Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 8, No. 3, 2020, pp. 537-552
DOI:10.22034/cmde.2020.29576.1424

Analytical approximations of one-dimensional hyperbolic equation with non-local integral conditions by reduced differential transform method

Seyyedeh Roodabeh Moosavi Noori
Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Guilan, P.O.Box 1914, Rasht, Iran.
E-mail: moosavinoori@gmail.com

Nasir Taghizadeh
Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Guilan, P.O.Box 1914, Rasht, Iran.
E-mail: taghizadeh@guilan.ac.ir

Jalil Manafian*
Department of Applied Mathematics,
Faculty of Mathematical Sciences,
University of Tabriz, Tabriz, Iran.
E-mail: j_manafianheris@tabrizu.ac.ir

Abstract In this work, an initial-boundary value problem with a non-classic condition for the one-dimensional wave equation is presented and the reduced differential transform method is applied to ascertain the solution of the problem. We will investigate a new kind of non-local boundary value equations in which are the solution of hyperbolic partial differential equations with a non-standard boundary characteristic. The advantage of this method is its simplicity in using, it solves the problem directly and straightforward without using perturbation, linearization, Adomian's polynomial or any other transformation and gives the solution in the form of convergent power series with simply determinable components. Also, the convergence of the method is proved and seven examples are tested to shows the competency of our study.

Keywords. Reduced differential transform method, Non-classic condition, Hyperbolic partial differential equation, Approximate solutions.

2010 Mathematics Subject Classification. 35K15, 35C05, 35E15.

1. Introduction

In the current study, we will examine the hyperbolic equation with a non-local constraint in the following boundary condition in which have been investigated in [21, 24]:

$$\theta_{tt} - \mu(x,t)\theta_{xx} = \lambda(x,t), \qquad 0 < x < l, \qquad 0 < t \leq T, \tag{1.1}$$

Received: 28 September 2018 ; Accepted: 12 May 2019.
$*$ corresponding.

by considering the following initial conditions

$$\theta(x,0) = \alpha(x), \qquad \theta_t(x,0) = \beta(x), \qquad 0 \le x \le l, \tag{1.2}$$

and Dirichlet boundary condition as bellows

$$\theta(0,t) = \gamma(t), \tag{1.3}$$

together with the nonlocal condition

$$\int_0^l \theta(x,t)dx = \chi(t), \qquad 0 < t \le T, \tag{1.4}$$

where λ, α, β, γ, and χ are known functions.
It is worth mentioning that α and β satisfy the compatibility conditions as below:

$$\alpha(0) = \gamma(0), \quad \beta(0) = \gamma'(0), \quad \int_0^l \alpha(x)dx = \chi(0), \quad \int_0^l \beta(x)dx = \chi'(0). \tag{1.5}$$

Keskin [17] introduced a reliable and effective method called the reduced differential transform method (RDTM) to look for exact solutions of partial differential equations. Keskin and Oturanc [18, 19, 20] improved the reduced differential transform method (RDTM) and showed that the RDTM procedure is very simple to achieve the exact solutions for a more class of the linear and nonlinear differential equations. This suggested technique is very effective and powerful in obtaining the approximate solutions also analytical solutions of many physical equations arising in applied sciences and engineering. In recent years, this effective method is widely used by many such as in [1, 23, 26, 28, 29, 31, 32] and by the references therein.
Various powerful numerical and semi-analytical methods for solving linear and nonlinear partial differential equations (PDEs) have proposed, some these methods which solve PDEs are including: the Adomian decomposition method [33], the homotopy analysis method [7, 8, 13, 14, 27], the homotopy perturbation method [9], the variational iteration method [10, 11, 12, 15], the Laplace Adomian decomposition method [22], the optimal homotopy and differential transform methods [25], the semi-analytical iterative technique [34], Exp-function method [5, 6, 36], the sine-Gordon expansion method [2, 3, 4, 16, 30, 35] and so on.
The important goal of this work is to utilize the reduced differential transform method (RDTM) to deal with the one-dimensional hyperbolic equation with integral conditions.
The rest of this study is presented in the bellow sections: In Section 2, we simply introduce the reduced differential transform method. In Section 3, we discuss the convergence of the considered method. In Section 4, the method is used to deal with the one-dimensional hyperbolic equation with non-local integral conditions. Finally, we offer some summaries and conclusions in Section 5.

2. Summary of the Method

We present some important definitions and operations of the reduced differential transform method in which can help to more understand of the stated method in this section. Now, assume that the function of two variables $\theta(x,t)$ will be described as a product of two different variable functions, i.e., $\theta(x,t) = \phi(x)\psi(t)$. The function $\theta(x,t)$ can be displayed due to the properties of the differential transform as follows:

$$\theta(x,t) = \left(\sum_{i=0}^{\infty} \Phi(i)x^i\right)\left(\sum_{j=0}^{\infty} \Psi(j)t^j\right) = \sum_{k=0}^{\infty} \Theta_k(x)t^k, \tag{2.1}$$

where $\Theta_k(x)$ is the converted function of the source function $\theta(x,t)$.

Definition 2.1. Let $\theta(x,t)$ is analytic and differentiated continuously function with regard to space x and time t, in the domain of interest, then the t-dimensional spectrum function is

$$\Theta_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\theta(x,t)\right]_{t=t_0}. \tag{2.2}$$

Definition 2.2. The reduced differential inverse transform of $\Theta_k(x)$ is determined as

$$\theta(x,t) = \sum_{k=0}^{\infty} \Theta_k(x)(t-t_0)^k. \tag{2.3}$$

Then, consolidating Eqs. (2.2) and (2.3) yields

$$\theta(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\theta(x,t)\right]_{t=t_0}(t-t_0)^k, \tag{2.4}$$

the help of the upper definitions, it will be discover that the conception of the RDTM comes from the power series expansion. As an example, the basic ideas of the RDTM, suppose that we have a nonlinear partial differential equation written in an operator form

$$L\theta(x,t) + R\theta(x,t) + N\theta(x,t) = \omega(x,t), \tag{2.5}$$

with considering the following initial condition

$$\theta(x,0) = \alpha(x), \tag{2.6}$$

where $L = \frac{\partial}{\partial t}$, R is a linear operator which has partial derivatives, N is a nonlinear operator and $\omega(x,t)$ is an inhomogeneous term.

After applying the RDTM definition, the following iteration formula can be stated as

$$(k+1)\Theta_{k+1}(x) = \Omega_k(x) - R\Theta_k(x) - N\Theta_k(x), \tag{2.7}$$

where $L\Theta_k(x)$, $R\Theta_k(x)$, $N\Theta_k(x)$ and $\Omega_k(x)$ are the reduced differential transform functions of $L\theta(x,t)$, $R\theta(x,t)$, $N\theta(x,t)$ and $\omega(x,t)$ respectively.

From initial condition (2.6), we have

$$\Theta_0(x) = \alpha(x). \tag{2.8}$$

To discover the remaining iteration, we plugging (2.8) into (2.7) and by simple reiterative calculation, we tend to get the subsequent $\Theta_k(x)$ values. Afterward, the inverse transformation of the set of values $\{\Theta_k(x)\}_{k=0}^n$ admits the n-terms approximation solution as bellows:

$$\tilde{\theta}_n(x,t) = \sum_{k=0}^{n} \Theta_k(x)(t-t_0)^k. \tag{2.9}$$

Thus, the final solution of the considered problem can be gained by

$$\theta(x,t) = \lim_{n\to\infty} \tilde{\theta}_n(x,t). \tag{2.10}$$

Table I contains the basic mathematical operations carried out by RDTM.

3. Convergence of method

The principal main of this section is to survey the convergence of the reduced differential transform method, according to other methods of RDTM stated in the previous section, when employed to Eq. (2.5). The sufficient conditions for convergence of the method and the error computation are addressed.
The fundamental point views of RDTM includes of ascertaining power series expansion for the solutions of nonlinear models with the initial time t_0,

$$\theta(x,t) = \sum_{k=0}^{\infty} a_k(x)(t-t_0)^k, \qquad t \in l, \tag{3.1}$$

where $l = (t_0, t_0+r)$, $r > 0$. The important results are proposed in the below theorems.

Theorem 3.1. *Suppose $\varphi_k(x,t) = a_k(x)(t-t_0)^k$, then the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$, stated in Eq (3.1), converges if $\exists\ 0 < \gamma < 1$ such that $\|\varphi_{k+1}\| \le \gamma\|\varphi_k\|$, $\forall k \in \mathbf{N} \cup \{0\}$.*

Theorem 3.1 is a specific case of Banach's fixed point theorem. We summarize the proof of Theorem 3.1 to investigate the truncation error of the series solution Eq. (3.1), as follows

Proof. Denote as $(C[l], \|.\|)$ the Banach space of all continuous functions on l with the norm $\|\varphi_k(x,t)\| = \|a_k(x)(t-t_0)^k\|$. Also assume that $\|a_0(x)\| < N_0$, where N_0 is a positive number. Define the sequence of partial sums $\{\Sigma_n\}_{n=0}^{\infty}$ as

$$\Sigma_n = \varphi_0 + \varphi_1 + \ldots + \varphi_n. \tag{3.2}$$

We want to present that $\{\Sigma_n\}_{n=0}^{\infty}$ is a Cauchy sequence in this Banach space. To reach this goal, we take

$$\|\Sigma_{n+1} - \Sigma_n\| = \|\varphi_{n+1}\| \le \gamma\|\varphi_n\| \le \ldots \le \gamma^{n+1}\|\varphi_0\| \le \gamma^{n+1} N_0. \tag{3.3}$$

For every $n, m \in \mathbf{N}$, $n \geq m$, we get

$$\begin{aligned}\|\Sigma_n - \Sigma_m\| &= \|(\Sigma_n - \Sigma_{n-1}) + (\Sigma_{n-1} - \Sigma_{n-2}) + \ldots + (\Sigma_{m+1} - \Sigma_m)\| \\ &\leq \|(\Sigma_n - \Sigma_{n-1})\| + \|(\Sigma_{n-1} - \Sigma_{n-2})\| + \ldots + \|(\Sigma_{m+1} - \Sigma_m)\| \\ &\leq \frac{1-\gamma^{n-m}}{1-\gamma}\gamma^{m+1}\|\varphi_0\|, \end{aligned} \tag{3.4}$$

and because $0 < \gamma < 1$, we achieve

$$\lim_{n,m\to\infty} \|\Sigma_n - \Sigma_m\| = 0. \tag{3.5}$$

Therefore, $\{\Sigma_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $(C[l], \|.\|)$. Afterward the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$, defined in Eq. (3.1), converges and it completes the proof.

□

If the series $\sum_{k=0}^{\infty} a_k(x)(t-t_0)^k$ converges then it is an exact solution of the nonlinear equation (2.5).

Theorem 3.2. *Suppose that the series solution $\sum_{k=0}^{\infty} \varphi_k(x,t)$, where $\varphi_k(x,t) = a_k(x)(t-t_0)^k$, converges to the solution $\theta(x,t)$. If the truncated series $\sum_{k=0}^{m} \varphi_k(x,t)$ is used as an approximation to the solution $\theta(x,t)$ and then the maximum absolute truncated error is computed as*

$$\left\|\theta(x,t) - \sum_{k=0}^{m} \varphi_k(x,t)\right\| \leq \frac{1}{1-\gamma}\gamma^{m+1}\|\varphi_0\|. \tag{3.6}$$

Proof. From Theorem 3.1, following inequality Eq. (3.4), we have

$$\|\Sigma_n - \Sigma_m\| \leq \frac{1-\gamma^{n-m}}{1-\gamma}\gamma^{m+1}\|\varphi_0\|, \tag{3.7}$$

for $n \geq m$. Also, since $0 < \gamma < 1$, we have $1 - \gamma^{n-m} < 1$, therefore, the inequality Eq.(3.7) can be changed to

$$\|\Sigma_n - \Sigma_m\| \leq \frac{1}{1-\gamma}\gamma^{m+1}\|\varphi_0\|. \tag{3.8}$$

It is clear when $n \to \infty$, $\Sigma_n \to \theta(x,t)$. Thus, inequality Eq. (3.6) is obtained and the Theorem is proved.

□

4. Applications

Here, seven examples are given to show the efficiency of the reduced differential transform method for solving equation (1.1) with the following conditions

Example 1: We first consider (1.1)-(1.4) with $T = 1$, $l = 1$, and

$$\mu(x,t) = 1, \qquad \lambda(x,t) = 0, \qquad \alpha(x) = x^2, \qquad \beta(x) = 0,$$
$$\gamma(t) = t^2, \qquad \chi(t) = t^2 + \frac{1}{3}. \tag{4.1}$$

According to the operations of differential transformation given in Table I for Eq. (1.1) we obtain the recurrent relation as bellows

$$(k+1)(k+2)\Theta_{k+2}(x) = \frac{\partial^2}{\partial x^2}\Theta_k(x). \tag{4.2}$$

Taking the initial conditions (4.1), we get

$$\Theta_0(x) = x^2, \qquad \Theta_1(x) = 0. \tag{4.3}$$

Substituting the above equation in Eq. (4.2), we drive the following results

$$\Theta_2(x) = 1, \qquad \Theta_3(x) = 0, \qquad \Theta_4(x) = 0, \qquad \cdots .$$

Hence, the solution in series form is as follow

$$\tilde{\theta}_n(x,t) = \sum_{k=0}^{\infty}\Theta_k(x)t^k = \Theta_0(x) + \Theta_1(x)t + \Theta_2(x)t^2 + \Theta_3(x)t^3 + ... = x^2 + t^2,$$

which converges efficiently to the exact solution

$$\theta(x,t) = x^2 + t^2.$$

Example 2: We take (1.1)-(1.4) with $T = 0.5$, $l = 1$, and

$$\begin{gathered}\mu(x,t) = 1, \quad \lambda(x,t) = 0, \quad \alpha(x) = 0, \quad \beta(x) = \pi\cos\pi x, \\ \gamma(t) = \sin\pi t, \quad \chi(t) = 0.\end{gathered} \tag{4.4}$$

According to the operations of differential transformation given in Table I for Eq. (1.1) and from initial conditions (4.4), we get to the bellow relation

$$\Theta_0(x) = 0, \qquad \Theta_1 = \pi\cos\pi x. \tag{4.5}$$

Plugging the above equation in Eq. (4.2), we drive the following results

$$\Theta_2(x) = 0, \quad \Theta_3(x) = -\frac{\pi^3}{3!}\cos\pi x, \quad \Theta_4(x) = 0, \quad \Theta_5(x) = \frac{\pi^5}{5!}\cos\pi x, \quad \cdots .$$

Hence, the solution in series form is as follow

$$\begin{aligned}\tilde{\theta}_n(x,t) &= \sum_{k=0}^{\infty}\Theta_k(x)t^k = \Theta_0(x) + \Theta_1(x)t + \Theta_2(x)t^2 + ... \\ &= \cos\pi x\Big(\pi t - \frac{(\pi t)^3}{3!} + \frac{(\pi t)^5}{5!} + ...\Big),\end{aligned}$$

which converges efficiently to the exact solution

$$\theta(x,t) = \cos\pi x\sin\pi t.$$

Example 3: Taking (1.1)-(1.4) with $T = 0.5$, $l = 1$, and

$$\begin{gathered}\mu(x,t) = 1, \quad \lambda(x,t) = 0, \quad \alpha(x) = \cos\pi x, \quad \beta(x) = 0, \\ \gamma(t) = 0, \quad \chi(t) = 0.\end{gathered} \tag{4.6}$$

According to the operations of differential transformation given in Table I for Eq. (1.1) and from initial conditions (4.6), reads as

$$\Theta_0(x) = \cos \pi x, \qquad \Theta_1(x) = 0. \tag{4.7}$$

Putting the above equation in Eq. (4.2), we drive the results as bellow

$$\Theta_2(x) = -\frac{\pi^2}{2!}\cos \pi x, \qquad \Theta_3(x) = 0, \qquad \Theta_4(x) = \frac{\pi^4}{4!}\cos \pi x, \qquad \cdots .$$

Hence, the solution in series form is as follow

$$\begin{aligned}\tilde{\theta}_n(x,t) &= \sum_{k=0}^{\infty} \Theta_k(x) t^k = \Theta_0(x) + \Theta_1(x)t + \Theta_2(x)t^2 + ... \\ &= \cos \pi x \Big(1 - \frac{(\pi t)^2}{2!} + \frac{(\pi t)^4}{4!} + ...\Big),\end{aligned}$$

which converges efficiently to the exact solution

$$\theta(x,t) = \cos \pi x \cos \pi t.$$

Example 4: We take (1.1)-(1.4) with $T = 1$, $l = 1$, and

$$\begin{gathered}\mu(x,t) = 1, \quad \lambda(x,t) = 0, \quad \alpha(x) = \sin \pi x, \quad \beta(x) = \pi \sin \pi x, \\ \gamma(t) = 0, \quad \chi(t) = \frac{2}{\pi}(\cos \pi t + \sin \pi t).\end{gathered} \tag{4.8}$$

According to the operations of differential transformation given in Table I for Eq. (1.1) and from initial conditions (4.8), will be as

$$\Theta_0(x) = \sin \pi x, \qquad \Theta_1(x) = \pi \sin \pi x. \tag{4.9}$$

Appending the above equation in Eq. (4.2), we drive the following results

$$\Theta_2(x) = -\frac{\pi^2}{2!}\sin \pi x, \quad \Theta_3(x) = -\frac{\pi^3}{3!}\sin \pi x, \quad \Theta_4(x) = \frac{\pi^4}{4!}\sin \pi x, \quad \cdots .$$

Hence, the solution in series form is as follow

$$\begin{aligned}\tilde{\theta}_n(x,t) &= \sum_{k=0}^{\infty} \Theta_k(x) t^k = \Theta_0(x) + \Theta_1(x)t + \Theta_2(x)t^2 + ... \\ &= \sin \pi x \Big(1 + \pi t - \frac{(\pi t)^2}{2!} - \frac{(\pi t)^3}{3!} + \frac{(\pi t)^4}{4!} + ...\Big),\end{aligned}$$

which converges efficiently to the exact solution

$$\theta(x,t) = \sin \pi x(\cos \pi t + \sin \pi t).$$

Example 5: We take (1.1)-(1.4) with $T = 1$, $l = 1$, and

$$\begin{gathered}\mu(x,t) = 1, \quad \lambda(x,t) = (x^2 - t^2)e^{xt}, \quad \alpha(x) = 1, \quad \beta(x) = x, \\ \gamma(t) = 1, \quad \chi(t) = \frac{e^t - 1}{t}.\end{gathered} \tag{4.10}$$

According to the operations of differential transformation given in Table I for Eq. (1.1) we gain the following recurrent relation

$$(k+1)(k+2)\Theta_{k+2}(x) = \frac{\partial^2}{\partial x^2}\Theta_k(x) + N_k(x), \tag{4.11}$$

and from initial conditions (4.10), we write

$$\Theta_0(x) = 1, \qquad \Theta_1(x) = x. \tag{4.12}$$

Substituting the above equation in Eq. (4.11), we drive the following results

$$\Theta_2(x) = \frac{x^2}{2!}, \quad \Theta_3(x) = \frac{x^3}{3!}, \quad \Theta_4(x) = \frac{x^4}{4!}, \quad \cdots .$$

Hence, the solution in series form is as follow

$$\begin{aligned}\tilde{\theta}_n(x,t) &= \sum_{k=0}^{\infty}\Theta_k(x)t^k = \Theta_0(x) + \Theta_1(x)t + \Theta_2(x)t^2 + ...\\ &= 1 + xt + \frac{(xt)^2}{2!} + \frac{(xt)^3}{3!} + ...,\end{aligned}$$

which converges efficiently to the exact solution

$$\theta(x,t) = e^{xt}.$$

Example 6: Taking (1.1)-(1.4) with $T = 1$, $l = 1$, and

$$\begin{gathered}\mu(x,t) = 1, \quad \lambda(x,t) = 2(x^2 - t^2), \quad \alpha(x) = \sinh x, \quad \beta(x) = \cosh x,\\ \gamma(t) = 0, \quad \chi(t) = \frac{t^2}{3} + \cosh(1+t).\end{gathered} \tag{4.13}$$

According to the operations of differential transformation given in Table I for Eq. (1.1) we ascertain the following recurrent relation

$$(k+1)(k+2)\Theta_{k+2}(x) = \frac{\partial^2}{\partial x^2}\Theta_k(x) + 2x^2\delta(k) - 2\delta(k-2), \tag{4.14}$$

and from initial conditions (4.13), we get

$$\Theta_0(x) = \sinh x, \qquad \Theta_1(x) = \cosh x. \tag{4.15}$$

Appending the above equation in Eq. (4.14), we drive the following results

$$\Theta_2(x) = x^2 + \frac{1}{2!}\sinh x, \quad \Theta_3(x) = \frac{1}{3!}\cosh x, \quad \Theta_4(x) = \frac{1}{4!}\sinh x, \quad \cdots .$$

Hence, the solution in series form is as follow

$$\begin{aligned}\tilde{\theta}_n(x,t) &= \sum_{k=0}^{\infty}\Theta_k(x)t^k = \Theta_0(x) + \Theta_1(x)t + \Theta_2(x)t^2 + ...\\ &= x^2t^2 + \sinh x\Big(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + ...\Big) + \cosh x\Big(t + \frac{t^3}{3!} + \frac{t^5}{5!} + ...\Big),\end{aligned}$$

which converges efficiently to the exact solution

$$\theta(x,t) = x^2t^2 + \sinh(x+t).$$

Example 7: We take (1.1)-(1.4) with $T=1$, $l=1$, and

$$\mu(x,t)=\frac{x^2}{2},\quad \lambda(x,t)=0,\quad \alpha(x)=x,\quad \beta(x)=x^2,$$
$$\gamma(t)=0,\quad \chi(t)=\frac{1}{2}+\frac{1}{3}\sinh t. \tag{4.16}$$

According to the operations of differential transformation given in Table I for Eq. (1.1) we achieve the recurrent relation as bellows

$$(k+1)(k+2)\Theta_{k+2}(x)=\frac{x^2}{2}\frac{\partial^2}{\partial x^2}\Theta_k(x), \tag{4.17}$$

and from initial conditions (4.16), reads as

$$\Theta_0(x)=x,\qquad \Theta_1(x)=x^2. \tag{4.18}$$

Substituting the above equation in Eq. (4.17), we drive the following results

$$\Theta_2(x)=0,\qquad \Theta_3(x)=\frac{x^2}{3!},\qquad \Theta_4(x)=0,\qquad \Theta_5(x)=\frac{x^2}{5!},\qquad \cdots.$$

Hereafter, the solution in series form is as follow

$$\begin{aligned}\tilde{\theta}_n(x,t)&=\sum_{k=0}^{\infty}\Theta_k(x)t^k=\Theta_0(x)+\Theta_1(x)t+\Theta_2(x)t^2+...\\&=x+x^2\Big(t+\frac{t^3}{3!}+\frac{t^5}{5!}+...\Big),\end{aligned}$$

which converges efficiently to the exact solution

$$\theta(x,t)=x+x^2\sinh t.$$

Comparing our consequence with the solutions ascertained in [21, 24] by HPM, we can see that the results are the same.

5. Conclusion

In this article, we successfully employed the reduced differential transform method to solve accurately the one-dimensional hyperbolic equation with integral conditions. The suggested technique, without involving the perturbation, linearization or discretization provides a solution in the form of convergent power series with daintily estimated components. RDTM can be applied the most of the biological, physical, engineering and other models as an alternative for obtaining reliable and fastest converge, useful approximations. On the other hand, RDTM is mighty of reducing the size of computational work compared to other traditional methods. The results illustrate that the RDTM is a very interesting and efficient semi-numerical-analytical method for successfully employed to achieve the exact and approximate solutions of the linear and nonlinear PDEs.

FIGURE 1. The 3D plot of equation (left) exact solution and (right) numerical solution to Example 1.

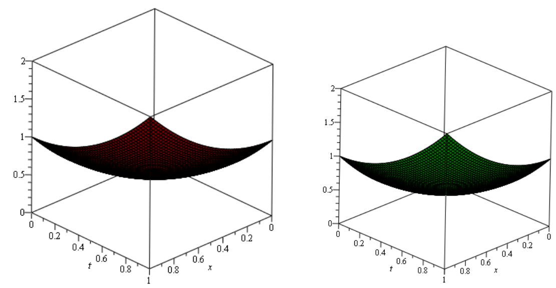

Table I. The fundamental operations of RDTM	
Original Form	**Transformed Form**
$\theta(x,t)$	$\Theta_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k}\theta(x,t)]_{t=0}$
$\theta(x,t) = \theta_1(x,t) \pm \theta_2(x,t)$	$\Theta_k(x) = \Theta_{1k}(x) \pm \Theta_{2k}(x)$
$\theta(x,t) = \lambda\theta_1(x,t)$	$\Theta_k(x) = \lambda\Theta_{1k}(x)$ (λ is a constant)
$\theta(x,t) = x^m t^n$	$\Theta_k(x) = x^m\delta(k-n), \qquad \delta(k) = \begin{cases} 1, & k=0 \\ 0, & k\neq 0 \end{cases}$
$\theta(x,t) = x^m t^n \theta_1(x,t)$	$\Theta_k(x) = x^m\Theta_{1,k-n}(x)$
$\theta(x,t) = \theta_1(x,t)\theta_2(x,t)$	$\Theta_k(x) = \sum_{r=0}^{k}\Theta_{2r}(x)\Theta_{1,k-r}(x) = \sum_{r=0}^{k}\Theta_{1r}(x)\Theta_{2,k-r}(x)$
$\theta(x,t) = \frac{\partial^r}{\partial t^r}\theta_1(x,t)$	$\Theta_k(x) = (k+1)\ldots(k+r)\Theta_{1,k+r}(x) = \frac{(k+r)!}{k!}\Theta_{1,k+r}(x)$
$\theta(x,t) = \frac{\partial}{\partial x}\theta_1(x,t)$	$\Theta_k(x) = \frac{\partial}{\partial x}\Theta_{1k}(x)$
$\theta(x,t) = e^{\lambda t}$	$\Theta_k(x) = \frac{\lambda^k}{k!}$
$\theta(x,t) = \sin(\omega t + \alpha x)$	$\Theta_k(x) = \frac{\omega^k}{k!}\sin(\frac{k\pi}{2} + \alpha x)$
$\theta(x,t) = \cos(\omega t + \alpha x)$	$\Theta_k(x) = \frac{\omega^k}{k!}\cos(\frac{k\pi}{2} + \alpha x)$

FIGURE 2. The 3D plot of equation (left) exact solution and (right) numerical solution to Example 2.

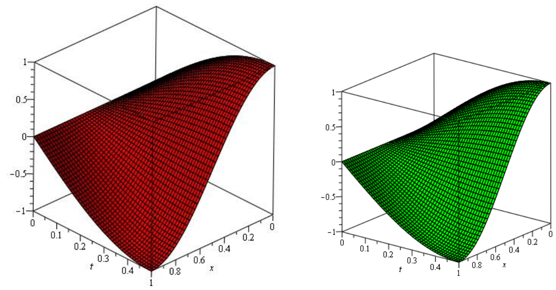

FIGURE 3. The 3D plot of equation (left) exact solution and (right) numerical solution to Example 3.

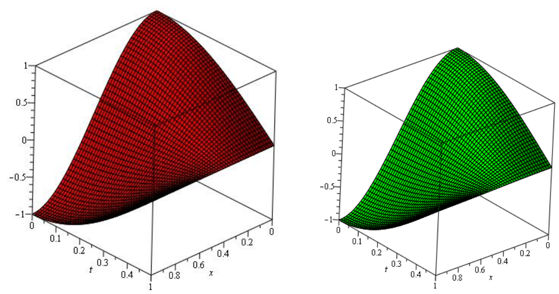

FIGURE 4. The 3D plot of equation (left) exact solution and (right) numerical solution to Example 4.

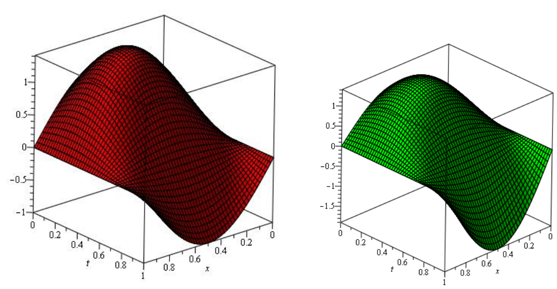

FIGURE 5. The 3D plot of equation (left) exact solution and (right) numerical solution to Example 5.

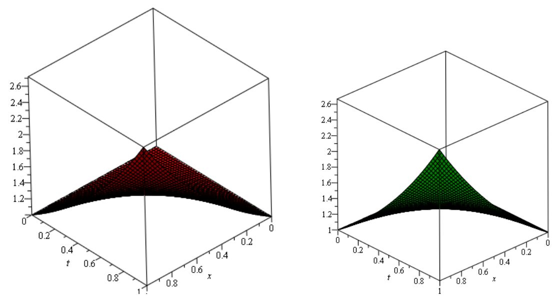

FIGURE 6. The 3D plot of equation (left) exact solution and (right) numerical solution to Example 6.

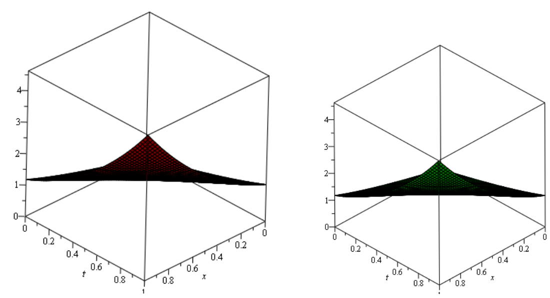

FIGURE 7. The 3D plot of equation (left) exact solution and (right) numerical solution to Example 7.

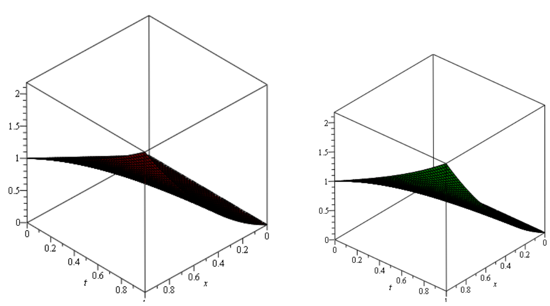

Table 1. The absolute error, $(n = 1)$, between the exact and the numerical solutions for example 1.

(x, t)	e_{RDTM}	e_{HPM}
(0.5,0.1)	0	0
(0.5,0.2)	0	0
(0.5,0.3)	0	0
(0.5,0.4)	0	0
(0.5,0.5)	0	0

Table 2. The absolute error, $(n = 3)$, between the exact and the numerical solutions for example 2.

(x, t)	e_{RDTM}	e_{HPM}
(0.3,0.1)	3.5127×10^{-8}	2.0071×10^{-7}
(0.3,0.2)	4.4839×10^{-6}	5.2822×10^{-6}
(0.3,0.3)	7.6092×10^{-5}	2.5487×10^{-4}
(0.3,0.4)	5.6463×10^{-4}	9.0886×10^{-4}
(0.3,0.5)	2.6596×10^{-3}	7.2582×10^{-3}

Table 3. The absolute error, $(n = 3)$, between the exact and the numerical solutions for example 3.

(x, t)	e_{RDTM}	e_{HPM}
(0.3,0.1)	7.8344×10^{-7}	9.7571×10^{-7}
(0.3,0.2)	4.9877×10^{-5}	5.2822×10^{-5}
(0.3,0.3)	5.6316×10^{-4}	6.5087×10^{-4}
(0.3,0.4)	3.1256×10^{-3}	6.0886×10^{-3}
(0.3,0.5)	1.1737×10^{-2}	3.2782×10^{-2}

Table 4. The absolute error, $(n = 3)$, between the exact and the numerical solutions for example 4.

(x, t)	e_{RDTM}	e_{HPM}
(0.3,0.1)	1.7772×10^{-8}	0.7572×10^{-7}
(0.3,0.2)	2.1721×10^{-6}	6.3422×10^{-6}
(0.3,0.3)	3.5279×10^{-5}	2.5087×10^{-5}
(0.3,0.4)	2.5001×10^{-4}	4.5354×10^{-4}
(0.3,0.5)	1.1218×10^{-3}	3.2782×10^{-3}

TABLE 5. The absolute error, $(n = 3)$, between the exact and the numerical solutions for example 5.

(x, t)	e_{RDTM}	e_{HPM}
(0.1,0.1)	3.3333×10^{-10}	2.7512×10^{-9}
(0.1,0.2)	6.6666×10^{-9}	8.3402×10^{-9}
(0.1,0.3)	3.3999×10^{-8}	8.5087×10^{-8}
(0.1,0.4)	1.0733×10^{-7}	4.5354×10^{-6}
(0.1,0.5)	2.6266×10^{-7}	5.2782×10^{-6}

TABLE 6. The absolute error, $(n = 4)$, between the exact and the numerical solutions for example 6.

(x, t)	e_{RDTM}	e_{HPM}
(1,0.1)	5.2657×10^{-10}	2.1412×10^{-9}
(1,0.2)	7.4762×10^{-10}	4.3472×10^{-9}
(1,0.3)	1.2994×10^{-9}	8.5687×10^{-8}
(1,0.4)	$.9382\times10^{-8}$	4.5354×10^{-7}
(1,0.5)	1.2192×10^{-7}	4.2799×10^{-6}

TABLE 7. The absolute error, $(n = 4)$, between the exact and the numerical solutions for example 7.

(x, t)	e_{RDTM}	e_{HPM}
(1,0.1)	1.9874×10^{-11}	3.1412×10^{-10}
(1,0.2)	4.0349×10^{-11}	4.3472×10^{-10}
(1,0.3)	6.1429×10^{-12}	8.5687×10^{-10}
(1,0.4)	7.1730×10^{-10}	4.2354×10^{-9}
(1,0.5)	5.3941×10^{-9}	4.9799×10^{-8}

References

[1] M. O. Al-Amr, *New applications of reduced differential transform method*, Alexandria Engineering Journal, *53* (2014), 243–247.

[2] H. M. Baskonus, *New acoustic wave behaviors to the Davey-Stewartson equation with power-law nonlinearity arising in fluid dynamics*, Nonlinear Dynamics, *86*(1) (2016), 177–183.

[3] H. M. Baskonus, *New Complex and Hyperbolic Function Solutions to the Generalized Double Combined Sinh-Cosh-Gordon Equation*, AIP Conf. Proc., *1798* (2017), 1–9.

[4] C. Cattani, T. A. Sulaiman, H. M. Baskonus, and H. Bulut, *On the soliton solutions to the Nizhnik-Novikov-Veselov and the Drinfel'd-Sokolov systems*, Optical and Quantum Electronics, *50*(3) (2018), 138.

[5] C. Cattani, T. A. Sulaiman, H. M. Baskonus, and H. Bulut, *Solitons in an inhomogeneous Murnaghan's rod*, European Physical Journal Plus, *133*(228) (2018), 1–12.

[6] A. Ciancio, H. M. Baskonus, T. A. Sulaiman, and H. Bulut, *New Structural Dynamics of Isolated Waves Via the Coupled Nonlinear Maccari's System with Complex Structure*, Indian Journal of Physics, *92*(10) (2018), 1281–1290.

[7] M. Dehghan, J. Manafian, and A. Saadatmandi, *Solving nonlinear fractional partial differential equations using the homotopy analysis method*, Numerical Methods for Partial Differential Equations, *26*(2) (2010), 448–479.

[8] M. Dehghan, J. Manafian, and A. Saadatmandi, *The solution of the linear fractional partial differential equations using the homotopy analysis method*, Zeitschrift für Naturforschung A, *65a* (2010), 935–949.

[9] M. Dehghan and J. Manafian, *The solution of the variable coefficients fourth order parabolic partial differential equations by homotopy perturbation method*, Zeitschrift für Naturforschung A, *64*(7-8) (2009), 420–430.

[10] M. Dehghan, J. Manafian, and A. Saadatmandi, *Application of semi-analytic methods for the Fitzhugh-Nagumo equation, which models the transmission of nerve impulses*, Mathematical Methods in the Applied Sciences, *33*(11) (2010), 1384–1398.

[11] M. Dehghan, J. Manafian, and A. Saadatmandi, *Application of semi-analytical methods for solving the Rosenau-Hyman equation arising in the pattern formation in liquid drops*, International Journal of Numerical Methods for Heat and Fluid Flow, *22*(6) (2012), 777–790.

[12] M. Dehghan and J. Manafian, *Study of the wave-breaking's qualitative behavior of the Fornberg-Whitham equation via quasi-numeric approaches*, International Journal of Numerical Methods for Heat and Fluid Flow, *21* (2011), 736–753.

[13] T. Hayat, M. Nawaz, and S. Obaidat, *Axisymmetric magnetohydrodynamic flow of micropolar fluid between unsteady stretching surfaces*, Applied Mathematics and Mechanics, *32*(3) (2011), 361–374.

[14] T. Hayat, A. Shafiq, M. Nawaz, and A. Alsaedi, *MHD axisymmetric flow of third grade fluid between porous disks with heat transfer*, Applied Mathematics and Mechanics, *33*(6) (2012), 749–764.

[15] J. H. He, *Variational approach to the sixth-order boundary value problems*, Applied Mathematics and Computation, *143* (2003), 537–538.

[16] O. A. Ilhan, T. A. Sulaiman, H. Bulut, and H. M. Baskonus, *On the New Wave Solutions to a Nonlinear Model Arising in Plasma Physics*, European Physical Journal Plus, *133*(27) (2018), 1–6.

[17] Y. Keskin, Ph.D Thesis, Selcuk University, in Turkish, 2010.

[18] Y. Keskin and G. Oturanc, *Reduced differential transform method for partial differential equations*, Int. J. Nonlin. Sci. Num., *10*(6) (2009), 741–749.

[19] Y. Keskin and G. Oturanc, *The reduced differential transform method: a new approach to fractional partial differential equations*, Nonlin. Sci. Lett. A., *1*(2) (2010), 207–217.

[20] Y. Keskin and G. Oturanc, *Application of reduced differential transformation method for solving Gas Dynamics equation*, Int. J. Contemp. Math. Sci., *22*(22) (2010), 1091–1096.

[21] X. Li, *Numerical solution of an initial-boundary value problem with nonlocal condition for the wave equation*, Mathematical Sciences, *2*(3) (2008), 281–292.

[22] J. Manafian, *Solving the integro-differential equations using the modified Laplace Adomian decomposition method*, Journal Mathematics of Extension, *6*(1) (2012), 65–79.

[23] M. S. Mohamed and K. A. Gepreel, *Reduced differential transform method for nonlinear integral member of Kadomtsev-Petviashvili hierarchy differential equations*, Journal of the Egyptian Mathematical Society, *25*(1) (2017), 1–7.

[24] S. T. Mohyud-Din, A. Yildirim, and Y. Kaplan, *Homotopy Perturbation Method for One-Dimensional Hyperbolic Equation with Integral Conditions*, Zeitschrift für Naturforschung A, *65a* (2010), 1077–1080.

[25] B. Parsa, M. M. Rashidi, O. Anwar Bég, and S. M. Sadri, *Semi-computational simulation of magneto-Hemodynamic flow in a semi-porous channel using optimal homotopy and differential transform methods*, Computers in Biology and Medicine, *43*(9) (2013), 1142–1153.

[26] R. K. Saeed and A. A. Mustafa, *Numerical solution of FisherKPP equation by using reduced differential transform method*, AIP Publishing, *1888* (2017), .

[27] M. Sajid and T. Hayat, *The application of homotopy analysis method to thin film flows of a third order fluid*, Chaos Solitons and Fractal, *38*(2) (2008), 506–515.

[28] V. K. Srivastava, M. K. Awasthi, and R. K. Chaurasia, *Reduced differential transform method to solve two and three dimensional second order hyperbolic telegraph equations*, Journal of King Saud University Engineering Sciences, *29*(2) (2017), 166–171.

[29] V. K. Srivastava, M. K. Awasthi, and S. Kumar, *Analytical approximations of two and three dimensional time-fractional telegraphic equation by reduced differential transform method*, Egyptian Journal of Basic and Applied Sciences, *1*(1) (2014), 60–66.

[30] T. A. Sulaiman, T. Akturk, H. Bulut, and H. M. Baskonus, *Investigation of various soliton solutions to the Heisenberg ferromagnetic spin chain equation*, Journal of Electromagnetic Waves and Applications, *32*(9) (2018), 1093–1105.

[31] N. Taghizadeh and S. R. Moosavi Noori, *Reduced differential transform method for solving parabolic-like and hyperbolic-like equations*, SeMA J., *74*(4) (2017), 559–567.

[32] N. Taghizadeh and S. R. Moosavi Noori, *Exact Solutions of the Cubic Nonlinear Schrodinger Equation with a Trapping Potential by Reduced Differential Transform Method*, Math. Sci. Lett., *5*(3) (2016), 1–5.

[33] M. Tatari and M. Dehghan, *The use of the Adomian decomposition method for solving multipoint boundary value problems*, Physics Scripta, *73* (2006), 672–676.

[34] H. Temimi and A. Ansari, *A semi analytical iterative technique for solving nonlinear problems*, Journal of Computational and Applied Mathematics, *61*(2) (2011), 203–210.

[35] G. Yel, H. M. Baskonus, and H. Bulut, *Novel archetypes of new coupled Konno-Oono equation by using sine-Gordon expansion method*, Optical and Quantum Electronics, *49*(285) (2017), 1–10.

[36] A. Yokus, H. M. Baskonus, T. A. Sulaiman, and H. Bulut, *Numerical simulation and solutions of the two-component second order KdV evolutionary system*, Numerical Methods for Partial Differential Equations, *34*(1) (2017), 211–227.