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Distribution of zeros of solutions of sixth order ($2 \leq n \leq 5$)-points boundary value problem in terms of semi-critical intervals

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Abstract In this paper, the issue of distribution of zeros of the solutions of linear homogenous differential equations (LHDE) have been investigated in terms of semi-critical intervals. We shall follow a geometric approach to state and prove some properties of LHDEs of the sixth order with (2, 3, 4, and 5 points) boundary conditions and with measurable coefficients. Moreover, the relations between semi-critical intervals of the LHDEs have been explored. Also, the obtained results have been generalized for the 5^{th} order differential equations.

Keywords. Linear differential equations; Distribution of zeros for the solution; Boundary value problems; Semi-oscillatory interval; Semi-critical interval.

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1. Introduction

In the 60s of the last century laws of distribution of zeros of solutions began to emerge in some studies. It was dealt with in one package with the question of research interval of applicability of theorems on differential inequalities, considering not only one differential equation and multi-point boundary value problems for the given equation, but more generally, multi-point boundary value problems for any given equation. This field of study attracts many researchers and it gains much more interest for its applications to functional equations [14], neutral differential equations [13, 15], differential equations with delay constant [21, 25, 26], and variable delay [24, 27].
The question of the laws of distribution of zeros of solutions of a linear differential equation touches upon many studies on the theory and practice of differential equations.

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In I. Mikusinsky[22], the following analogue of the Sturm theorem is obtained: If the solutions $u(t)$ and $v(t)$ of the equation $x^{(n)} + g(t)x = 0$ satisfy the conditions

$$u(\alpha) = u^{'}(\alpha) = ... = u^{(n-2)}(\alpha) = 0, u^{(n-1)}(\alpha) = 1, u(\beta) = 0,$$

$$v(\gamma) = v^{'}(\gamma) = ... = v^{(n-2)}(\gamma) = 0, v^{(n-1)}(\gamma) = 1, \alpha \leq \gamma \leq \beta,$$

then the solution $v(t)$ does not have zeros in $(\gamma, \beta]$. Also, A. K. Kondratiev[19], considering the same equation $x^{(n)} + g(t)x = 0$ for values of $n = 3$ and $n = 4$ at constant coefficient $g(t)$, proved the following theorem on alternation of zeros of solutions:

- if $n = 3$, then between two consecutive zeros of one solution there are at most two zeros of the other;
- if $n = 4$ and $g(t) \geq 0$, then between two consecutive zeros of one solution lies not more than four zeros of the other, where four zeros can lie;
- if $n = 4$ and $g(t) \geq 0$, then between two successive zeros of one solution lie not more than three zeros of the other.

M. Akhundov. and A. T. Toraev[1], found a generalization of the result of Kondratiev for the equation $x^{'''} + g_1(t)x^{'} + g_2(t)x = 0$, where $g_1(t)$ and $g_2(t)$ is constant-sign. U. Levin[20], showed that the theorem of A. K. Kondratiev is also valid for equations of the form $x^{'''} + g_1(t)x^{''} + g_2(t)x^{'} = 0$ and $x^{(IV)} + (g(t)x^{'})^{'} = 0$.
After the publication of [12], where the laws of distribution of zeros of solutions for a LHDE of the third order of the general form for $n = 3$ in terms of semi-critical intervals established, a large number of papers have appeared [2, 3, 4, 12, 16, 23]. They studied with one or other degrees of completeness of the problem of the distribution of zeros of solutions of equations of nth order at $n \geq 4$ with summable coefficients besides the continuous ones.
The authors of [5, 6, 7, 8, 9, 10, 11, 17, 18] investigated LHDEs of the (fifth, sixth) order, they used the analytic approach to prove the properties of the distribution zeros of their solutions.
In this paper, we shall rather use the geometric approach to state and prove some properties of LHDE of the sixth order with (2, 3, 4, and 5 points) boundary conditions. Main results in this study are;
$r_{51}(s) \geq r_{111111}(s), r_{312}(s) \geq r_{111111}(s), r_{2121}(s) \geq r_{111111}(s)$ and $r_{21111}(s) \geq r_{111111}(s)$.

2. Concepts and Terminology

Consider the equation

$$L\,[y] = y^{(6)} - g_5\,(x)\,y^{(5)} - g_4(x)\,y^{(4)} - g_3\,(x)\,y^{'''} - g_2\,(x)\,y^{''} - g_1\,(x)\,y^{'} - g_0\,(x)\,y = 0. \tag{2.1}$$

Assume that the coefficients $g_k\,(x)$ are measurable and continues on $[a, b]$ satisfying the conditions

$$y^{(k_j)}\,(\xi_j) = A_{j,k_j},\ k_j = 0,\ \dots,\ p_j - 1,\ j = 1,\ 2,\ \dots,\ m,\ \sum_{j=0}^{m} p_j = 6,$$
$$m \leq 6 \tag{2.2}$$

where m is the number of points ξ_j, p_j is the number of conditions at the points ξ_j.

Problem (2.1) and (2.2) is called $\ll$ ($p_1 p_2 \dots p_m$ – problem) $\gg$.

Definition 2.1. [10] For each fixed point $\alpha \in [a, b)$, there exists a nonzero interval $[\alpha, \beta)$, in which any non-trivial solution of equation (2.1) has no more than 5 zero, taking into account their multiplicities. This interval is called the semi-oscillation for equation (2.1). The maximum intervals of semi-oscillation with a common origin in α is denoted by $[\alpha, r(\alpha))$.

Definition 2.2. [9] The interval $[\alpha, \mu)$, in which the given problem has a unique solution, is called the semi-critical interval of this problem. The maximum intervals of semi - critical with a common origin in α is denoted by $[\alpha,\ \ r_{p_1 p_2 \dots . p_k}(\alpha))$, k =2, 3, 4, 5.

The concept of the semi-critical interval is directly related to the distribution of zeros of the solution of equation (2.1).

We decipher the definitions of the maximal semi-critical intervals of some boundary value problems.

The interval $[s, r_{51}(s))$ is called such an interval in which any non-trivial solution (for the equation (1)) that has a zero at ξ_1 of multiplicity five and has no more zeros to the right of ξ_1, where s $\leq \xi_1 <\ r_{51}\,(s) <\ \xi_2$.

In the interval $[s, r_{42}(s))$, non-trivial solution (for the equation (1)) that has a zero at ξ_1 of multiplicity four, can not have a double zero to the right of ξ_1, where s $\leq \xi_1 <\ r_{42}\,(s) <\ \xi_2$.

A non-trivial solution (for the equation (1)) that has a zero at $\xi_1$of multiplicity three and zero $\xi_2 > \xi_1$ can not have a zero $\xi_3 > \xi_2$ of multiplicity higher than the second in the interval $[s,\ \ r_{312}(s))$, where s $\leq \xi_1 < \xi_2 <\ r_{312}\,(s) <\ \xi_3$.

A non-trivial solution (for equation (1)) that has zeros ξ_1, ξ_3 of at least multiplicity of two and a simple zero ξ_2 where $\xi_3 > \xi_2 > \xi_1$ of any multiplicity, can not have in the interval $[s,\ \ r_{2121}(s))$ zeros $\xi_4 > \xi_3$, where s $\leq \xi_1 < \xi_2 < \xi_3 <\ r_{2121}\,(s) <\ \xi_4$.

A non-trivial solution (for the equation (1)) that has zero ξ_1 not lower second multiplicity and three simple zeroes $\xi_2,\ \xi_3,\ \xi_4$, where $\xi_4 > \xi_3 > \xi_2 > \xi_1$ of any multiplicity, can not have in $[s, r_{21111}(s))$ zeros $\xi_5 > \xi_4$.

In the interval $[s, r_{111111}(s))$ a nontrivial solution can not have six different simple zeros.

Let us clarify justifications of listed assertions. On the example of the problem $\ll$(51 – problem) $\gg$ to have a unique solution, It is necessary and sufficient that the determinant

$$\triangle(\xi_1,\xi_2)=\begin{vmatrix} u_0(\xi_1) & u_1(\xi_1) & u_2(\xi_1) & u_3(\xi_1) & u_4(\xi_1) & u_5(\xi_1)\\ u_0^{'}(\xi_1) & u_1^{'}(\xi_1) & u_2^{'}(\xi_1) & u_3^{'}(\xi_1) & u_4^{'}(\xi_1) & u_5^{'}(\xi_1)\\ u_0^{''}(\xi_1) & u_1^{''}(\xi_1) & u_2^{''}(\xi_1) & u_3^{''}(\xi_1) & u_4^{''}(\xi_1) & u_5^{''}(\xi_1)\\ u_0^{'''}(\xi_1) & u_1^{'''}(\xi_1) & u_2^{'''}(\xi_1) & u_3^{'''}(\xi_1) & u_4^{'''}(\xi_1) & u_5^{'''}(\xi_1)\\ u_0^{(4)}(\xi_1) & u_1^{(4)}(\xi_1) & u_2^{(4)}(\xi_1) & u_3^{(4)}(\xi_1) & u_4^{(4)}(\xi_1) & u\,_5^{(4)}(\xi_1)\\ u_0(\xi_2) & u_1(\xi_2) & u_2(\xi_2) & u_3(\xi_2) & u_4(\xi_2) & u_5(\xi_2) \end{vmatrix}$$
$$\neq 0,$$

where $u_i(\xi_1),\ i=0,\ 1,\ 2,\ 3,\ 4,\ 5$ is the fundamental system of solutions of equation (2.1), was different from zero. But $\Delta(\xi_1,x)$ is a solution of equation (2.1) and at the point ξ_1 this solution has a five-multiple zero. Thus, the condition does not vanish (is not vanishing) at $x\in[\xi_1,\omega)$.

The Function $\Delta(\xi_1,x)$ is a condition for the existence and uniqueness of the solution of problem (2.1), (2.2) for any $\xi_2\in[\xi_1,\omega)$.

In a similar way, one can make sure of other cases.

The present paper considers the laws of the distribution of zeros. The main results of the distribution of zeros are the following:

The interval $[s,\ r_{111111}(s))$ is the intersection of intervals
$[s,\ r_{51}(s)),\ [s,\ r_{42}(s)),\ [s,\ r_{33}(s)),\ [s\ ,\ r_{24}(s)), [s,\ r_{15}(s)), [s,\ r_{312}(s)),$
$[s,\ r_{411}(s)),\ [s,\ r_{321}(s)), [s,\ r_{231}(s)),\ [s,\ r_{213}(s)), [s,\ r_{141}(s),(s)),$
$[s,\ r_{114}(s)), [s,\ r_{123}(s)),\ [s,\ r_{132}(s)), [s,\ r_{2121}(s)),\ [s,\ r_{3111}(s)),$
$[s,\ r_{2112}(s)), [s,\ r_{2211}(s)),\ [s,\ r_{1113}(s)), [s,\ r_{1131}(s)),\ [s,\ r_{1311}(s)),$
$[s,\ r_{1221}(s)),\ [s,\ r_{1212}(s)),\ [s,\ r_{1122}(s)), [s,\ r_{21111}(s)),$
$[s,\ r_{12111}(s)), [s,\ r_{11211}(s)), [s,\ r_{11121}(s)),\ [s,\ r_{11112}(s)).$
and is equal to the smallest of the intervals
$r_{111111}(s)\le min\,[r_{51}(s), r_{42}(s), r_{33}(s),\ r_{24}(s), r_{15}(s),\ r_{411}(s), r_{312}(s),$
$r_{321}(s), r_{231}(s), r_{213}(s), r_{141}(s), r_{114}(s), r_{123}(s),\ r_{132}(s), r_{3111}(s), r_{2112}(s),$
$r_{2121}(s),\ r_{2211}(s), r_{1113}(s), r_{1131}(s), r_{1311}(s), r_{1221}(s), r_{1212}(s), r_{1122}(s),$
$r_{21111}(s), r_{12111}(s), r_{11211}(s), r_{11121}(s), r_{11112}(s)].$

To prove the assertions formulated above, consider the following auxiliary lemmas.

Lemma 2.3. [7] *Let* $v_1(x),\ v_2(x)$ *be a pair of not identically equal to zero, twice continuously differentiable functions such that*

$v_1(x)\neq cv_2(x),\ (c=const),\ v_1(\alpha)=v_1(\beta)=0,\ v_2(x)\neq 0\ in\ [\alpha,\ \beta].$

Then there exists a linear combination

$$u(x)=\ c_1v_1(x)\ +\ c_2v_2(x),\ \left(c_1^2+c_2^2>0\right),$$

for which the point ξ *is a zero of multiplicity two, that is*

$$u(\xi)=u'(\xi)=0,\text{where }\xi\in(\alpha,\ \beta).$$

Lemma 2.4. [7] *Let* $v_1(x),\ \ v_2(x)$ *be a pair of not identically equal to zero, twice continuously differentiable functions such that,*
$v_1(x)\neq cv_2(x),\ (c=const),\ v_1^{(k)}(\xi)=v_2^{(k)}(\xi)=0,\ k=0,\ 1.$

Then, there exists a linear combination

$$u(x)=c_1v_1(x)+c_2v_2(x),\quad \left(c_1^2+c_2^2>0\right),$$

for which the point ξ *is a zero of multiplicity three, that is*

$$u^{(i)}(\xi) = 0,\ i= 0,\ 1,\ 2.$$

Lemma 2.5. [7] *Let* $v_1(x),\ v_2(x)$ *– be twice continuously differentiable functions, satisfying the conditions* $v_1(x).v_2(x) > 0,\ \ \alpha < x < \beta,\ \ v_1^{(k)}(\xi) = v_2^{(k)}(\xi),\ \ \ k= 0,\ 1;$ $v_1^{''}(\xi) \neq v_2^{''}(\xi),\ \ \xi \in (\alpha, \beta);\ v_1(x) \neq v_2(x)\ \ \ \ (x\ = \xi);$ *then for any* $\varepsilon > 0$ *there is a linear combination*

$u(x) = v_1(x) - cv_2(x),\ \ (\mathrm{c}\ = const),$

such that

$u(\xi_1) = u(\xi_2) = 0,$ *where* $\alpha < \xi_1 < \xi < \xi_2 < \beta$ *and* $max\,[|\xi - \xi_1|,\ \ |\xi - \xi_2|] < \varepsilon.$

Lemma 2.6. [8] *Non-trivial solutions* $v_1(x)$ *and* $v_2(x)$ *of equation* (2.1) *are linearly dependent if* $v_i^{(k)}(\xi) = 0,\ \ \ k =\ 0,\ 1,\ 2;\ i= 1,\ 2.$

Lemma 2.7. [8] *Let u(x), v(x) – be a pair of non-trivial solutions of equation* (2.1) *such that* $u^{(k)}(\alpha) = 0, k= 0,\ 1,\ 2,\ 3,\ 4;\ v(\alpha) = 0.$ *If* $u(x) \neq 0$ *in* $(\alpha,\ \ \beta + \varepsilon)$*, then for any* $\varepsilon > 0$ *and for some constant c, the difference* $cu(x) - v(x)$ *vanishes (goes to zero) at the points* $\beta_i \in (\alpha,\ \ \beta + \varepsilon)$*, whose number is equal to* $p + q$*, where p is the number of odd zeros of the solution v(x) in* $(\alpha,\ \beta]$ *and q is the number of those* $\alpha_i \in (\alpha,\ \beta]$*, that*

$$v(\alpha_i) = v^{'}(\ \alpha_i) = 0,\ \ \ u^{'''}(\alpha)\, v^{''}(\alpha_i) > 0.$$

3. Main Results

In this section, according to the mentioned conditions in the definitions above, we will prove the following

$r_{111111}(s) \leq min\,[r_{51}(s), r_{42}(s), r_{33}(s),\ r_{24}(s), r_{15}(s),\ r_{411}(s), r_{312}(s),$ $r_{321}(s), r_{231}(s),\ r_{213}(s), r_{141}(s), r_{114}(s), r_{123}(s),\ r_{132}(s), r_{3111}(s), r_{2112}(s),$ $r_{2121}(s),\ r_{2211}(s), r_{1113}(s),\ r_{1131}(s), r_{1311}(s), r_{1221}(s), r_{1212}(s), r_{1122}(s),$ $r_{21111}(s), r_{12111}(s), r_{11211}(s), r_{11121}(s), r_{11112}(s)].$

Theorem 3.1. $r_{51}(s) \geq r_{111111}(s).$

Proof. Assume that $r_{51}(s) < r_{111111}(s)$. Thus, two points are existed, $\alpha,\ \beta \in [s,\ \ \ r_{111111}(s))$,and a solution $u(x)$ of equation (2.1) which obeys $u(\alpha) = u^{'}(\alpha) = u^{''}(\alpha) = u^{'''}(\alpha) = u^{(IV)}(\alpha) = u(\beta) = 0,\ u(x) > 0$ in $(\alpha,\ \beta)$, where

$$s \leq \alpha < r_{51}(s) < \beta < r_{111111}(s)$$

and either $u^{'}(\beta) \neq 0$, $u^{'}(\beta) = u^{''}(\beta) = 0$ or $u^{'}(\beta) = 0,\ u^{''}(\beta) > 0,$

Accordingly, we consider two cases:

Case 1: Assume β is a zero of odd multiplicity. We can assume that $\alpha > s$. Then, there exists a unique solution $v(x)$ of equation (2.1) in the interval $[s, r_{111111}(s))$ such that

$$v(\xi_i) = u(\xi_i),\ i = 0,\ 1,\ 2,\ 3,\ 4,\ v(\xi_5) < 0,$$

where

$$s < \xi_0 < \alpha = \xi_1 < \xi_2 < \xi_3 < \xi_4 < r_{51}(s) < \beta = \xi_5 < r_{111111}(s)$$

. Now by Lemma (2.5), the curves $u(x),\ v(x)$ have no tangencies of even order in points with abscissas $\xi_0,\ \xi_2,\ \xi_3,\ \xi_4$. So that either, $u'(\xi_0) < v'(\xi_0)$, or $u'(\xi_0) > v'(\xi_0)$, since by the condition r_{51}(s) $< \beta$, the curves $u(x)$ and $v(x)$ have no five fold tangency at the point with abscissa ξ_0.

If $u'(\xi_0) < v'(\xi_0),\ \ v'(\xi_1) \neq 0, v(\xi_i) = u(\xi_i), i = 0,\ 1,\ 2,\ 3,\ 4,$ then $u'(\xi_3) > v'(\xi_3)$,
where $s < \xi_0 < \alpha = \xi_1 < \xi_2 < \xi_3 < \xi_4 < \xi_5 < r_{51}(s) < \beta = \xi_6 < r_{111111}(s),\ v(\xi_6) < 0$.
We note that the curves u(x) and $v(x)$ have no tangencies of even order at points with abscissas $\xi_0,\ \xi_2,\ \xi_3,\ \xi_4$,$\xi_5$, but the difference $u(x) - v(x)$ has six zeros $\xi_0, \xi_1, \xi_2, \xi_3, \xi_4$, and ξ_5, in the interval $[s, r_{111111}(s))$, which is impossible.

If

$$u'(\xi_0) < v'(\xi_0), v'(\xi_0) \neq 0$$

, then, $u'(\xi_4) < v'(\xi_4)$, hence $v(x) > u(x)$ in some right half - neighborhood point ξ_4. But, since $v(\xi_5)$<0, then the difference $u(x) - v(x)$ has a zero at some point $\xi \in (\xi_4,\ \xi_5)$, which is impossible, because this difference already has zeros at the fifth points ξ_0 ,$\xi_1, \xi_2, \xi_3, \xi_4$, in the interval $[s, r_{111111}(s))$.

If $u'(\xi_0) > v'(\xi_0)$, or $v'(\xi_0) < 0$, $v(\xi_i) = u(\xi_i),\ i = 0,\ 1,\ 2,\ 3,\ 4,\ \ v(\xi_5) < 0$

then the curves $u(x)$ and $v(x)$ do not have a common point in the interval $[\beta, r_{111111}(s))$, which means $u(x) - v(x) \neq 0$ in the interval $[\beta, r_{111111}(s))$. Now, choosing a point $\xi \in (\beta, r_{111111}(s))$ such that $u(\xi) < 0$, yields that the linear combination

$$y(x) = u(x) - \frac{u(\xi)}{v(\xi)} v(x)$$

, has a zero at the point ξ and five points ‘ξ_0,‘ξ_1,‘ξ_2,‘ξ_3,‘ξ_4, in interval $[s,\ r_{111111}(s))$, which is impossible.

Finally, if

$$v'(\alpha) = 0, v''(\alpha) < 0,$$

then the difference $u(x) - v(x)$ has zeros at the points with abscissas $\xi_0,\ \xi_2,\ \xi_3,\ \xi_4$, and a double zero at the point $\alpha = \xi_1$.

But, since the solution $w(x)$ has five-multiple zero at the point ξ_0, and has no zeros in the interval $(\xi_0, \xi_4 + \varepsilon),\ \varepsilon < r_{51} - \xi_4$, then, by Lemma (2.5) the linear combination

$cw(x) - [u(x) - v(x)]$ (for some constant value c) has six zeros in the interval $(\xi_0, \xi_4 + \varepsilon) \subset [s, r_{111111}(s))$, which is impossible.

So, the case β being a zero of odd multiplicity of the solution $u(x)$ leads to the desired result, $r_{51}(s) \geq r_{111111}(s)$.

Case 2: Assume β is a zero of an even multiplicity. Then, there exist two points $\alpha,\ \beta \in [s,\ \ r_{111111}(s))$ and a solution $u(x)$ of equation (2.1) such that

$u(\alpha) = u'(\alpha) = u''(\alpha) = u'''(\alpha) = u(\beta) = u'(\beta) = 0,\ u(x) > 0$, in $(\alpha,\ \beta)$,

where

$$s \leq \alpha < r_{51}(s) < \beta < r_{111111}(s), u''(\alpha) > 0, \ u''(\beta) > 0.$$

Consider the solution $v(x)$ satisfying boundary conditions

$$u(\xi_i) = v(\xi_i), \ \ i = 0, \ 1, \ 2, \ 3, \ 4, v'(\alpha) > 0, \ v(\beta) > 0,$$

where

$$s < \xi_0 < \alpha = \xi_1 < \xi_2 < \xi_3 < \xi_4 < r_{51}(s) < \beta = \xi_5 < r_{111111}(s),$$

and the curves $u(x)$ and $v(x)$ do not have a common points in the interval $[\beta, r_{111111}(s))$, it means $u(x) - v(x) \neq 0$ in the interval $[\beta, r_{111111}(s))$. Then, choosing the point $\xi \in (\beta, r_{111111}(s))$, where $v(\xi) > 0$, the difference

$$y(x) = u(x) - \frac{u(\xi)}{v(\xi)} v(x)$$

, has a zero at the point ξ and five zeros in points ξ_0, ‘ξ_1, ‘ξ_2, ‘ξ_3, ‘ξ_4 , where‘ξ_1, ‘ξ_2, ‘ξ_3, ‘ξ_4, are the shifted positions of the points $\xi_1, \ \xi_2, \ \xi_3, \ \ \xi_4$, for c $\to \frac{u(\xi)}{v(\xi)}$, which is impossible.

Consider the solution $v(x)$ such that

$$u(\xi_i) = v(\xi_i), \ \ i = 0, \ 1, \ 2, \ 3, \ 4, v(\xi_5) > 0,$$

where

$$s < \xi_0 < \alpha = \xi_1 < \xi_2 < \xi_3 < \xi_4 < r_{51}(s) < \beta = \xi_5 < r_{111111}(s).$$

As mentioned above via Lemma (2.5) the curves u(x) and $v(x)$ have no tangencies at the abscissa points $\xi_1, \ \xi_2, \ \xi_3, \ \ \xi_4$. Therefore, if $v'(\alpha) < 0$ or $v'(\alpha) = 0$, and $v''(\alpha) < 0$,

then the contradiction with the assumption is obvious, for then $u'(\xi_4) > v'(\xi_4)$.

Hence, the difference $u(x) - v(x)$ has zero at some point $\eta \in (\xi_4, \ \xi_5)$, since by construction $v(\xi_5) > 0$, which is impossible, because this difference already has five zeros at the points $\xi_0, \ \xi_1, \ \ \xi_2, \ \xi_3, \ \xi_4 \in [s, r_{111111}(s))$.

If $u(\xi_i) = v(\xi_i), \ \ i = 0, \ 1, \ 2, \ 3, \ 4, \ v(\xi_5) > 0, \ \ v(\alpha) = v'(\alpha) = 0, \ v''(\alpha) < 0$, then the difference $u(x) - v(x)$ has zeros at the abscissas $\xi_0, \ \xi_2, \ \xi_3, \ \xi_4$ points and a double zero at the point $\alpha = \xi_1$.

But, since the solution $w(x)$ has three-multiple zero at the point ξ_0 and no zeros in the region $(\xi_0, \ \xi_4 + \varepsilon), \ \ \varepsilon < r_{51} - \xi_4$.

Thereby, depending on Lemma (2.5), there would be a linear combination $cw(x) - [u(x) - v(x)]$ for some constant value c, associated with six zeros in the region

$(\xi_0, \ \xi_4 + \varepsilon) \subset [s, r_{111111}(s))$, which is impossible. This leads to $r_{51}(s) \geq r_{111111}(s)$. Hence, the theorem is proved. □

Theorem 3.2. $r_{312}(s) \geq r_{111111}(s)$

Proof. Assume that, $r_{312}(s) < r_{111111}(s)$, which means the existence of a solution $u(x)$ of equation (2.1) having three zeros $\alpha, \ \ \beta, \ \ \gamma$ in the interval $[s, r_{111111}(s))$ such that

$$u(\alpha) = u'(\alpha) = u''(\alpha) = u(\beta) = u(\gamma) = u'(\gamma),$$

where

$$u^{'''}(\alpha)\, u^{''}(\gamma) \neq 0.$$

Since $u(x) \neq 0$ in the interval $(\alpha,\ \beta) \cup (\beta,\ \gamma)$.

Since $u(x) < 0$ in the interval $(\gamma,\ \mathrm{r}_{1111111}(\mathrm{s}))$, consider the solution $v(x)$ satisfying the boundary conditions $u(\xi_i) = v(\xi_i)$, i=0, 1, 2, 3, 4, $v(\xi_5) < 0$,

where

s$< \xi_0 < \alpha = \xi_1 < \xi_2 < \beta < \xi_3 < \xi_4 < r_{3121}(s) < \gamma = \xi_5 < r_{111111}(s)$. By virtue of Lemma (2.3) and Theorem (3.1), the curves $u(x)$, $v(x)$ have no tangencies at the points $\xi_1,\ \xi_2,\ \xi_3,\ \xi_4$, with abscissas in the interval $[s, r_{111111}(s))$.

Therefore, if $v^{'}(\alpha) < 0$ and $v(\xi_5) < 0$, then putting $c = \frac{u(\xi)}{v(\xi)}$ where $\gamma < \xi < r_{111111}(s)$ it is easy to see that the difference $u(x) - v(x)$ has six zeros ξ_0, ‘ξ_1, ‘ξ_2, ‘ξ_3, ‘ξ_4 , and ξ, in the interval $[s, r_{111111}(s))$ where ‘ξ_1, ‘ξ_2, ‘ξ_3, ‘ξ_4, are the shifted position of the points ξ_1, ξ_2, ξ_3, ξ_4, for $c = \frac{u(\xi)}{v(\xi)}$, which is impossible.

The impossibility of the inequality $v^{'}(\alpha) > 0$ is almost obvious.

If

$$v^{'}(\alpha) = 0,\ v^{''}(\alpha) > u^{''}(\alpha).$$

, then, $v^{'}(\xi_4) > u^{'}(\xi_4)$, consequently, by virtue of the condition v $(\xi_5) < 0$, the difference $u(x) - v(x)$ has a zero $\xi \in (\xi_4,\ \xi_5)$, which is impossible, because the solution of the form

$$y(x) = u(x) - v(x)$$

has six zeros $\xi_0,\ \xi_1,\ \xi_2,\ \xi_3,\ \xi_4,\ \ \xi$, in the interval $[s,\ r_{111111}(s))$.

Similarly, contradiction can be obtained in case of the following condition_

$$v(\alpha) = v^{'}(\alpha) = v^{''}(\alpha) = v^{'''}(\alpha) = v^{(IV)}(\alpha) = 0.$$

Finally, if $v^{'}(\alpha) = 0$ and $v^{''}(\alpha) < u^{''}(\alpha)$, then, by virtue of Lemma (2.2) the linear combination $u(x) - v(x)$ (at some constant c) having three-multiple zero at the point ξ_0 and zeros ‘ξ_1, ‘ξ_2, ‘ξ_3, which are the shifted positions of the points ξ_1, ξ_2, ξ_3, for $c \to \frac{u^{''}(\xi_0)}{v^{''}(\xi_0)}$. This contradicts Theorem (3.1). The theorem is proved. □

Theorem 3.3. $r_{2121}(s) \geq r_{111111}(s)$.

Proof. Assume that

$$r_{2121}(s) < r_{111111}(s),$$

then, there is a pair of non-trivial solutions $u(x)$ and $v(x)$ of equation (2.1) such that

$$u(a_1) = u^{'}(a_1) = u(a_2) = u(a_3) = u^{'}(a_3) = u(a_4) = 0,$$

and

$$u^{(k)}(a_1) = 0, \quad k = 0,\ 1,\ 2, \quad \mathrm{sgn}\ v^{'''}(a_1) = \mathrm{sgn}\ u^{''}(a_3), \quad v(a_4) > 0,$$

where s $< a_1 < a_2 < a_3 < r_{2121}(s) < a_4 < r_{111111}(s)$.

Owing to Lemma (2.7) and Theorem (3.1),

$$u^{''}(a_3) \neq 0, \quad sgnu^{''}(a_1) = -sgnu^{''}(a_3), u^{'}(a_2)u^{'}(a_4) < 0$$

which means that $u^{'}(a_2) = -sgnu^{'}(a_4)$, $v(x) \neq 0$ in the interval $(a_1, a_4+\varepsilon)$.

It is easy to check that the difference

$$u(x) - \frac{u(\xi)}{v(\xi)} v(x), \quad (a_3 < \xi < a_4),$$

has six zeroes in the interval $[s, r_{111111}(s))$, which is impossible. This contradiction proves the theorem, $r_{2121}(s) \geq r_{111111}(s)$. □

Theorem 3.4. $r_{21111}(s) \geq r_{111111}(s)$.

Proof. We obtain the validity of the assertion of the Theorem, using Lemma (2.3) and Theorem (3.3). Indeed, if one assumes that

$$r_{21111}(s) < r_{111111}(s),$$

then some solution $u(x)$ of the equation (2.1) has in the interval $[s, r_{111111}(s))$ five consecutive zeros a_1, a_2, a_3, a_4, a_5, the first of which double zero, such that

$$u(a_1) = u^{'}(a_1) = u(a_2) = u(a_3) = u(a_4) = u(a_5) = 0,$$

note that $u^{'}(a_5) > 0$, means that sgn $u^{'}(a_2) = -$sgn $u^{'}(a_5)$.

If $v(x)$ is a solution of equation (2.1) such that
$v^{(k)}(a_1) = 0, \quad k = 0, 1, 2, \; u(\xi_i) = v(\xi_i), \; i = 1, 2, 3,$
where

$$s < a_1 = \xi_1 < \xi_2 < a_2 < a_3 < \xi_3 < a_4 < r_{211111}(s) < a_5 < r_{111111}(s),$$

$$\text{sgn } v^{'''}(a_1) = sgn\ u^{''}(a_1), v^{'}(\xi_2) < u^{'}(\xi_2) \ and \ v^{'}(\xi_3) > u^{'}(\xi_3),$$

then, by virtue of Lemma (2.1) there exists a linear combination $z(x) = c_1 u(x) + c_2 v(x)$, ($c_1^2 + c_2^2 > 0$) that has zeros at the points with abscissas ‘ξ_2, ‘ξ_3, where ‘ξ_2, ‘ξ_3 are the shifted positions of the points ξ_2, ξ_3, and a double zero at the point η, (*where* $\eta \in (a_2, a_3)$) and also has a double zero at the point a_1. This means that

$$z(a_1) = z^{'}(a_1) = z(‘\xi_2) = z(\eta) = z^{'}(\eta) = z(‘\xi_3) = 0.$$

Then, we have arrived to a contradiction with theorem (3.3) $\ll$(2121 − problem) $\gg$.

This contradiction proves the theorem of $r_{21111}(s) \geq r_{111111}(s)$. In the same way, we could prove the remaining formulates, thereby, we extracted the following results
$r_{111111}(s) = min[\, r_{51}(s), r_{42}(s), r_{33}(s), r_{24}(s), r_{15}(s), r_{411}(s), r_{312}(s),$
$r_{321}(s), r_{231}(s), r_{213}(s), r_{141}(s), r_{114}(s), r_{123}(s), r_{132}(s), r_{3111}(s), r_{2112}(s),$
$r_{2121}(s), r_{2211}(s), r_{1113}(s), r_{1131}(s), r_{1311}(s), r_{1221}(s), r_{1212}(s), r_{1122}(s),$
$r_{21111}(s), r_{12111}(s), r_{11211}(s), r_{11121}(s), r_{11112}(s)]$.

□

4. Conclusion

This study is an investigation of the distribution of zeros of non-trivial solutions of a linear homogeneous differential equation of sixth order in terms of semi-critical intervals of boundary value problems. It also includes the description of the behavior trend of the estimated intervals of uniqueness of the solutions. Basically, we have obtained new results (Theorems 3.1, 3.2, 3.3, 3.4). Using these theorems, we have established the limiting relations between the lengths of semi - critical intervals of the uniqueness of solutions of (two, three, four and five points) boundary value problems with fixed points and the description of their estimated behavior. Further, we proved that the interval [s, $r_{111111}(s)$) is the intersection of the following intervals

[s, $r_{51}(s)$),[s, $r_{312}(s)$), [s, $r_{2121}(s)$), [s, $r_{21111}(s)$).

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References

[1] A. M. Akhundov and A. T. Toraev, *On solutions of the differential equation*. Known. AN Turkm. SSR, ser. fizmatite. techn. chemical. and geol sciences, *1* (1964), 21-23.

[2] R. G. Aliev and S. h Gamidov, *Boundary value problems for ordinary differential equations*. Educational allowance, Makhachkala, *78*, 1992.

[3] R. G. Aliev*On the question of distribution of zeros of solutions of linear differential equations of fourth order*, Recnts.of Sci.Asp.Conf, Kazan Inst, 1962.

[4] R. G. Aliev,*On some properties of solutions of the fourth-order ordinary differential equation*, News of universities. Mathematics,*2*(45)(1965),3-6.

[5] S. A. S. Al-Joufi, *Limit relations between the semi-critical intervals solutions of multipoint boundary value problems ofor linear fifth-order differential equation*, Herald of Dagestan State University (DSU),*6*(2012), 61-66.

[6] S. A. S. Al-Joufi, *On the distribution of zeros of solutions and their first derivatives of the fifth order linear differential equation. V All-Russian scientific - practical conference*, "The scientific initiative of foreign students and graduate students of Russian universities" Collection of reports. (April 25-27). Tomsk,*1*(2012), 316-318.

[7] S. A. S. Al-Joufi,*On the relations between the intervals of uniqueness of solutions of boundary value problems for a fifth-order equation*, Actual problems of mathematics and related questions (materials of the International Conference "Mukhtar Readings", (DSTU) . Makhachkala, (2012), 19-23.

[8] S. A. S. Al-Joufi and A. H. Kathim,*The laws of the distribution of zeros of solutions of a linear differential equation of order 5*, Herald of Dagestan State University (DSU),*1*(2012), 79-86.

[9] S. A. S. Al-Joufi,*On the zeros of solutions of linear fifth-order differential equation*, Actual problems of mathematics and related questions (materials of the International Conference "Mukhtar Readings", (DSTU). Makhachkala, (2012),23-26.

[10] S. A. S. Al-Joufi, *On the laws of the distribution of zeros of non-trivial solutions of a linear differential equation in the terminology of conjugate simi-critical intervals*. Functional differential equations and their applications (materials of the VI International Scientific Conference, dedicated to the 80th anniversary of the DSU).(September 23-26) Makhachkala, (2013), 41-47.

[11] S. A. S. Al-Joufi, *Method transfer to the limit, for resolve the problems of non- Vallée-Poussin type*, Actual problems of mathematics and related questions; (DSTU). Makhachkala, (2017), 11-19.

[12] N. V. Azbelev and B. Tsalyuk, *On the problem of the distribution of the zeros of solutions of a fourth-order linear differential equation*, Mateem. Sat, *51*(4)(1960), 475-486.

[13] A. F. Baker and A. Hassan El-Morshedy, *The distribution of zeros of all solutions of first order neutral differential equations*, Applied Mathematics and Computation, New York,USA, *259*(2015), 777-789.

[14] A. Domoshnitsky, M . DrakhlinI, and P. Stavroulakis, *Distribution of zeros of solutions to functional equations* Math. Comput. Modelling, *42*(2005), 193-205.

[15] L. Hui, H. Zhenlai,and S. Shurong, *The distribution of zeros of oscillatory solutions for second order nonlinear neutral delay differential equations*, Applied Mathematics Letters, *63*(2017),14-20.
[16] I. N. Inozemtseva, *On the distribution of zeros of solutions of a linear differential equation of order n*, Pro.Tikhnich Math. Tambov, *2*(1968), 27-29.
[17] A. H. Kathim, *The Distribution of the Solution Zeros for the Boundary Problems of Four Points of Linear Homogeneous Differential Equations of the Sixth Order*, British Journal of Mathematics & Computer Science, *18*(6)(2016), 1-13.
[18] A. H. Kathim and W. M. Taha,*The distribution of zeros of the solutions of linear homogeneous differential equations of the sixth order using semi-critical intervals*, Tikrit Journal of Pure Science, *21*(3)(2016), 162-165.
[19] A. K. Kondratiev, *On oscillation of the solutions of the equation*$y^{(n)} + p(x)y = 0$, Proceedings of Moscow. Math. ob-va, *10*(1961), 419-436.
[20] A. U. Levin, *On the distribution of zeros of solutions of a linear differential equation*, DAN, *156*(6)(1964), 1281-1284.
[21] F. X. Liang, *The distribution of zeros of solutions of first-order delay differential equations*,Journal of Mathematical Analysis and Applications, *186*(2)(1994), 383-392.
[22] I. Mikusinski, *Sur lequation* $X^{(n)} + A(t)X = 0$, Ann.Polon.Math *1*(26)(1955), 207-221.
[23] A. C. Peterson, *A theorem of Aliev*, Proc.Amer.Math, *23*(2)(1969), 364-366.
[24] H. S. Saker and A. M. Arahet, *Distributions of Zeros of Solutions for Third Order Differential Equations with Variable Coefficients*, Mathematical Problems in Engineering, (2015), 1-9.
[25] W. Shan and Z. Niu, Y. Guo, *The distribution of zeros of solutions for a class of neutral delay differential equations*, Science Direct Applied Mathematics and Computation, *186*(2007), 1137-1150.
[26] H. Wu, L. Erbe, *On the distance between consecutive zeros of solutions of first order delay differential equations*, Appl.Math.Comput *219*(2013), 8622-8631.
[27] B. G. Zhang and Y. Zhou, *The distribution of zeros of solutions of differential equations with a variable delay*, J. Math. Anal. Appl, *256*(1)(2001), 216-228.