x.$$

Similarly, approximate solutions for $N=5$ are obtained as

$$\begin{aligned} y_{1,5}(x)=&-0.00569482927687x^5+0.040951805279x^4-0.184340222499x^3\\ &+0.666542960004x^2-2x+1 \end{aligned}$$

and

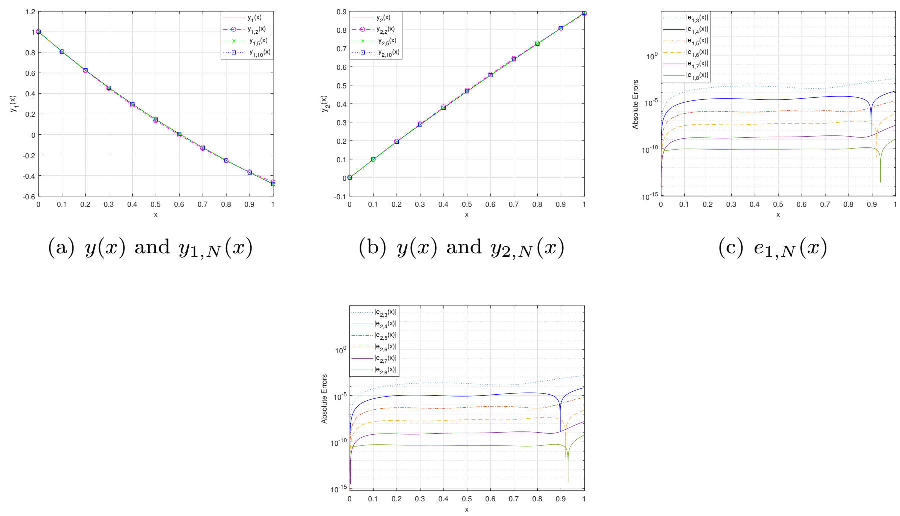
$$\begin{aligned} y_{2,5}(x)=&\,0.00275727613666x^5-0.0189435481094x^4+0.0736556629851x^3\\ &-0.166605390222x^2+x-1.80411241502e-16. \end{aligned}$$

Thus, we obtain the actual error functions for $N=5$ as

$$\begin{aligned} e_{1,5}(x)&=y_1(x)-y_{1,5}(x)\\ &=e^{-x}+3e^{-x/3}-3-(-0.00569482927687x^5+0.040951805279x^4\\ &-0.184340222499x^3+0.666542960004x^2-2x+1) \end{aligned}$$

and

$$\begin{aligned} e_{2,5}(x)&=y_2(x)-y_{2,5}(x)\\ &=-\tfrac{e^{-x}}{2}+\tfrac{3e^{-x/3}}{2}-1+x-(0.00275727613666x^5-0.0189435481094x^4\\ &+0.0736556629851x^3-0.166605390222x^2+x-1.80411241502e-16). \end{aligned}$$



(a) $y(x)$ and $y_{1,N}(x)$

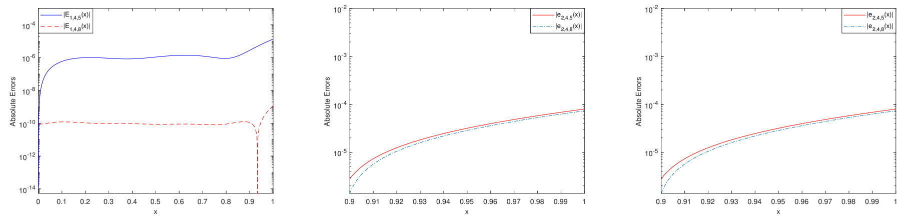
(b) $y(x)$ and $y_{2,N}(x)$

(c) $e_{1,N}(x)$

(d) $e_{2,N}(x)$

FIGURE 1. Comparison of exact solutions and approximate solutions for $N = 2, 5, 10$ for Eq.(4.3) and comparison of numerical results of actual absolute errors for various $N$ values for Eq.(4.3)

The exact solutions, the approximate solutions for $N = 5$, and the actual absolute errors of Eq.(4.3) are shown in Table 1. The Laguerre collocation method is compared with other methods for $N = 5$ in Table 2. From this, it can be seen that better results are obtained by the Laguerre collocation method. Actual, estimated, and improved absolute errors are compared in Table 3 and thus we can say that for actual absolute errors, as the value of N increases errors decrease. In addition, the numerical results of actual and estimated absolute errors are much closer to each other, and the improved absolute errors are better than the actual and estimated absolute errors. With these explications, it can be said that the method is effective. In Figure 1 (a) - (b), we show the numerical results of the exact solution and the obtained approximate solution for $N = 2, 5, 10$ of Eq. (4.3). The actual absolute errors of Eq.(4.3) are compared in Figure 1 (c) - (d). From here, as we increase the value of $N$, it is seen that the errors decrease. In Figure 2 (a), we show the improved absolute errors of $y_1(x)$ solution of Eq. (4.3) for $N = 4$, $M = 5, 8$. In Figure 2 (b), we show the estimated absolute error of $y_2(x)$ solution of Eq. (4.3) for $N = 4$, $M = 5, 8$. In Figure 2 (c), we show the actual absolute error and the improved absolute error of $y_2(x)$ solution of Eq. (4.3) for $N = 5$, $M = 6$.




(a) $E_{1,N,M}(x)$ (b) $e_{2,N,M}(x)$ (c) $e_{2,N}(x)$ and $E_{2,N,M}(x)$

FIGURE 2. Comparison of numerical results of improved absolute errors of $y_1(x)$ solution for $N = 4$, $M = 5, 8$, estimated absolute errors of $y_2(x)$ solution for $N = 4$, $M = 5, 8$ and actual and improved absolute errors of $y_2(x)$ solution for $N = 5$, $M = 6$ for Eq.(4.3)

TABLE 1. Comparison of exact solution $y_1(x)$ and approximate solutions for $N = 5$ for Eq.(4.3)

| $x_i$ | Exact Solutions $y_1(x_i)$ | Approximate Solutions $N = 5$, $y_{1,5}(x_i)$ | Absolute Errors $N = 5$, $e_{1,5}(x_i)$ |
|---|---|---|---|
| 0 | 1 | 1 | 0 |
| 0.2 | 0.625251708172835 | 0.625250697163263 | 1.0110e-06 |
| 0.4 | 0.295840003164482 | 0.295839150524119 | 8.5264e-07 |
| 0.6 | 0.005003895327973 | 0.00500250158139457 | 1.3937e-06 |
| 0.8 | -0.252886020788833 | -0.252886921731829 | 9.0094e-07 |
| 1 | -0.482526627107190 | -0.482540286492454 | 1.3659e-05 |
| $x_i$ | $y_2(x_i)$ | $N = 5$, $y_{2,5}(x_i)$ | $N = 5$, $e_{2,5}(x_i)$ |
| 0 | 0 | -1.80411241501588e-16 | 1.8041e-16 |
| 0.2 | 0.018730753077982 | 0.193895602346391 | 5.0134e-07 |
| 0.4 | 0.070320046035639 | 0.377600379671574 | 4.2412e-07 |
| 0.6 | 0.148811636094026 | 0.553691004682296 | 6.9311e-07 |
| 0.8 | 0.249328964117222 | 0.7242284766452 | 4.5116e-07 |
| 1 | 0.367879441171442 | 0.890864000790473 | 6.7555e-06 |

**Example 4.3.** As the third, let's consider the system of fourth-order variable-coefficient linear differential equations

$$\begin{cases} y_1^{(4)}(x) - cos(x)y_2^{(2)}(x) + xy_3^{(1)}(x) - y_1(x) = xe^x + cos^2(x) \\ y_2^{(4)}(x) + sin(x)y_1^{(3)}(x) + cos(x)y_1(x) - cos(x)y_3(x) = cos(x)(1 - e^x) \\ e^{-x}y_3^{(4)}(x) + y_2^{(2)}(x) - cos(x)y_1^{(1)}(x) + y_2(x) = sin^2(x) \end{cases} \tag{4.5}$$

with the conditions

$$\begin{array}{llll} y_1(0) = 0, & y_1^{(1)}(0) = 1, & y_1^{(2)}(0) = 0, & y_1^{(3)}(0) = -1, \\ y_2(0) = 1, & y_2^{(1)}(0) = 0, & y_2^{(2)}(0) = -1, & y_2^{(3)}(0) = 0, \\ y_3(0) = 1, & y_3^{(1)}(0) = 1, & y_3^{(2)}(0) = 1, & y_3^{(3)}(0) = 1, \end{array} \tag{4.6}$$

TABLE 2. Comparison of numerical results of actual errors with other methods for $N = 5$ for Eq.(4.3)

| | Chebyshev [1] | LTM [4] | Bessel [36] | Laguerre |
|---|---|---|---|---|
| $x_i$ | $e_{1,5}(x_i)$ | $e_{1,5}(x_i)$ | $e_{1,5}(x_i)$ | $e_{1,5}(x_i)$ |
| 0.1 | 4.510522e-5 | 6.7614e-5 | 5.9187e-007 | 5.9187e-07 |
| 0.2 | 7.985043e-5 | 8.4949e-5 | 1.0110e-006 | 1.0110e-06 |
| 0.5 | 9.719089e-5 | 3.18972e-3 | 1.0978e-006 | 1.0978e-06 |
| 0.8 | 8.006002e-5 | 5.20283e-3 | 9.0094e-007 | 9.0094e-07 |
| 1.0 | 1.067677e-4 | 1.193776e-2 | 1.3659e-005 | 1.3659e-05 |
| $x_i$ | $e_{2,5}(x_i)$ | $e_{2,5}(x_i)$ | $e_{2,5}(x_i)$ | $e_{2,5}(x_i)$ |
| 0.1 | 2.247723e-5 | 8.4086e-6 | 2.9327e-007 | 2.9327e-07 |
| 0.2 | 3.984701e-5 | 1.9575e-5 | 5.0134e-007 | 5.0134e-07 |
| 0.5 | 4.890662e-5 | 2.242e-4 | 5.4596e-007 | 5.4596e-07 |
| 0.8 | 4.064222e-5 | 4.647e-4 | 4.5116e-007 | 4.5116e-07 |
| 1.0 | 5.390356e-5 | 4.710e-4 | 6.7555e-006 | 6.7555e-06 |

and the exact solutions

$$y_1(x) = sin(x), \quad y_2(x) = cos(x) \quad \text{and} \quad y_3(x) = e^x.$$

Here,

$$\begin{aligned}
&k = 3, \quad m = 4, \quad g_1(x) = xe^x + cos^2(x), \quad g_2(x) = cos(x)(1 - e^x), \quad g_3(x) = sin^2(x),\\
&P^0_{1,1}(x) = -1, \quad P^0_{1,2}(x) = 0, \quad P^0_{1,3}(x) = 0,\\
&P^0_{2,1}(x) = cos(x), \quad P^0_{2,2}(x) = 0, \quad P^0_{2,3}(x) = -cos(x),\\
&P^0_{3,1}(x) = 0, \quad P^0_{3,2}(x) = 1, \quad P^0_{3,3}(x) = 0,\\
&P^1_{1,1}(x) = 0, \quad P^1_{1,2}(x) = 0, \quad P^1_{1,3}(x) = x,\\
&P^1_{2,1}(x) = 0, \quad P^1_{2,2}(x) = 0, \quad P^1_{2,3}(x) = 0,\\
&P^1_{3,1}(x) = -cos(x), \quad P^1_{3,2}(x) = 0, \quad P^1_{3,3}(x) = 0,
\end{aligned}$$

$$\begin{aligned}
&P^2_{1,1}(x) = 0, \quad P^2_{1,2}(x) = -cos(x), \quad P^2_{1,3}(x) = 0,\\
&P^2_{2,1}(x) = 0, \quad P^2_{2,2}(x) = 0, \quad P^2_{2,3}(x) = 0,\\
&P^2_{3,1}(x) = 0, \quad P^2_{3,2}(x) = 1, \quad P^2_{3,3}(x) = 0,\\
&P^3_{1,1}(x) = 0, \quad P^3_{1,2}(x) = 0, \quad P^3_{1,3}(x) = 0,\\
&P^3_{2,1}(x) = sin(x), \quad P^3_{2,2}(x) = 0, \quad P^3_{2,3}(x) = 0,\\
&P^3_{3,1}(x) = 0, \quad P^3_{3,2}(x) = 0, \quad P^3_{3,3}(x) = 0,\\
&P^4_{1,1}(x) = 1, \quad P^4_{1,2}(x) = 0, \quad P^4_{1,3}(x) = 0,\\
&P^4_{2,1}(x) = 0, \quad P^4_{2,2}(x) = 1, \quad P^4_{2,3}(x) = 0,\\
&P^4_{3,1}(x) = 0, \quad P^4_{3,2}(x) = 0, \quad P^4_{3,3}(x) = e^{-x}.
\end{aligned}$$

TABLE 3. Comparison of actual, estimated, and improved absolute errors for $N = 3, 5, 7$ and $M = 4, 5, 6, 7, 8$ for Eq.(4.3)

| $x_i$ | Actual Abs. Errors $\lvert e_{1,3}(x_i)\rvert$ | Estimated Abs. Errors $\lvert e_{1,3,5}(x_i)\rvert$ | Improved Abs. Errors $\lvert E_{1,3,4}(x_i)\rvert$ |
|---|---|---|---|
| 0 | 1.9516e-18 | 6.0185e-36 | 1.9516e-18 |
| 0.2 | 3.2982e-04 | 3.2881e-04 | 2.0715e-05 |
| 0.4 | 5.0742e-04 | 5.0657e-04 | 1.9419e-05 |
| 0.6 | 3.0431e-04 | 3.0291e-04 | 2.4041e-05 |
| 0.8 | 6.2354e-04 | 6.2264e-04 | 3.9209e-05 |
| 1 | 3.3007e-03 | 3.2871e-03 | 1.5281e-04 |
| | $\lvert e_{1,5}(x_i)\rvert$ | $\lvert e_{1,5,7}(x_i)\rvert$ | $\lvert E_{1,5,6}(x_i)\rvert$ |
| 0 | 3.3307e-16 | 1.5407e-33 | 3.3307e-16 |
| 0.2 | 1.0110e-06 | 1.0095e-06 | 4.1070e-08 |
| 0.4 | 8.5264e-07 | 8.5090e-07 | 4.1562e-08 |
| 0.6 | 1.3937e-06 | 1.3918e-06 | 5.0201e-08 |
| 0.8 | 9.0094e-07 | 8.9879e-07 | 8.3187e-08 |
| 1 | 1.3659e-05 | 1.3626e-05 | 5.6070e-07 |
| | $\lvert e_{1,7}(x_i)\rvert$ | $\lvert e_{1,7,8}(x_i)\rvert$ | $\lvert E_{1,7,8}(x_i)\rvert$ |
| 0 | 2.5260e-12 | 6.0397e-31 | 2.5260e-12 |
| 0.2 | 1.4761e-09 | 1.4299e-09 | 4.6185e-11 |
| 0.4 | 1.7442e-09 | 1.6865e-09 | 5.7704e-11 |
| 0.6 | 1.9778e-09 | 1.9070e-09 | 7.0793e-11 |
| 0.8 | 2.1495e-09 | 2.0640e-09 | 8.5469e-11 |
| 1 | 3.3211e-08 | 3.4372e-08 | 1.1607e-09 |
| $x_i$ | $\lvert e_{2,3}(x_i)\rvert$ | $\lvert e_{2,3,5}(x_i)\rvert$ | $\lvert E_{2,3,4}(x_i)\rvert$ |
| 0 | 1.1384e-18 | 2.2569e-36 | 1.1384e-18 |
| 0.2 | 1.5318e-04 | 1.5268e-04 | 1.0103e-05 |
| 0.4 | 2.3929e-04 | 2.3887e-04 | 9.5581e-06 |
| 0.6 | 1.5178e-04 | 1.5108e-04 | 1.1860e-05 |
| 0.8 | 2.9797e-04 | 2.9752e-04 | 1.9323e-05 |
| 1 | 1.5029e-03 | 1.4962e-03 | 7.3101e-05 |
| | $\lvert e_{2,5}(x_i)\rvert$ | $\lvert e_{2,5,7}(x_i)\rvert$ | $\lvert E_{2,5,6}(x_i)\rvert$ |
| 0 | 1.8041e-16 | 1.3867e-32 | 1.8041e-16 |
| 0.2 | 5.0134e-07 | 5.0060e-07 | 2.0479e-08 |
| 0.4 | 4.2412e-07 | 4.2325e-07 | 2.0740e-08 |
| 0.6 | 6.9311e-07 | 6.9212e-07 | 2.5065e-08 |
| 0.8 | 4.5116e-07 | 4.5008e-07 | 4.1518e-08 |
| 1 | 6.7555e-06 | 6.7389e-06 | 2.7910e-07 |
| | $\lvert e_{2,7}(x_i)\rvert$ | $\lvert e_{2,7,8}(x_i)\rvert$ | $\lvert E_{2,7,8}(x_i)\rvert$ |
| 0 | 1.3696e-12 | 9.9224e-31 | 1.3696e-12 |
| 0.2 | 7.3734e-10 | 7.1436e-10 | 2.2985e-11 |
| 0.4 | 8.7153e-10 | 8.4278e-10 | 2.8752e-11 |
| 0.6 | 9.8849e-10 | 9.5318e-10 | 3.5302e-11 |
| 0.8 | 1.0746e-09 | 1.0319e-09 | 4.2646e-11 |
| 1 | 1.6585e-08 | 1.7165e-08 | 5.8014e-10 |

Now, let's find solutions truncated Laguerre series expansions in the form

$$y_{i,N}(x) = \sum_{n=0}^{5} a_{i,n} L_n(x), \quad i = 1, 2, 3$$

by Laguerre polynomials for $N = 5$. Here, the set of the collocation points for $N = 5$ is computed as

$$\{x_0 = 0, x_1 = \frac{1}{5}, x_2 = \frac{2}{5}, x_3 = \frac{3}{5}, x_4 = \frac{4}{5}, x_5 = 1\}$$

and thus the system of fundamental matrix equation of the problem is written as

$$\left\{ \mathbf{P}_0 \mathbf{L} + \mathbf{P}_1 \mathbf{L} \bar{\mathbf{M}} \right\} \mathbf{A} = \mathbf{G}.$$

The augmented matrix for this system is calculated from Matlab. The matrix form of the conditions is also calculated. Then, instead of the last twelve line of the augmented matrix, the matrix form of the conditions is written and so that the new augmented matrix $[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}]$ is obtained. Hence, this system is solved and the Laguerre coefficient matrix $\mathbf{A}$ is obtained. Finally, the matrix $\mathbf{A}$ is substituted into the (2.1)

$$y_{j,N}(x) = \mathbf{L}(x)\mathbf{A}_j, \quad j = 1, 2, 3$$

and thus we obtain approximate solutions as

$$\begin{aligned} y_{1,5}(x) = {} & 0.00827514752066x^5 - 5.42101086243e - 20x^4 - 0.166666666667x^3 \\ & -2.60208521397e - 18x^2 + x - 7.3035415877e - 19, \end{aligned}$$

$$\begin{aligned} y_{2,5}(x) = {} & -0.000829945789168x^5 + 0.0416666666667x^4 - 4.74338450462e - 19x^3 \\ & -0.5x^2 - 3.36102673471e - 18x + 1 \end{aligned}$$

and

$$\begin{aligned} y_{3,5}(x) = {} & 0.0092284748542x^5 + 0.0416666666667x^4 + 0.166666666667x^3 + 0.5x^2 \\ & +x + 1. \end{aligned}$$

Thus, for $N = 5$ the actual error functions are obtained as

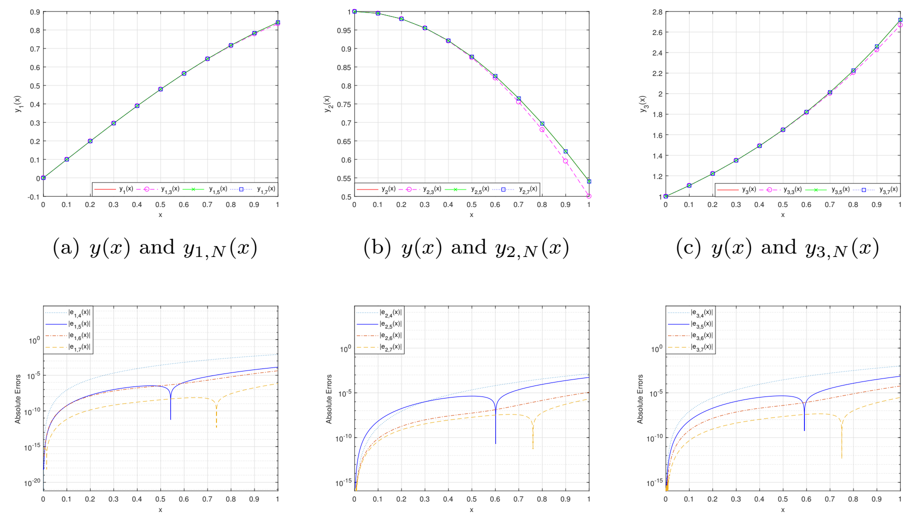
$$\begin{aligned} e_{1,5}(x) &= y_1(x) - y_{1,5}(x) \\ &= sin(x) - (0.00827514752066x^5 - 5.42101086243e - 20x^4 \\ & -0.166666666667x^3 - 2.60208521397e - 18x^2 + x - 7.3035415877e - 19), \end{aligned}$$

$$\begin{aligned} e_{2,5}(x) &= y_2(x) - y_{2,5}(x) \\ &= cos(x) - (-0.000829945789168x^5 + 0.0416666666667x^4 \\ & -4.74338450462e - 19x^3 - 0.5x^2 - 3.36102673471e - 18x + 1) \end{aligned}$$

and

$$\begin{aligned} e_{3,5}(x) &= y_3(x) - y_{3,5}(x) \\ &= e^x - (0.0092284748542x^5 + 0.0416666666667x^4 + 0.166666666667x^3 \\ & +0.5x^2 + x + 1). \end{aligned}$$

Now, exact solutions, approximate solutions, improved approximate solutions, and absolute errors are displayed in tables and graphs and compared with other methods available in the literature.



(a) $y(x)$ and $y_{1,N}(x)$ (b) $y(x)$ and $y_{2,N}(x)$ (c) $y(x)$ and $y_{3,N}(x)$

(d) $e_{1,N}(x)$ (e) $e_{2,N}(x)$ (f) $e_{3,N}(x)$

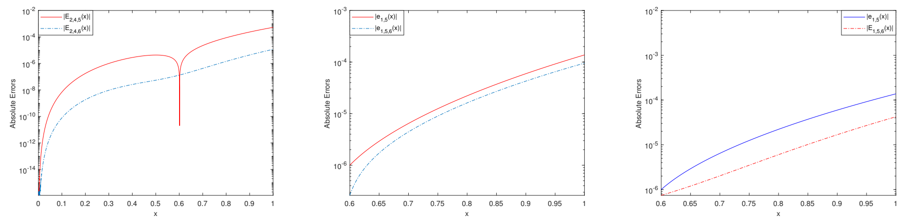
FIGURE 3. Comparison of numerical results of exact solutions $y_1(x)$, $y_2(x)$ and $y_3(x)$ and approximate solutions for $N = 3, 5, 7$ for Eq.(4.5) and comparison of numerical results of actual absolute errors for Eq.(4.5)

When we examine the actual absolute errors in Table 6, it can be seen that as the $N$ value increases, smaller errors are obtained. From this, it is seen that approximate solutions are closer to the exact solution with increasing $N$ values. By looking at the numerical results of actual, estimated and improved absolute errors, it seems that they are very close to each other. From this, it is seen that the residual error estimation method is effective.

The actual absolute errors of Eq.(4.5) are compared in Figure 3 (d) - (f). In Figure 4 (a), we show the improved absolute errors of $y_2(x)$ solution of Eq. (4.5) for $N = 4$, $M = 5, 6$. In Figure 4 (b), we show the actual and estimated absolute error of $y_1(x)$ solution of Eq. (4.5) for $N = 5$, $M = 6$. In Figure 4 (c), we show the actual absolute error and the improved absolute error of $y_1(x)$ solution of Eq. (4.5) for $N = 5$, $M = 6$. From Figures 3 (a) - (c), it can be seen that approximate solutions are closer to the exact solution with increasing $N$ values. From Figures 3 (d) - (f), it can be seen that as the $N$ value increases, the errors become smaller. From Figures 4 (a) - (c), it is seen that the method is effective.

**Example 4.4.** [35] Finally, we consider the modeling of pollution of a system of lakes by a system of differential equations

$$\begin{cases} y_1^{'}(x) = \frac{38}{1180} y_3(x) - \frac{20}{2900} y_1(x) - \frac{18}{2900} y_1(x) + sin(x) + 1 \\ y_2^{'}(x) = \frac{18}{2900} y_1(x) - \frac{18}{850} y_2(x) \\ y_3^{'}(x) = \frac{20}{2900} y_1(x) + \frac{18}{850} y_2(x) - \frac{38}{1180} y_3(x) \end{cases} \tag{4.7}$$




(a) $E_{2,N,M}(x)$ (b) $e_{1,N}(x)$ and $e_{1,N,M}(x)$ (c) $e_{1,N}(x)$ and $E_{1,N,M}(x)$

FIGURE 4. Comparison of numerical results of improved absolute errors for $N = 4$ and $M = 5, 6$ of solution $y_2(x)$, actual and estimated absolute errors for $N = 5$ and $M = 6$ of solution $y_1(x)$ and actual and improved absolute errors for $N = 5$ and $M = 6$ of solution $y_1(x)$ for Eq.(4.5)

TABLE 4. Comparison of numerical results of actual absolute errors with Bessel collocation method for $N = 5$ for Eq.(4.5)

| $x_i$ | Laguerre $\lvert e_{1,5}(x_i)\rvert$ | Bessel [36] $\lvert e_{1,5}(x_i)\rvert$ | Laguerre $\lvert e_{2,5}(x_i)\rvert$ | Bessel [36] $\lvert e_{2,5}(x_i)\rvert$ | Laguerre $\lvert e_{3,5}(x_i)\rvert$ | Bessel [36] $\lvert e_{3,5}(x_i)\rvert$ |
|---|---|---|---|---|---|---|
| 0 | 7.3035e-19 | 0 | 1.3010e-18 | 0 | 2.4395e-19 | 0 |
| 0.2 | 1.6081e-08 | 1.6081e-008 | 1.7676e-07 | 1.7676e-007 | 1.9495e-07 | 1.9495e-007 |
| 0.4 | 2.7146e-07 | 2.7146e-007 | 2.8260e-06 | 2.8260e-006 | 3.1353e-06 | 3.1353e-006 |
| 0.6 | 1.0021e-06 | 1.0021e-006 | 1.5149e-07 | 1.5149e-007 | 1.1942e-06 | 1.1942e-006 |
| 0.8 | 2.2176e-05 | 2.2176e-005 | 8.8001e-05 | 8.8001e-005 | 1.1694e-04 | 1.1694e-004 |
| 1 | 1.3750e-04 | 1.3750e-004 | 5.3442e-04 | 5.3442e-004 | 7.2002e-04 | 7.2002e-004 |

with the conditions

$$y_1(0) = 0, \quad y_2(0) = 0 \quad \text{and} \quad y_3(0) = 1. \tag{4.8}$$

Now, let's find solutions truncated Laguerre series expansions in the form

$$y_{i,N}(x) = \sum_{n=0}^{3} a_{i,n} L_n(x), \quad i = 1, 2, 3$$

by Laguerre polynomials for $N = 3$. Here, the set of the collocation points for $N = 3$ is computed as

$$\{x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1\}$$

and thus the system of fundamental matrix equation of the problem is written as

$$\left\{\mathbf{L}\bar{\mathbf{M}} - \mathbf{P}_0\mathbf{L}\right\}\mathbf{A} = \mathbf{G}.$$

The augmented matrix for this system is calculated from Matlab. The matrix form of the conditions is also calculated. Then, instead of the last three line of the augmented

TABLE 5. Comparison of numerical results of the exact solutions, approximate solutions and actual absolute errors for $N = 5$ for Eq.(4.5)

| $x_i$ | Exact Solutions $y_1(x_i) = sin(x)$ | Approximate Solutions $N = 5, \quad y_{1,5}(x_i)$ | Absolute Errors $N = 5, \quad e_{1,5}(x_i)$ |
|---|---|---|---|
| 0 | 0 | -7.3035415877e-19 | 7.3035415877e-19 |
| 0.2 | 0.198669330795061 | 0.198669314713871 | 1.60811899907465e-08 |
| 0.4 | 0.389418342308651 | 0.389418070843924 | 2.71464727030768e-07 |
| 0.6 | 0.564642473395035 | 0.564643475471134 | 1.00207609898817e-06 |
| 0.8 | 0.717356090899523 | 0.717378267006066 | 2.21761065429815e-05 |
| 1 | 0.841470984807897 | 0.84160848085366 | 1.374960457630750e-04 |
| $x_i$ | $y_2(x_i) = cos(x)$ | $N = 5, \quad y_{2,5}(x_i)$ | $N = 5, \quad e_{2,5}(x_i)$ |
| 0 | 1 | 1 | 0 |
| 0.2 | 0.980066577841242 | 0.980066401084014 | 1.76757227965396e-07 |
| 0.4 | 0.921060994002885 | 0.921058168021786 | 2.82598109901944e-06 |
| 0.6 | 0.825335614909678 | 0.825335463415439 | 1.51494239042371e-07 |
| 0.8 | 0.696706709347165 | 0.696794710030486 | 8.80006833209324e-05 |
| 1 | 0.54030230586814 | 0.540836720877532 | 5.344150093919971e-04 |
| $x_i$ | $y_3(x_i) = e^x$ | $N = 5, \quad y_{3,5}(x_i)$ | $N = 5, \quad e_{3,5}(x_i)$ |
| 0 | 1 | 1 | 0 |
| 0.2 | 1.22140275816017 | 1.22140295311196 | 1.94951789822539e-07 |
| 0.4 | 1.49182469764127 | 1.49182783291586 | 3.13527459017138e-06 |
| 0.6 | 1.82211880039051 | 1.82211760620474 | 1.19418577004815e-06 |
| 0.8 | 2.22554092849247 | 2.22542398664041 | 1.169418520601260e-04 |
| 1 | 2.71828182845905 | 2.7175618081879 | 7.200202711499240e-04 |

matrix, the matrix form of the conditions is written and so that the new augmented matrix $[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}]$ is obtained. Hence, this system is solved and the Laguerre coefficient matrix $\mathbf{A}$ is obtained. Finally, the matrix $\mathbf{A}$ is substituted into the (2.1)

$$y_{j,N}(x) = \mathbf{L}(x)\mathbf{A}_j, \quad j = 1, 2, 3$$

and thus we obtain approximate solutions as

$$y_{1,3}(x) = -0.0559691071449x^3 + 0.511169853145x^2 + x - 3.52365706058e - 19$$

$$y_{2,3}(x) = 0.000913154523724x^3 + 0.00314419625898x^2 - 4.23516473627e - 22x$$
$$+3.1763735522e - 22$$

and

$$y_{3,3}(x) = 0.00102656683731x^3 + 0.00349268868257x^2 + 4.23516473627e - 22$$

Now, in order to measure the accuracy of the obtained approximate solutions, we have considered the absolute residuals of these solutions. These absolute residuals are depicted in Table (7), (8), and (9). When these tables are examined, it can be said

TABLE 6. Comparison of numerical results of actual, estimated, and improved absolute errors for $N = 4, 5$ and $M = 5, 6, 7$ for Eq.(4.5)

| $x_i$ | Actual Absolute Errors | Improved Absolute Errors | |
|---|---|---|---|
| | $\|e_{1,4}(x_i)\|$ | $\|E_{1,4,5}(x_i)\|$ | $\|E_{1,4,6}(x_i)\|$ |
| 0 | 0 | 0 | 0 |
| 0.2 | 2.6641e-06 | 1.6081e-08 | 1.2270e-08 |
| 0.4 | 8.5009e-05 | 2.7146e-07 | 1.6502e-07 |
| 0.6 | 6.4247e-04 | 1.0021e-06 | 7.4185e-07 |
| 0.8 | 2.6894e-03 | 2.2176e-05 | 6.0155e-06 |
| 1 | 8.1377e-03 | 1.3750e-04 | 4.2206e-05 |
| | Actual Absolute Errors | Estimated Absolute Errors | |
| | $\|e_{1,5}(x_i)\|$ | $\|e_{1,5,6}(x_i)\|$ | $\|e_{1,5,7}(x_i)\|$ |
| 0 | 7.3035e-19 | 8.4703e-19 | 2.6021e-18 |
| 0.2 | 1.6081e-08 | 1.2270e-08 | 1.3847e-10 |
| 0.4 | 2.7146e-07 | 1.6502e-07 | 1.7048e-09 |
| 0.6 | 1.0021e-06 | 7.4185e-07 | 6.4708e-09 |
| 0.8 | 2.2176e-05 | 6.0155e-06 | 2.5663e-08 |
| 1 | 1.3750e-04 | 4.2206e-05 | 6.7982e-07 |
| | Actual Absolute Errors | Improved Absolute Errors | |
| | $\|e_{2,4}(x_i)\|$ | $\|E_{2,4,5}(x_i)\|$ | $\|E_{2,4,6}(x_i)\|$ |
| 0 | 0 | 8.8162e-39 | 1.7347e-18 |
| 0.2 | 8.8825e-08 | 1.7676e-07 | 1.8352e-09 |
| 0.4 | 5.6727e-06 | 2.8260e-06 | 2.5960e-08 |
| 0.6 | 6.4385e-05 | 1.5149e-07 | 1.3224e-07 |
| 0.8 | 3.5996e-04 | 8.8001e-05 | 1.4377e-06 |
| 1 | 1.3644e-03 | 5.3442e-04 | 1.1721e-05 |
| | Actual Absolute Errors | Estimated Absolute Errors | |
| | $\|e_{2,5}(x_i)\|$ | $\|e_{2,5,6}(x_i)\|$ | $\|e_{2,5,7}(x_i)\|$ |
| 0 | 1.3010e-18 | 1.3010e-18 | 1.0408e-17 |
| 0.2 | 1.7676e-07 | 1.8352e-09 | 7.4583e-10 |
| 0.4 | 2.8260e-06 | 2.5960e-08 | 8.9777e-09 |
| 0.6 | 1.5149e-07 | 1.3224e-07 | 3.2897e-08 |
| 0.8 | 8.8001e-05 | 1.4377e-06 | 6.1163e-08 |
| 1 | 5.3442e-04 | 1.1721e-05 | 2.0777e-06 |
| | Actual Absolute Errors | Improved Absolute Errors | |
| | $\|e_{3,4}(x_i)\|$ | $\|E_{3,4,5}(x_i)\|$ | $\|E_{3,4,6}(x_i)\|$ |
| 0 | 0 | 5.1428e-39 | 1.5613e-17 |
| 0.2 | 2.7582e-06 | 1.9495e-07 | 1.4268e-08 |
| 0.4 | 9.1364e-05 | 3.1353e-06 | 1.9153e-07 |
| 0.6 | 7.1880e-04 | 1.1942e-06 | 8.7973e-07 |
| 0.8 | 3.1409e-03 | 1.1694e-04 | 7.8338e-06 |
| 1 | 9.9485e-03 | 7.2002e-04 | 5.8121e-05 |
| | Actual Absolute Errors | Estimated Absolute Errors | |
| | $\|e_{3,5}(x_i)\|$ | $\|e_{3,5,6}(x_i)\|$ | $\|e_{3,5,7}(x_i)\|$ |
| 0 | 2.4395e-19 | 1.2197e-18 | 2.7756e-17 |
| 0.2 | 1.9495e-07 | 1.4268e-08 | 9.1174e-10 |
| 0.4 | 3.1353e-06 | 1.9153e-07 | 1.0922e-08 |
| 0.6 | 1.1942e-06 | 8.7973e-07 | 3.9578e-08 |
| 0.8 | 1.1694e-04 | 7.8338e-06 | 1.0377e-07 |
| 1 | 7.2002e-04 | 5.8121e-05 | 3.0472e-06 |

TABLE 7. Comparison of the solutions $y_{1,N}(x)$ and the residual functions $R_{1,N}(x)$ with Bessel method for $N = 3$ of system (4.7)

| | PM | Bessel [35] | PM | Bessel [35] |
|---|---|---|---|---|
| $x_i$ | $y_{1,3}(x_i)$ | $y_{1,3}(x_i)$ | $R_{1,3}(x)$ | $R_{1,3}(x)$ |
| 0 | 0 | 0 | 0 | 0 |
| 0.2 | 0.2199990413 | 0.2199990413 | 1.9603e-03 | 1.9603e-003 |
| 0.4 | 0.4782051536 | 0.4782051536 | 1.1016e-03 | 1.1016e-003 |
| 0.6 | 0.7719318200 | 0.7719318200 | 1.6179e-03 | 1.6179e-003 |
| 0.8 | 1.0984925232 | 1.0984925232 | 7.3601e-03 | 7.3601e-003 |
| 1 | 1.4552007460 | 1.4552007460 | 3.1884e-02 | 3.1884e-002 |

TABLE 8. Comparison of the solutions $y_{2,N}(x)$ and the residual functions $R_{2,N}(x)$ with Bessel method for $N = 6$ of system (4.7)

| | PM | Bessel [35] | PM | Bessel [35] |
|---|---|---|---|---|
| $x_i$ | $y_{2,6}(x_i)$ | $y_{2,6}(x_i)$ | $R_{2,6}(x)$ | $R_{2,6}(x)$ |
| 0 | 0 | 0 | 0 | 0 |
| 0.2 | 1.3210e-04 | 0.1321001e-3 | 6.0769e-10 | 6.0769e-010 |
| 0.4 | 5.5974e-04 | 0.5597438e-3 | 5.5439e-10 | 5.5439e-010 |
| 0.6 | 1.3279e-03 | 0.1327949e-2 | 8.3158e-10 | 8.3158e-010 |
| 0.8 | 2.4776e-03 | 0.2477595e-2 | 2.4308e-09 | 2.4308e-009 |
| 1 | 4.0437e-03 | 0.4043723e-2 | 1.1899e-07 | 1.1899e-007 |

TABLE 9. Numerical results of the solutions $y_{3,N}(x)$ and the residual functions $R_{3,N}(x)$ for $N = 3, 7$ of system (4.7)

| $x_i$ | $y_{3,3}(x_i)$ | $R_{3,3}(x)$ | $y_{3,7}(x)$ | $R_{3,7}(x)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0.2 | 1.4792e-04 | 4.9742e-06 | 1.4685e-04 | 1.6801e-11 |
| 0.4 | 6.2453e-04 | 2.8424e-06 | 6.2258e-04 | 4.9448e-12 |
| 0.6 | 1.4791e-03 | 4.2636e-06 | 1.4778e-03 | 7.4151e-12 |
| 0.8 | 2.7609e-03 | 1.9897e-05 | 2.7585e-03 | 6.7220e-11 |
| 1 | 4.5193e-03 | 8.8826e-05 | 4.5043e-03 | 3.6589e-09 |

that the method has given good results when the analytical solution is not known and almost the same results are obtained with the Bessel method. Additionally, we can say that for residual functions, errors decrease as the value of N increases in Table (9), where their values are seen to decrease greatly with the number N of collocation points.

## 5. Conclusion

Mathematical models of many important problems encountered in the fields of physics and engineering are differential equations and their systems. Numerical methods are needed when it is often difficult to find exact solutions, or when there are no exact solutions. So we need to find approximate solutions. In this study, a collocation method based on Laguerre polynomials is presented. In addition, error estimation based on residual function is presented. Laguerre polynomial solutions are improved with the help of the estimated error function. Numerical examples are performed regarding the proposed method. In addition, measurements are also made using the accuracy of the solutions (residual function). Approximate solutions and residual corrections are shown in tables and graphs. Numerical examples are also compared with other methods and very good results were obtained with the Laguerre collocation method.

It is understood from the results that better results can be obtained if increasing the $N$ cutting limit. Also, as the N value increase, the computational errors that errors caused by the program decreases. For this reason, it may not be very useful to increase the number of terms too much. Since if it is chosen the big enough $N$ value, then it can be obtained very close solutions to the exact solution. In addition, if the exact solution is a polynomial, when the cutting limit of the method is N, the exact solution is obtained as in Example 1. Furthermore, it can be said from Table (9) that the method gives reliable results even if analytical solutions are not known.

Approximate solutions in the method can be easily calculated using computer programs such as Matlab, Maple and Mathematica. In this work, the codes written in Matlab program are run and approximate solutions are calculated in a short time. This method can be applied by developing for other systems of equations.

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