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Existence of solutions for a class of critical Kirchhoff type problems involving Caffarelli-Kohn-Nirenberg inequalities

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Abstract In this paper, we study the existence of a nontrival weak solution for a class of Kirchhoff type problems with singular potentials and critical exponents. The proofs are essentially based on an appropriated truncated argument, Caffarelli-Kohn-Nirenberg inequalities, combined with a variant of the concentration compactness principle. We also get a priori estimates of the obtained solution.

Keywords. Kirchhoff type problems, Caffarelli-Kohn-Nirenberg inequalities, Critical exponents, Mountain pass theorem.

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1. Introduction

In this paper, we are interested in the existence of solutions for a class of Kirchhoff type problems of the form

$$\begin{cases} -M\left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right)\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) \\ \qquad\qquad = \lambda |x|^{-bp} f(x,u) + |x|^{-cp^*_{a,c}}|u|^{p^*_{a,c}-2}u, \quad x\in\Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $0 \in \Omega$, $0 \le a < \frac{N-p}{p}$, $1 < p < N$, $a \le b, c < a+1$, $p^*_{a,c} = \frac{Np}{N-(1+a-c)p}$, $f \in C(\Omega\times\mathbb{R},\mathbb{R})$, λ is a positive parameter, $M \in C(\mathbb{R}^+_0, \mathbb{R}^+_0)$ is increasing and satisfies the following condition:

(M_0) There exists $m_0 > 0$ such that

$$M(t) \ge m_0, \quad \forall t \in \mathbb{R}^+_0 := [0,+\infty).$$

It should be noticed that if $a = b = 0$ and $c = 0$ then problem (1.1) becomes the p-Kirchhoff type problem with critical growth

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p\,dx\right)\Delta_p u = \lambda f(x,u) + |u|^{p^*-2}u, \quad x\in\Omega, \\ u = 0, \quad x\in\partial\Omega, \end{cases} \tag{1.2}$$

where $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent.

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Since the first equation in (1.2) contains an integral over Ω, it is no longer a pointwise identity; therefore it is often called a nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [10]. Problem (1.2) is related to the stationary version of the Kirchhoff equation which is presented by Kirchhoff in 1883, see [18] for details.

In recent years, Kirchhoff type equations have been studied in many papers, we refer to some interesting papers [1, 4, 8, 9, 15, 17, 23, 25], in which the authors have used different methods to get the existence of solutions for the problems with subcritical growth. Because of the presence of the critical exponent p^*, problem (1.2) creats many difficulties in applying variational methods. These come from the fact that the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is not compact and thus the Palais-Smale condition fails, we refer to [2, 12, 16, 22, 26]. In recent papers [6, 11, 13, 14] the authors have considered a class of Kirchhoff type problems with singular potentials involving Caffarelli - Kohn - Nirenberg inequalities [27]. There, some existence and multiplicity results for the appropriated problems have been obtained by using variational methods in the subcritical case. In this paper, we will study the existence of nontrival solutions for problem (1.1) with singular potential and critical growth. By condition (M_0), the Kirchhoff function $M(t)$ may be unbounded. This causes some mathematical difficulties which make the study of such problems (1.1) and (1.2) particularly interesting. For this reason, we need a truncation on $M(t)$ as in (2.1). In order to overcome the lack of compactness, we use the weighted version of the Concentration Compactness Principle due to Xuan [28]. Applying the mountain pass theorem [3], we show that problem (1.1) has at least one nontrival weak solution u_λ, provided the parameter λ is large enough. Moreover, we prove that the norm of the obtained solution u_λ tends to zero when $\lambda \to +\infty$.

We start by recalling some useful results in [5, 7, 27]. We have known that for all $u \in C_0^\infty(\mathbb{R}^N)$, there exists a constant $C_{a,c} > 0$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{-cp^*_{a,c}} |u|^{p^*_{a,c}}\, dx \right)^{\frac{p}{p^*_{a,c}}} \leq C_{a,c} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p\, dx, \tag{1.3}$$

where

$$-\infty < a < \frac{N-p}{p}, \quad a \leq c \leq a+1, \quad p^*_{a,c} = \frac{Np}{N-(1+a-c)p}.$$

Let $W_0^{1,p}(\Omega, |x|^{-ap})$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{a,p} = \left(\int_\Omega |x|^{-ap} |\nabla u|^p\, dx \right)^{\frac{1}{p}}.$$

Then $W_0^{1,p}(\Omega, |x|^{-ap})$ is a reflexive and separable Banach space. From the boundedness of Ω and the standard approximation argument, it is easy to see that (1.3) holds for any $u \in W_0^{1,p}(\Omega, |x|^{-ap})$ in the sense that

$$\left(\int_\Omega |x|^{-\alpha} |u|^l\, dx \right)^{\frac{p}{l}} \leq C_{a,c} \int_\Omega |x|^{-ap} |\nabla u|^p\, dx, \tag{1.4}$$

for $1 \le l \le p^* = \frac{Np}{N-p}$, $\alpha \le (1+a)l + N\left(1-\frac{l}{p}\right)$, that is, the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^l(\Omega, |x|^{-\alpha})$ is continuous, where $L^l(\Omega, |x|^{-\alpha})$ is the weighted $L^l(\Omega)$ space with the norm

$$|u|_{l,\alpha} := |u|_{L^l(\Omega,|x|^{-\alpha})} = \left(\int_\Omega |x|^{-\alpha}|u|^l\,dx\right)^{\frac{1}{l}}.$$

The best constant of the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^l(\Omega, |x|^{-\alpha})$ will be denoted by $S_{a,c}$, which is characterized by (see [7, 19])

$$S_{a,c} = \inf_{u\in X\setminus\{0\}} \frac{\int_\Omega |x|^{-ap}|\nabla u|^p\,dx}{\left(\int_\Omega |x|^{-cp^*_{a,c}}|u|^{p^*_{a,c}}\,dx\right)^{\frac{p}{p^*_{a,c}}}} > 0. \tag{1.5}$$

In fact, we have the following compact embedding result which is an extension of the classical Rellich-Kondrachov compactness theorem.

Lemma 1.1 (see [27], Compactness embedding theorem)**.** *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary and that $0 \in \Omega$, where $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, $1 \le l < \frac{Np}{N-p}$ and $\alpha < (1+a)l + N\left(1-\frac{l}{p}\right)$. Then the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^l(\Omega, |x|^{-\alpha})$ is compact.*

In the rest of this section, we recall the weighted version of the Concentration Compactness Principle due to Xuan [28], the readers can see the original papers by Lions [20, 21] for the non-singular case.

Proposition 1.2 (see [28])**.** *Let $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, $a \le c \le a+1$, $p^*_{a,c} = \frac{Np}{N-(1+a-c)p}$, and let $\mathcal{M}^+(\mathbb{R})$ be the space of positive bounded measures on $\mathbb{R}^N$. Suppose that $\{u_n\} \subset W_0^{1,p}(\Omega, |x|^{-ap})$ is a sequence such that*

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } W_0^{1,p}(\Omega, |x|^{-ap}),\\ ||x|^{-a}|\nabla u_n||^p &\rightharpoonup \mu \text{ in } \mathcal{M}^+(\mathbb{R}^N),\\ ||x|^{-c}|u_n||^{p^*_{a,c}} &\rightharpoonup \nu \text{ in } \mathcal{M}^+(\mathbb{R}^N),\\ u_n(x) &\to u(x) \text{ a.e. on } \mathbb{R}^N. \end{aligned}$$

Then there are the following statements:

- (i) *There exists some at most countable set J, a family $\{x_j : j \in J\}$ of distinct points in $\mathbb{R}^N$ and a family $\{\nu_j : j \in J\}$ of positive numbers such that*

$$\nu = ||x|^{-c}|u||^{p^*_{a,c}} + \sum_{j\in J}\nu_j\delta_{x_j},$$

  where δ_{x_j} is the Dirac unitary mass concentrated at $x_j \in \mathbb{R}^N$.
- (ii) *The following inequality holds*

$$\mu \ge ||x|^{-a}|\nabla u||^p + \sum_{j\in J}\mu_j\delta_{x_j}$$

for some family $\{\mu_j : j \in J\}$ of positive numbers satisfying $S_{a,c}\nu_j^{\frac{p}{p^*_{a,c}}} \leq \mu_j$ for all $j \in J$, where the constant $S_{a,c}$ is given by (1.5). In particular, $\sum_{j\in J} \nu_j^{\frac{p}{p^*_{a,c}}} < +\infty$.

2. Main result

In this section, we shall state and prove the main result of the paper. We use the letters C_i to denote positive constants whose values are changed from line to line. In order to state the main result of the paper, we introduce the following hypotheses:

(F_1) There exist $C > 0$ and $p < q < \min\left\{\frac{Np}{N-p}, \frac{p(N-bp)}{N-(a+1)p}\right\}$ such that

$$|f(x,t)| \leq C(1 + |t|^{q-1}),$$

for all $(x,t) \in \Omega \times \mathbb{R}$;

(F_2) $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-1}} = 0$ uniformly in $x \in \Omega$;

(F_3) There exists $p < \theta < \min\left\{\frac{Np}{N-p}, \frac{p(N-bp)}{N-(a+1)p}\right\}$ such that

$$0 < \theta F(x,t) := \theta \int_0^t f(x,s)ds \leq f(x,t)t,$$

for all $x \in \Omega$ and $t > 0$.

There are many functions satisfying the conditions (F_1)-(F_3). A typical example of such functions is given by

$$f(x,t) = \sum_{i=1}^{k} \gamma_i(x)|t|^{q_i-1},$$

where $k \in \mathbb{N}^*$, $p < q_i < \min\left\{\frac{Np}{N-p}, \frac{p(N-bp)}{N-(a+1)p}\right\}$ and $\gamma_i : \overline{\Omega} \to (0, +\infty)$ is a continuous function, $i = 1, 2, ..., k$.

Definition 2.1. We say that $u \in X = W_0^{1,p}(\Omega, |x|^{-ap})$ is a weak solution of problem (1.1) if

$$\begin{aligned}
&M\left(\int_\Omega |x|^{-ap}|\nabla u|^p\, dx\right) \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx \\
&-\lambda \int_\Omega |x|^{-bp} f(x,u)v\, dx - \int_\Omega |x|^{-cp^*_{a,c}}|u|^{p^*_{a,c}-2}uv\, dx = 0, \quad \forall v \in X.
\end{aligned}$$

Theorem 2.2. *Assume that the conditions (M_0) and (F_1)-(F_3) are satisfied. Then there exists $\lambda^* > 0$ such that, for all $\lambda \geq \lambda^*$, problem (1.1) has a positive solution. Moreover, if u_λ is a solution of problem (1.1) then $\lim_{\lambda\to+\infty} \|u_\lambda\|_{a,p} = 0$.*

Here we are assuming, without loss of generality, that the Kirchhoff function $M(t)$ is unbounded. Contrary case, the truncation on $M(t)$ is not necessary. From (M_0), given $r \in \mathbb{R}$ such that $m_0 < r < \frac{\theta}{p} m_0$, there exists $t_0 > 0$ such that $M(t_0) = r$. We set

$$M_r(t) := \begin{cases} M(t), & 0 \le t \le t_0, \\ r, & t \ge t_0. \end{cases} \tag{2.1}$$

From (M_0) and (2.1) we get

$$M_r(t) \le r, \quad \forall t \ge 0. \tag{2.2}$$

As we shall see, the proof of Theorem 2.2 is based on a careful study of the solutions of the following auxiliary problem

$$\begin{cases} -M_r\left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right) \operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) \\ \qquad\qquad = \lambda |x|^{-bp} f(x,u) + |x|^{-cp^*_{a,c}}|u|^{p^*_{a,c}-2}u, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{2.3}$$

where f, a, b, c, and λ are as in Section 1. We shall prove the following auxiliary result.

Theorem 2.3. *Assume that the conditions (M_0) and $(F_1)-(F_3)$ are satisfied. Then, there exists $\lambda_0 > 0$ such that for all $\lambda \ge \lambda_0$ and all $r \in \left(m_0, \frac{\theta}{p} m_0\right)$, problem (2.3) has a positive solution.*

Because we want to find a positive solution, we can assume that $f(x,t) = 0$ for all $x \in \Omega$ and $t \le 0$. A function $u \in X$ is said to be a weak solution of problem (2.3) if

$$M_r\left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right)\int_\Omega |x|^{-ap}\nabla u \cdot \nabla v\,dx - \lambda\int_\Omega |x|^{-bp} f(x,u)v\,dx$$
$$-\int_\Omega |x|^{-cp^*_{a,c}}|u|^{p^*_{a,c}-2}uv\,dx = 0$$

for all $v \in X$. Hence, we shall look for weak solutions of (2.3) by finding critical points of the C^1- functional $I_{r,\lambda} : X \to \mathbb{R}$ given by the formula

$$I_{r,\lambda}(u) = \frac{1}{p}\widehat{M}_r\left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right) - \lambda\int_\Omega |x|^{-bp}F(x,u)\,dx$$
$$-\frac{1}{p^*_{a,c}}\int_\Omega |x|^{cp^*_{a,c}}|u|^{p^*_{a,c}}\,dx,$$

where $\widehat{M}(t) = \int_0^t M(s)\,ds$. Note that

$$I'_{r,\lambda}(u)(v) = M_r\left(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\right)\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx$$
$$-\lambda\int_\Omega |x|^{-bp}f(x,u)v\,dx - \int_\Omega |x|^{-cp^*_{a,c}}|u|^{p^*_{a,c}-2}uv\,dx,$$

for all $v \in X$. Moreover, if the critical point is nontrival, by the maximum principle (see [24]), we conclude that it is a positive solution of the problem.

We say that a sequence $\{u_n\} \subset X$ is a Palais-Smale sequence for the functional $I_{r,\lambda}$ at level $c_{r,\lambda} \in \mathbb{R}$ if

$$I_{r,\lambda}(u_n) \to c_{r,\lambda} \text{ and } I'_{r,\lambda}(u_n) \to 0 \text{ in } X^*,$$

where X^* is the dual space of X. If every Palais-Smale sequence of $I_{r,\lambda}$ has a strong convergent subsequence, then one says that $I_{r,\lambda}$ satisfies the Palais-Smale condition ((PS) condition for short), see [3].

Lemma 2.4. *For all $\lambda > 0$, there exist positive constants ρ and γ such that $I_{a,\lambda}(u) \geq \gamma > 0$ for all $u \in X$ with $\|u\|_{a,p} = \rho$.*

Proof. From (F_2) for each $\epsilon > 0$, there exists $\delta > 0$ such that
$$|f(x,t)| < \epsilon |t|^{p-1}, \quad \forall |t| < \delta \text{ and all } x \in \Omega.$$
Hence, by (F_1), for each $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that
$$|f(x,t)| \leq \epsilon |t|^{p-1} + C_\epsilon |t|^{q-1}, \quad \forall t \in \mathbb{R} \text{ and all } x \in \Omega.$$
This leads to the fact that
$$|F(x,t)| \leq \frac{\epsilon}{p}|t|^p + \frac{C_\epsilon}{q}|t|^q, \quad \forall t \in \mathbb{R} \text{ and all } x \in \Omega. \tag{2.4}$$
By Lemma 1.1, there exist two positive constants C_1, C_2 such that
$$C_1 \int_\Omega |x|^{-bp}|u|^p \, dx \leq \int_\Omega |x|^{-ap}|\nabla u|^p \, dx$$
and
$$C_2 \int_\Omega |x|^{-bp}|u|^q \, dx \leq \int_\Omega |x|^{-ap}|\nabla u|^p \, dx,$$
for all $u \in X$.

Hence, by (M_0) and (2.4), for all $u \in X$, we get
$$\begin{aligned}
I_{r,\lambda}(u) &= \frac{1}{p}\widehat{M}_r\left(\int_\Omega |x|^{-ap}|\nabla u|^p \, dx\right) - \lambda \int_\Omega |x|^{-bp} F(x,u)\, dx \\
&\quad - \frac{1}{p^*_{a,c}} \int_\Omega |x|^{-cp^*_{a,c}} |u|^{p^*_{a,c}} \, dx \\
&\geq \frac{m_0}{p}\|u\|^p_{a,p} - \lambda \int_\Omega |x|^{-bp}\left(\frac{\epsilon}{p}|u|^p + \frac{C_\epsilon}{q}|u|^q\right) dx \\
&\quad - \frac{1}{p^*_{a,c} S_{a,c}^{\frac{p^*_{a,c}}{p}}} \|u\|^{p^*_{a,c}}_{a,p} \\
&\geq \frac{m_0}{p}\|u\|^p_{a,p} - \lambda \frac{\epsilon}{pC_1}\|u\|^p_{a,p} - \lambda \frac{C_\epsilon}{qC_2}\|u\|^q_{a,p} - \frac{1}{p^*_{a,c} S_{a,c}^{\frac{p^*_{a,c}}{p}}} \|u\|^{p^*_{a,c}}_{a,p}.
\end{aligned}$$
For $\lambda > 0$, let $\epsilon = \frac{m_0 C_1}{2\lambda}$, we get
$$\begin{aligned}
I_{r,\lambda}(w) &\geq \frac{m_0}{2p}\|u\|^p_{a,p} - \lambda \frac{C_\epsilon}{qC_2}\|u\|^q_{a,p} - \frac{1}{p^*_{a,c} S_{a,c}^{\frac{p^*_{a,c}}{p}}} \|u\|^{p^*_{a,c}}_{a,p} \\
&= \|u\|^p_{a,p}\left(\frac{m_0}{2p} - \lambda \frac{C_\epsilon}{qC_2}\|u\|^{q-p}_{a,p} - \frac{1}{p^*_{a,c} S_{a,c}^{\frac{p^*_{a,c}}{p}}} \|u\|^{p^*_{a,c}-p}_{a,p}\right).
\end{aligned}$$

Since $p < q < \min\left\{\frac{Np}{N-p}, \frac{p(N-bp)}{N-(a+1)p}\right\} \leq \frac{Np}{N-p} \leq \frac{Np}{N-(1+a-c)p} = p^*_{a,c}$, there exist positive constants ρ and γ such that $I_{r,\lambda}(u) \geq \gamma > 0$ for all $u \in X$ with $\|u\|_{a,p} = \rho$. $\square$

Lemma 2.5. *For all $\lambda > 0$, there exists $e \in X$ with $\|e\|_X > \rho$ such that $I_{r,\lambda}(e) < 0$.*

Proof. From (F_3), we have

$$\int_{t_0}^{t} \frac{\theta}{s}\, ds \leq \int_{t_0}^{t} \frac{f(x,s)}{F(x,s)}\, ds, \quad \forall t > t_0,$$

so that

$$F(x,t) \geq C_3 t^\theta, \quad \forall t > t_0,$$

where $C_3 > 0$. Hence, by the continuity of f, there exists $C_4 > 0$ such that

$$F(x,t) \geq C_3 t^\theta - C_4, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{2.5}$$

Fix $u_0 \in C_0^\infty(\Omega)$ with $u_0 \geq 0$ and $\|u_0\|_{a,p} = 1$. Using (2.2) and (2.5), for all $t > 0$ large enough, we have

$$\begin{aligned} I_{r,\lambda}(tu_0) &= \frac{1}{p}\widehat{M}_r\left(\int_\Omega |x|^{-ap}|\nabla tu_0|^p\, dx\right) - \lambda\int_\Omega |x|^{-bp}F(x,tu_0)\, dx \\ &\quad - \frac{1}{p^*_{a,c}}\int_\Omega |x|^{-cp^*_{a,c}}|tu_0|^{p^*_{a,c}}\, dx \\ &\leq \frac{r}{p}t^p - \lambda C_3 t^\theta \int_\Omega |x|^{-bp}|u_0|^\theta\, dx + \lambda C_4 \int_\Omega |x|^{-bp}\, dx \\ &\quad - \frac{t^{p^*_{a,c}}}{p^*_{a,c}}\int_\Omega |x|^{-cp^*_{a,c}}|u_0|^{p^*_{a,c}}\, dx. \end{aligned}$$

Since $\theta > p$ and $\int_\Omega |x|^{-bp}\, dx < +\infty$, there exists a positive constant $t_* > 0$ large enough such that $I_{r,\lambda}(t_* u_0) < 0$. Thus, the result follows by considering $e = t_* u_0$. $\square$

Using a version of the Mountain pass theorem due to Ambrosetti and Rabinowitz without (PS) condition (see [3]), there exists a sequence $\{u_n\} \subset X$ such that

$$I_{r,\lambda}(u_n) \to c_{r,\lambda}, \quad I'_{r,\lambda}(u_n) \to 0 \text{ as } n \to \infty,$$

where $c_{r,\lambda} = \inf_{\eta\in\Gamma}\max_{t\in[0,1]} I_{r,\lambda}(\eta(t))$ and

$$\Gamma = \{\eta \in C([0,1], X) : \ \eta(0) = 0, \ \eta(1) = e\}.$$

Lemma 2.6. *It holds that*

$$\lim_{\lambda\to+\infty} c_{r,\lambda} = 0.$$

Proof. Since the functional $I_{r,\lambda}$ has the Mountain pass geometry (see Lemma 2.4 and Lemma 2.5), it follows that there exists $t_\lambda > 0$ verifying $I_{r,\lambda}(t_\lambda u_0) = \max_{t\geq 0} I_{r,\lambda}(tu_0)$, where u_0 is the function given by Lemma 2.5.
From this, we infer that $\frac{d}{dt}I_{r,\lambda}(t_\lambda u_0)(t_\lambda u_0) = 0$ or

$$0 = M_r\left(\|t_\lambda u_0\|^p_{a,p}\right)\int_\Omega |x|^{-ap}|\nabla t_\lambda u_0|^p\, dx - \lambda\int_\Omega |x|^{-bp}f(x,t_\lambda u_0)t_\lambda u_0\, dx$$

$$- t_\lambda^{p^*_{a,c}} \int_\Omega |x|^{-cp^*_{a,c}} |u_0|^{p^*_{a,c}} \, dx.$$

Hence,

$$t_\lambda^p M_r(|t_\lambda|^p) = \lambda \int_\Omega |x|^{-bp} f(x, t_\lambda u_0) t_\lambda u_0 \, dx + t_\lambda^{p^*_{a,c}} \int_\Omega |x|^{-cp^*_{a,c}} |u_0|^{p^*_{a,c}} \, dx. \quad (2.6)$$

From (2.2), (2.6) and (F_3), we get

$$a \geq t_\lambda^{p^*_{a,c}-p} \int_\Omega |x|^{-cp^*_{a,c}} |u_0|^{p^*_{a,c}} \, dx,$$

which implies that $\{t_\lambda\}$ is bounded. Thus, there exist a sequence $\lambda_n \to +\infty$ and $t_1 \geq 0$ such that $t_{\lambda_n} \to t_1$ as $n \to \infty$. Consequently, there is $C_3 > 0$ such that

$$t_{\lambda_n}^p M_r(t_{\lambda_n}^p) \leq C_3, \quad \forall n \in \mathbb{N},$$

and $\forall n \in \mathbb{N}$,

$$\lambda_n \int_\Omega |x|^{-bp} f(x, t_{\lambda_n} u_0) t_{\lambda_n} u_0 \, dx + t_{\lambda_n}^{p^*_{a,c}} \int_\Omega |x|^{-cp^*_{a,c}} |u_0|^{p^*_{a,c}} \, dx \leq C_3. \quad (2.7)$$

If $t_1 > 0$, by (2.7) and the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_\Omega |x|^{-bp} f(x, t_{\lambda_n} u_0) t_{\lambda_n} u_0 \, dx = \int_\Omega |x|^{-bp} f(x, t_1 u_0) t_1 u_0 \, dx > 0,$$

and thus (2.7) leads to

$$\lim_{n\to\infty} \left(\lambda_n \int_\Omega |x|^{-bp} f(x, t_{\lambda_n} u_0) t_{\lambda_n} u_0 \, dx + t_{\lambda_n}^{p^*_{a,c}} \int_\Omega |x|^{-cp^*_{a,c}} |u_0|^{p^*_{a,c}} \, dx \right) = +\infty,$$

which is an absurd. Thus, we conclude that $t_1 = 0$.

Now, let us consider the path $\eta_*(t) = te$ for $t \in [0,1]$, which belongs to Γ, to get the following estimate

$$0 < c_{r,\lambda} \leq \max_{t\in[0,1]} I_{r,\lambda}(\eta_*(t)) = I_{R,\lambda}(t_\lambda u_0) \leq \frac{1}{p} \widehat{M}_r(t_\lambda^p).$$

In this way,

$$\lim_{\lambda\to+\infty} \widehat{M}_r(t_\lambda^p) = 0,$$

which helps us to get $\lim_{\lambda\to+\infty} c_{r,\lambda} = 0$. □

Lemma 2.7. *Let $\{u_n\} \subset X$ be a sequence such that*

$$I_{r,\lambda}(u_n) \to c_{r,\lambda}, \quad I'_{r,\lambda}(u_n) \to 0 \text{ as } n \to \infty. \quad (2.8)$$

Then $\{u_n\}$ is bounded.

Proof. Assuming by contradiction that $\{u_n\}$ is not bounded in X, up to a subsequence if it is necessary, we have $\|u_n\|_{a,p} \to +\infty$ as $n \to \infty$. It follows from (2.8), (M_0) and (F_3) that for n large enough

$$\begin{aligned} &1 + c_{r,\lambda} + \|u_n\|_{a,p} \\ &\geq I_{r,\lambda}(u_n) - \frac{1}{\theta} I'_{r,\lambda}(u_n)(u_n) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p}\widehat{M}_r(\|u_n\|^p_{a,p}) - \lambda\int_\Omega |x|^{-bp}F(x,u_n)\,dx - \frac{1}{p^*_{a,c}}\int_\Omega |x|^{-cp^*_{a,c}}|u_n|^{p^*_{a,c}}\,dx \\
&\qquad - \frac{1}{\theta}M_r(\|u_n\|^p_{a,p})\int_\Omega |x|^{-ap}|\nabla u_n|^p\,dx + \frac{\lambda}{\theta}\int_\Omega |x|^{-bp}f(x,u_n)u_n\,dx \\
&\qquad + \frac{1}{\theta}\int_\Omega |x|^{-cp^*_{a,c}}|u_n|^{p^*_{a,c}}\,dx \\
&\geq \left(\frac{m_0}{p} - \frac{r}{\theta}\right)\|u_n\|^p_{a,p} - \frac{\lambda}{\theta}\int_\Omega |x|^{-bp}\left(f(x,u_n)u_n - \theta F(x,u_n)\right)dx \\
&\qquad + \left(\frac{1}{\theta} - \frac{1}{p^*_{a,c}}\right)\int_\Omega |x|^{-cp^*_{a,c}}|u_n|^{p^*_{a,c}}\,dx \\
&\geq \left(\frac{m_0}{p} - \frac{r}{\theta}\right)\|u_n\|^p_{a,p}.
\end{aligned}
$$

Since $r < \frac{m_0}{p}\theta$ and $\theta < p^*_{a,c}$, the sequence $\{u_n\}$ is bounded in X. □

Proof of Theorem 2.3. From Lemma 2.4, we have

$$\lim_{\lambda\to+\infty} c_{r,\lambda} = 0. \tag{2.9}$$

Therefore, there exists $\lambda_0 > 0$ such that

$$c_{r,\lambda} < \left(\frac{1}{\theta} - \frac{1}{p^*_{a,c}}\right)(S_{a,c}m_0)^{\frac{N}{(1+a-c)p}}, \tag{2.10}$$

for all $\lambda \geq \lambda_0$, where $S_{a,c}$ is given by (2.4). Now, fix $\lambda \geq \lambda_0$ and let us show that problem (2.3) admits a nontrival solution. From Lemmas 2.4 and 2.5, there exists a bounded sequence $\{u_n\} \subset X$ verifying

$$I_{r,\lambda}(u_n) \to c_{r,\lambda}, \quad I'_{r,\lambda}(u_n) \to 0 \text{ as } n \to \infty. \tag{2.11}$$

Hence, up to subsequences, we may assume that $\{u_n\}$ converges weakly to $u \in X$. Then $\{u_n\}$ converges strongly to u in $L^l(\Omega, |x|^{-\alpha})$ for $1 \leq l < \frac{Np}{N-p}$ and $\alpha < (1+a)l + N\left(1 - \frac{l}{p}\right)$ and $u_n(x) \to u(x)$ a.e. $x \in \Omega$.

From the concentration-compactness principle stated in Proposition 1.2, there exist non-negative measures μ and ν and a countable family $\{x_j : j \in J\} \subset \Omega$ such that

$$||x|^{-a}|\nabla u_n||^p \rightharpoonup \mu, \quad ||x|^{-c}|u_n||^{p^*_{a,c}} \rightharpoonup \nu, \tag{2.12}$$

where

$$
\begin{aligned}
&\nu = ||x|^{-c}|u||^{p^*_{a,c}} + \sum_{j\in J}\nu_j\delta_{x_j}, \\
&\mu \geq ||x|^{-a}|\nabla u||^p + \sum_{j\in J}\mu_j\delta_{x_j},
\end{aligned}
$$

$$S_{a,c}\nu_j^{\frac{p}{p^*_{a,c}}} \leq \mu_j, \quad \forall j \in J. \tag{2.13}$$

We shall prove that $\{u_n\}$ converges strongly to u in $L^{p^*_{a,c}}(\Omega, |x|^{-cp^*_{a,c}})$ by showing that $J = \emptyset$. Arguing by contradiction, assume that $J \neq \emptyset$ and fix $j \in J$. Consider

$\phi_j \in C_0^\infty(\Omega,[0,1])$ such that $\phi_j \equiv 1$ on $B_1(0)$, $\phi_j \equiv 0$ on $\Omega \backslash B_2(0)$ and $|\nabla\phi_j| \le 2$. Defining $\phi_{j,\epsilon} = \phi_j(\frac{x-x_j}{\epsilon})$, where $\epsilon > 0$, we have that $\{\phi_{j,\epsilon} u_n\}$ is bounded in X. Thus $I'_{r,\lambda}(u_n)(\phi_{j,\epsilon}u_n) \to 0$ as $n \to \infty$, that is,

$$\begin{aligned}
&M_r(\|u_n\|_{a,p}^p)\int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(\phi_{j,\epsilon}u_n)\,dx\\
&-\lambda \int_\Omega |x|^{-bp} f(x,u_n)\phi_{j,\epsilon}u_n\,dx\\
&-\int_\Omega |x|^{-cp^*_{a,c}}|u_n|^{p^*_{a,c}-2}u_n\phi_{j,\epsilon}u_n\,dx \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Hence,

$$\begin{aligned}
&M_r(\|u_n\|_{a,p}^p)\int_\Omega |x|^{-ap}u_n|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\phi_{j,\epsilon}\,dx\\
&= -M_r(\|u_n\|_{a,p}^p)\int_\Omega |x|^{-ap}|\nabla u_n|^p\phi_{j,\epsilon}\,dx + \lambda\int_\Omega |x|^{-bp}f(x,u_n)u_n\phi_{j,\epsilon}\,dx\\
&\quad + \int_\Omega |x|^{-cp^*_{a,c}}|u_n|^{p^*_{a,c}}\phi_{j,\epsilon}\,dx + o(1).
\end{aligned} \tag{2.14}$$

Since the support of $\phi_{j,\epsilon}$ is $B_{2\epsilon}(x_j)$, using the Hölder inequality and the Dominated Convergence Theorem we obtain

$$\begin{aligned}
&\left|\int_\Omega |x|^{-ap}u_n|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\phi_{j,\epsilon}\,dx\right|\\
&\le \int_{B_{2\epsilon}(x_j)} |x|^{-ap}\nabla u_n|^{p-1}|u_n\nabla\phi_{j,\epsilon}|\,dx\\
&= \int_{B_{2\epsilon}(x_j)} (|x|^{-a(p-1)}\nabla u_n|^{p-1})(|x|^{-a}|u_n\nabla\phi_{j,\epsilon}|)\,dx\\
&\le \left(\int_{B_{2\epsilon}(x_j)} |x|^{-ap}|\nabla u_n|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_{B_{2\epsilon}(x_j)} |x|^{-ap}|u_n\nabla\phi_{j,\epsilon}|^p\,dx\right)^{\frac{1}{p}}\\
&\le C_4\left(\int_{B_{2\epsilon}(x_j)} |x|^{-ap}|u_n|^p|\nabla\phi_{j,\epsilon}|^p\,dx\right)^{\frac{1}{p}}\\
&\le C_4\left(\int_{B_{2\epsilon}(x_j)} |x|^{-cp^*_{a,c}}|u_n|^{p^*_{a,c}}\,dx\right)^{\frac{1}{p^*_{a,c}}}\\
&\qquad\times\left(\int_{B_{2\epsilon}(x_j)} |x|^{-\frac{N(a-c)}{1+a-c}}|\nabla\phi_{j,\epsilon}|^{\frac{N}{1+a-c}}\,dx\right)^{\frac{1+a-c}{N}}\\
&\le \overline{C}_4\left(\int_{B_{2\epsilon}(x_j)} |x|^{-cp^*_{a,c}}|u_n|^{p^*_{a,c}}\,dx\right)^{\frac{1}{p^*_{a,c}}}\\
&\to 0 \text{ as } n\to 0 \text{ and } \epsilon\to 0.
\end{aligned}$$

Since $\{u_n\}$ is bounded, we may assume that $\|u_n\|_{a,p} \to t_2 \geq 0$ as $n \to \infty$. Observing that $M(t)$ is continuous, we then have $M(\|u_n\|_{a,p}^p) \to M(t_2^p) \geq m_0 > 0$ as $n \to \infty$. Hence,

$$M_r(\|u_n\|_{a,p}^p) \int_\Omega |x|^{-ap} u_n |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \phi_{j,\epsilon}\, dx \to 0 \text{ as } n \to \infty. \tag{2.15}$$

Similarly, we have

$$\int_\Omega |x|^{-bp} f(x, u_n) \phi_{j,\epsilon} u_n\, dx \to 0 \text{ as } n \to \infty \text{ and } \epsilon \to 0. \tag{2.16}$$

From (2.14)-(2.16), letting $n \to \infty$ in (2.14) we get

$$\int_\Omega \phi_{j,\epsilon} d\nu \geq M_r(t_2^p) \int_\Omega \phi_{j,\epsilon} d\mu + o_\epsilon(1).$$

Now, letting $\epsilon \to 0$ we deduce that

$$\phi_j(0)\nu_j = \phi_j(0)\mu_j M_r(t_2^p)$$

and thus

$$\nu_j \geq \mu_j m_0. \tag{2.17}$$

Combining (2.13) and (2.17) we obtain

$$\nu_j \geq (S_{a,c} m_0)^{\frac{N}{(1+a-c)p}}. \tag{2.18}$$

Now we shall prove that (2.18) cannot occur, and therefore the set $J = \emptyset$. Indeed, arguing by contradiction, let us suppose that $\nu_j \geq (S_{a,c} m_0)^{\frac{N}{(1+a-c)p}}$ for some $j \in J$. Since $\{u_n\}$ is a $(PS)_{c_{r,\lambda}}$ for the functional $I_{r,\lambda}$, from the conditions (F_3) and (M_0), and $m_0 < r < \frac{\theta}{p} m_0$ we have

$$\begin{aligned}
c_{r,\lambda} &= I_{r,\lambda}(u_n) - \frac{1}{\theta} I'_{r,\lambda}(u_n)(u_n) + 0_n(1) \\
&\geq \frac{1}{p} \widehat{M}_r(\|u_n\|_{a,p}^p) - \frac{1}{\theta} M_r(\|u_n\|_{a,p}^p)\|u_n\|_{a,p}^p \\
&\qquad + \left(\frac{1}{\theta} - \frac{1}{p^*_{a,c}}\right) \int_\Omega |x|^{-cp^*_{a,c}} |u_n|^{p^*_{a,c}}\, dx + o_n(1) \\
&\geq \left(\frac{m_0}{p} - \frac{r}{\theta}\right) \|u_n\|_{a,p}^p + \left(\frac{1}{\theta} - \frac{1}{p^*_{a,c}}\right) \int_\Omega |x|^{-cp^*_{a,c}} |u_n|^{p^*_{a,c}}\, dx + o_n(1) \\
&\geq \left(\frac{1}{\theta} - \frac{1}{p^*_{a,c}}\right) \int_\Omega |x|^{-cp^*_{a,c}} |u_n|^{p^*_{a,c}}\, dx + o_n(1) \\
&\geq \left(\frac{m_0}{p} - \frac{R}{\theta}\right) \|u_n\|_{a,p}^p + \left(\frac{1}{\theta} - \frac{1}{p^*_{a,c}}\right) \int_\Omega |x|^{-cp^*_{a,c}} |u_n|^{p^*_{a,c}}\, dx + o_n(1) \\
&\geq \left(\frac{1}{\theta} - \frac{1}{p^*_{a,c}}\right) \int_\Omega |x|^{-cp^*_{a,c}} |u_n|^{p^*_{a,c}} \phi_{j,\epsilon}\, dx + o_n(1).
\end{aligned}$$

Letting $n \to \infty$ and $\epsilon \to 0$ we get

$$c_{r,\lambda} \geq \left(\frac{1}{\theta} - \frac{1}{p^*_{a,c}}\right)\nu_j \geq \left(\frac{1}{\theta} - \frac{1}{p^*_{a,c}}\right)(S_{a,c}m_0)^{\frac{N}{(1+a-c)p}},$$

which contradicts (2.10). Thus, $J = \emptyset$ and $u_n \to u$ in $L^{p^*_{a,c}}(\Omega, |x|^{-cp^*_{a,c}})$ as $n \to \infty$.

Now, we prove that $u_n \to u$ in X as $n \to \infty$. From (2.1) and the boundedness of $\|u_n - u\|_{a,p}$ we have $I'_{r,\lambda}(u_n)(u_n - u) \to 0$ as $n \to \infty$ or

$$\begin{aligned} M_r(\|u_n\|^p_{a,p}) &\int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)\,dx \\ &- \lambda \int_\Omega |x|^{-bp} f(x, u_n)(u_n - u)\,dx \\ - \int_\Omega &|x|^{-cp^*_{a,c}}|u_n|^{p^*_{a,c}-2}u_n(u_n - u)\,dx \to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.19}$$

On the other hand, by Lemma 1.1 we have

$$\begin{aligned} &\left|\int_\Omega |x|^{-bp} f(x, u_n)(u_n - u)\,dx\right| \\ &\leq \int_\Omega |x|^{-bp}|f(x, u_n)||u_n - u|\,dx \\ &\leq C\int_\Omega |x|^{-bp}(1 + |u_n|^{q-1})|u_n - u|\,dx \\ &\leq C\left(\int_\Omega |x|^{-bp}\,dx\right)^{\frac{q-1}{q}}\left(\int_\Omega |x|^{-bp}|u_n - u|^q\,dx\right)^{\frac{1}{q}} \\ &\qquad + C\left(\int_\Omega |x|^{-bp}|u_n|^q\,dx\right)^{\frac{q-1}{q}}\left(\int_\Omega |x|^{-bp}|u_n - u|^q\,dx\right)^{\frac{1}{q}} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

and

$$\begin{aligned} &\left|\int_\Omega |x|^{-cp^*_{a,c}}|u_n|^{p^*_{a,c}-2}u_n(u_n - u)\,dx\right| \\ &\leq \int_\Omega |x|^{-cp^*_{a,c}}|u_n|^{p^*_{a,c}-1}|u_n - u|\,dx \\ &\leq \left(\int_\Omega |x|^{-cp^*_{a,c}}|u_n|^{p^*_{a,c}}\,dx\right)^{\frac{p^*_{a,c}-1}{p^*_{a,c}}}\left(\int_\Omega |x|^{-cp^*_{a,c}}|u_n - u|^{p^*_{a,c}}\,dx\right)^{\frac{1}{p^*_{a,c}}} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence, by (2.19) we obtain

$$M_r(\|u_n\|^p_{a,p})\int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)\,dx \to 0 \text{ as } n \to \infty.$$

By the condition (M_0) we have

$$\int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)\,dx \to 0 \text{ as } n \to \infty.$$

On the other hand, since $\{u_m\}$ converges weakly to u in X, we have

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u(\nabla u_m - \nabla u)\,dx = 0.$$

Hence,

$$\lim_{m\to\infty}\int_\Omega |x|^{-ap}\left(|\nabla u_m|^{p-2}\nabla u_m - |\nabla u|^{p-2}\nabla u\right)(\nabla u_m - \nabla u)\,dx = 0.$$

or

$$\lim_{m\to\infty}\int_\Omega \left(|\nabla v_m|^{p-2}\nabla v_m - |\nabla v|^{p-2}\nabla v\right)(\nabla v_m - \nabla v)\,dx = 0, \tag{2.20}$$

where $\nabla v_m = |x|^{-a}\nabla u_m$ and $\nabla v = |x|^{-a}\nabla u$.

Let us recall that the following inequalities hold

$$\begin{aligned} &(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta) \geq C_5(|\xi| + |\eta|)^{p-2}|\xi-\eta|^2 \text{ if } 1<p<2,\\ &(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta) \geq C_6|\xi-\eta|^p \text{ if } p \geq 2, \end{aligned} \tag{2.21}$$

for all $\xi, \eta \in \mathbb{R}^N$, where C_5 and C_6 are positive constants.

If $1<p<2$, using the Hölder inequality, by (2.20), (2.21) we have

$$\begin{aligned} 0 &\leq \|u_n - u\|_{a,p}^p = \||\nabla v_n - \nabla v|\|_{L^p(\Omega)}^p\\ &\leq \int_\Omega |\nabla v_n - \nabla v|^p(|\nabla v_n| + |\nabla v|)^{\frac{p(p-2)}{2}}(|\nabla v_n| + |\nabla v|)^{\frac{p(2-p)}{2}}\,dx\\ &\leq \left(\int_\Omega |\nabla v_n - \nabla v|^2(|\nabla v_n| + |\nabla v|)^{p-2}\,dx\right)^{\frac{p}{2}}\\ &\qquad\times\left(\int_\Omega (|\nabla v_n| + |\nabla v|)^p\,dx\right)^{\frac{2-p}{2}}\\ &\leq \frac{1}{C_5^{\frac{p}{2}}}\left(\int_\Omega (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)(\nabla v_n - \nabla v)\,dx\right)^{\frac{p}{2}}\\ &\qquad\times\left(\int_\Omega (|\nabla v_n| + |\nabla v|)^p\,dx\right)^{\frac{2-p}{2}}\\ &\leq \overline{C}_5\left(\int_\Omega (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)(\nabla v_m - \nabla v)\,dx\right)^{\frac{p}{2}}\\ &\to 0 \text{ as } n\to\infty, \end{aligned}$$

where $\overline{C}_5$ is a positive constant. If $p \geq 2$, one has

$$\begin{aligned} 0 &\leq \|u_n - u\|_{a,p}^p = \||\nabla v_n - \nabla v|\|_{L^p(\Omega)}^p\\ &\leq \frac{1}{C_6}\int_\Omega (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)(\nabla v_n - \nabla v)\,dx \end{aligned}$$

$\to 0$ as $n \to \infty$,

Therefore, we conclude that $\{u_n\}$ converges strongly to u in X and u is a nontrival solution of problem (2.3). $\square$

Proof of Theorem 2.2. Let λ_0 be as in Theorem 2.3 and, for $\lambda \geq \lambda_0$, let u_λ be the nontrival solution of problem (1.1) found in Theorem 2.3. We claim that there exists $\lambda^* \geq \lambda_0$ such that $\|u_\lambda\|_{a,p}^p \leq t_0$ for all $\lambda \geq \lambda^*$. From this we have $M_r(\|u_\lambda\|_{a,p}^p) = M(\|u_\lambda\|_{a,p}^p)$ and thus u_λ is a solution of problem (1.1).

If this claim does not hold, there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that $\|u_\lambda\|_{a,p}^p \geq t_0$. Hence, we have

$$\begin{aligned} c_{\lambda_n} &\geq \frac{1}{p}\widehat{M}(\|u_n\|_{a,p}^p) - \frac{1}{\theta}M(\|u_n\|_{a,p}^p)\|u_n\|_{a,p}^p \\ &\geq \left(\frac{m_0}{p} - \frac{r}{\theta}\right)\|u_n\|_{a,p}^p \\ &\geq \left(\frac{m_0}{p} - \frac{r}{\theta}\right)t_0, \end{aligned}$$

which is an absurd.

Finally, in order to show that $\lim_{\lambda\to+\infty}\|u_\lambda\|_{a,p} = 0$, it suffices to note that, from (M_0) and (F_3), we have

$$\begin{aligned} c_\lambda &\geq \frac{1}{p}\widehat{M}(\|u_\lambda\|_{a,p}^p) - \frac{1}{\theta}M(\|u_\lambda\|_{a,p}^p)\|u_\lambda\|_{a,p}^p \\ &\geq \frac{m_0}{p}\|u_\lambda\|_{a,p}^p - \frac{1}{\theta}M(t_0)\|u_\lambda\|_{a,p}^p \\ &= \left(\frac{m_0}{p} - \frac{r}{\theta}\right)\|u_\lambda\|_{a,p}^p. \end{aligned}$$

From Lemma 2.6 again we have $\lim_{\lambda\to+\infty} c_\lambda = 0$. Since $m_0 < r < \frac{\theta}{p}m_0$, the proof of Theorem 2.2 is now completed. $\square$

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References

[1] M. Allaoui, *Existence results for a class of $p(x)$-Kirchhoff problems*, Studia Sci. Math. Hungarica, *54*(3) (2016), 316–331. .

[2] C.O. Alves, F. J. S. A. Corrêa, and G. M. Figueiredo, *On a class of nonlocal elliptic problems with critical growth*, Differ. Equ. Appl., *2*(3) (2010), 409-417. .

[3] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical points theory and applications*, J. Funct. Anal., *14*(4) (1973), 349-381. .

[4] M. Avci, B. Cekic, and R. A. Mashiyev, *Existence and multiplicity of the solutions of the $p(x)$-Kirchhoff type equation via genus theory*, Math. Methods Appl. Sci., *34*(14) (2011), 1751-1759. .

[5] L. Caffarelli, R. Kohn, and L. Nirenberg, *First order interpolation inequalities with weights*, Composito Math., *53*(3) (1984), 259-275.

[6] G. Caristi, S. Heidarkhani, A. Salari, and S. A. Tersian, *Multiple solutions for degenerate nonlocal problems*, Appl. Math. Letters, *84* (2018), 26-33. .

[7] F. Catrina and Z. Q. Wang, *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existstence (and non existstence) and symmetry of extremal functions*, Comm. Pure Appl. Math., *54*(2) (2001), 229-258.

[8] C. Y. Chen, Y. C. Kuo, and T. F. Wu, *The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions*, J. Differ. Equ., *250* (2011), 1876-1908. .

[9] X. Cheng and G. Dai, Positive solutions for p-Kirchhoff type problems on $\mathbb{R}^N$, Math. Methods Appl. Sci., *38*(12) (2015), 2650-2662. .

[10] M. Chipot and B. Lovat, *Some remarks on nonlocal elliptic and parabolic problems*, Nonlinear Anal. (TMA), *30*(7) (1997), 4619-4627. .

[11] J. Chu, S. Heidarkhani, K.I. Kou, and A. Salari, *Weak solutions and energy estimates for a degenerate nonlocal problem involving sub-linear nonlinearities*, J. Korean Math. Soc., *54*(5) (2017), 1573-1594. .

[12] N. T. Chung and P. H. Minh, *Kirchhoff type problems involving p-biharmonic operators and critical exponents*, J. Appl. Anal. Comput., *7*(2) (2017), 659-669. .

[13] N. T. Chung and H. Q. Toan, *Existence and multiplicity of solutions for a class of degenerate nonlocal problems*, Electronic J. Differ. Equ., *2013*(148) (2013), 1-13.

[14] N. T. Chung and H. Q. Toan, *On a class of degenerate nonlocal problems with sign-changing nonlinearities*, Bull. Malaysian Math. Sci. Soc., *37*(4) (2014), 1157-1167.

[15] G. Dai and R. Hao, *Existence of solutions for a $p(x)$-Kirchhoff-type equation*, J. Math. Anal. Appl., *359* (2009), 275-284. .

[16] G. M. Figueiredo, *Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument*, J. Math. Anal. Appl., *401*(2) (2013), 706-713. .

[17] A. Ghanmi, *Nontrivial solutions for Kirchhoff-type problems involving the $p(x)$-Laplace operator*, Rocky Mountain J. Math., *48*(4) (2018), 1145-1158. .

[18] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, Germany, 1883.

[19] S. Liang and J. Zhang, *Multiplicity of solutions to the weighted critical quasilinear problems*, Proc. Edinburgh Math. Soc., *55* (2012), 181-195. .

[20] P. L. Lions, *The concentration compactness principle in the calculus of variations, the limit case* (I), Rev. Mat. Iberoamericana, *1*(1) (1985), 145-201. .

[21] P. L. Lions, *The concentration compactness principle in the calculus of variations, the limit case* (II), Rev. Mat. Iberoamericana, *1*(2) (1985), 45-121. .

[22] D. Naimen, *The critical problem of Kirchhoff type elliptic equations in dimension four*, J. Differ. Equ., *257*(4) (2014), 1168-1193. .

[23] J. Sun and T. F. Wu, *Existence and multiplicity of solutions for an indefinite Kirchhoff-type equation in bounded domains*, Proc. Royal Soc. Edinburgh Sect. A: Mathematics, *146*(2) (2016), 435-448. .

[24] P. Ubilla, J. Sanchez, L. Iturriaga, and F. Brock, *Existence of positive solutions for p-Laplacian problems with weights*, Comm. Pure Appl. Anal., *5*(4) (2006), 941-952. .

[25] F. Wang, M. Avci, and Y. An, *Existence of solutions for fourth order elliptic equations of Kirchhoff type*, J. Math. Anal Appl., *409*(1) (2014), 140-146. .

[26] Q. L. Xie, X. P. Wu, and C. L. Tang, *Existence and multiplicity of solutions for Kirchhoff type problem with critical exponent*, Comm. Pure Appl. Anal., *12*(6) (2013), 2773-2786. .

[27] B. J. Xuan, *The eigenvalue problem of a singular quasilinear elliptic equation*, Electron. J. Differ. Equ., *2004*(16) (2004), 1-11.

[28] B. J. Xuan, *The solvability of quasilinear Brezis-Nirenberg-type problems with singular weights*, Nonlinear Anal., *62*(4) (2005), 703-725. .